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NEW HYBRID GENERALIZED WEAKLY CONTRACTIVE MAPPINGS

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Abstract. The aim of this paper, we introduce the notion of new hybrid generalized weakly contractive mappings in a complete metric spaces and prove the existence and unique common fixed point for this mappings. In addition, an example given to illustrate the main result. Finally, we give some applications of our results to some fixed point results.

Keywords: hybrid generalized weakly contractive mappings; metric spaces; common fixed point.

2010 AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

The existence of solution for some real world problems has been checked in various branches of mathematics, such as, differential equations, integral equations, functional analysis, etc. and one has introduced solutions for this problems via fixed point theory. Furthermore, the application of fixed point theory is not only limited to mathematics, but also occur in various sciences, such as, computer science, physics, chemistry, biology, economics etc. Especially, the branch

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of economic which apply techniques of fixed point theory approaches to solve some equilibrium problems in game theory.

The concept of weak contraction mappings in Hilbert spaces was introduced by Alber et al [1] in 1977. Weak contraction principle states that all weak contraction mapping on a complete Hilbert space has a unique fixed point. In 2001, Rhoades [2] studied weak contraction principle in metric spaces. In addition, the weak contraction principle was studied by various researcher see in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Furthermore, in 1984, Khan et al. [16] introduced the concept of altering distance functions on metric spaces. Later, Choudhury et al. [17] obtained a generalization of the weak contraction principle in metric spaces by using altering distance functions. In 1994, Matthews [18] introduced the notion of partial metric spaces and extended Banach's contraction principle to partial metric spaces. In particular, Abdeljawad [19] obtained the result of the weak contraction principle in partial metric spaces. Recently, Cho [20] introduced the notion of generalized weakly contractive mappings in metric spaces and prove a fixed point theorem for generalized weakly contractive mappings on complete metric spaces. Later, in 2020, Xue [21] introduced the notion of hybrid generalized weakly contractive mappings and proved the existence and unique common fixed point for this mappings.

Motivated and inspired by the work of Cho [20] and Xue [21], we introduce the notion of a new hybrid generalized weakly contractive mappings in a complete metric spaces and prove the existence and unique common fixed point for this mappings.

2. Preliminaries

In this section, we give some definitions and Lemma for use in this paper as follows. Let X be a metric space. A function $f : X \to [0,\infty)$ is called lower semicontinuous if, for all $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

We denote $\mathscr{F}$ the sets of functions $F : [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(1) $F(0) = 0$ and $F(t) > 0$ for each $t > 0$;

(2) F is continuous.

Also, we denote Ψ and Φ the sets of functions $\psi, \phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions, respectively

(1) $\psi(t) = \phi(t) = 0$ if and only if $t = 0$;
(2) $\psi(t), \phi(t) > 0$ for all $t > 0$;
(3) $\liminf_{\tau\to t} \psi(\tau)$ and $\limsup_{\tau\to t} \phi(\tau)$ exist for all $t > 0$.

Let

$$\Psi = \{\psi : [0,\infty) \to [0,\infty) \mid \psi \text{ is continuous and } \psi(t) = 0 \leftrightarrow t = 0\}.$$

Also, we denote

$$\Phi = \{\phi : [0,\infty) \to [0,\infty) \mid \phi \text{ is continuous and } \phi(t) = 0 \leftrightarrow t = 0\}.$$

Lemma 2.1. [22] *If a sequence $\{x_n\}$ in X is not Cauchy, then there exist $\varepsilon > 0$ and two subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is smallest index for which $m(k) > n(k) > k$,*

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{2.1}$$

and

$$d(x_{m(k)-1}, x_{n(k)}) < \varepsilon. \tag{2.2}$$

Moreover, suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then we have:

(1) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
(2) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon$;
(3) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$;
(4) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

3. Main Results

From the work of Cho [20], let X be a metric space with metric d, let $T : X \to X$ and let $\varphi : X \to [0,\infty)$ be a lower semicontinuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi)), \tag{3.1}$$

where $\psi \in \Psi, \phi \in \Phi$,

$$\begin{aligned} m(x,y,d,T,\varphi) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), \\ &\quad d(y,Ty)+\varphi(y)+\varphi(Ty), \frac{1}{2}[d(x,Ty)+\varphi(x)+\varphi(Ty) \\ &\quad +d(y,Tx)+\varphi(y)+\varphi(Tx)]\} \end{aligned}$$

and

$$l(x,y,d,T,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty)\}.$$

Also, from the work of Xue [21], let (X,d) be a complete metric space and $S,T : X \to X$ be two self-mappings satisfying

$$\varphi(F(d(Sx,Ty))) \leq \psi(F(M(x,y))), \tag{3.2}$$

for all $x,y \in X$, where

(1) $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\}$;
(2) $F \in \mathscr{F}, \psi \in \Psi, \varphi \in \Phi$ with $\varphi(t) > \psi(t)$ for $t > 0$;
(3) $\liminf_{\tau\to t} \varphi(t) > \limsup_{\tau\to t} \psi(t)$ for $t > 0$.

Motivated and inspired by (3.1) and (3.2), we introduce the notion of new hybrid generalized weakly contractive mappings in a complete metric spaces and prove the existence and unique common fixed point for this mappings as follows. Let (X,d) be a complete metric space and let $\psi, \phi : X \to [0,\infty)$ be a lower semicontinuous function. Then $S,T : X \to X$ are called a new hybrid generalized weakly contractive mappings if it satisfies following condition:

$$\psi(F(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty))) \leq \psi(F(m_{S,T}(x,y,\varphi))) - \phi(F(l_{S,T}(x,y,\varphi))), \tag{3.3}$$

where $F \in \mathscr{F}, \psi \in \Psi, \varphi \in \Phi$,

$$\begin{aligned} m_{S,T}(x,y,\varphi) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Sx)+\varphi(x)+\varphi(Sx), \\ &\quad d(y,Ty)+\varphi(y)+\varphi(Ty), \frac{1}{2}[d(x,Ty)+\varphi(x)+\varphi(Ty) \\ &\quad +d(y,Sx)+\varphi(y)+\varphi(Sx)]\} \end{aligned} \tag{3.4}$$

and

$$(3.5)\qquad \begin{aligned} l_{S,T}(x,y,\varphi) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Sx)+\varphi(x)+\varphi(Sx), \\ &\quad d(y,Ty)+\varphi(y)+\varphi(Ty)\}. \end{aligned}$$

Theorem 3.1. *Let (X,d) be a complete metric space and S,T be two new hybrid generalized weakly contractive mappings* (3.3) *satisfying,*

(i) $F\in\mathscr{F},\psi\in\Psi,\varphi\in\Phi$ *with* $\varphi(t)>\psi(t)$ *for* $t>0$;

(iii) $\liminf_{\tau\to t}\varphi(t)>\limsup_{\tau\to t}\psi(t)$ *for* $t>0$.

Then there exists a unique common fixed point of S and T.

Proof. Let $x_0\in X$ be a fixed point and define a sequence $\{x_n\}$ as follows $x_{2n+2}=Tx_{2n+1}$ and $x_{2n+1}=Sx_{2n}$ for all $n\geq 0$. If there exists N such that $x_{2N+1}=Sx_{2N}=x_{2N}$ and $x_{2N+2}=Tx_{2N+1}=x_{2N+1}$. The proof is finished. Now, we assume that $x_{2n}\neq x_{2n+1}$ for all $n=0,1,2,....$ From (3.4) with $x=x_{2n-1}$ and $y=x_{2n}$ we have

$$\begin{aligned} & m_{S,T}(x_{2n-1},x_{2n},\varphi) \\ = \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}), d(x_{2n-1},Sx_{2n-1})+\varphi(x_{2n-1})+\varphi(Sx_{2n-1}), \\ & d(x_{2n},Tx_{2n})+\varphi(x_{2n})+\varphi(Tx_{2n}), \frac{1}{2}[d(x_{2n-1},Tx_{2n})+\varphi(x_{2n-1})+\varphi(Tx_{2n}) \\ & +d(x_{2n},Sx_{2n-1})+\varphi(x_{2n})+\varphi(Sx_{2n-1})]\}. \end{aligned}$$

Since

$$\begin{aligned} & \frac{1}{2}[d(x_{2n-1},Tx_{2n})+\varphi(x_{2n-1})+\varphi(Tx_{2n})+d(x_{2n},Tx_{2n-1})+\varphi(x_{2n})+\varphi(Tx_{2n-1})] \\ = \; & \frac{1}{2}[d(x_{2n-1},x_{2n+1})+\varphi(x_{2n-1})+\varphi(x_{2n+1})+d(x_{2n},x_{2n})+\varphi(x_{2n})+\varphi(x_{2n})] \\ \leq \; & \frac{1}{2}[d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n})+d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})] \\ \leq \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n})+d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\}, \end{aligned}$$

we get

$$
\begin{aligned}
(3.6)\qquad & m_{S,T}(x_{2n-1},x_{2n},\varphi) \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n-1},Sx_{2n-1})+\varphi(x_{2n-1})+\varphi(Sx_{2n-1}), \\
& d(x_{2n},Tx_{2n})+\varphi(x_{2n})+\varphi(Tx_{2n})\} \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}), \\
& d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\} \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\}.
\end{aligned}
$$

For the term of $l_{S,T}(x,y,\varphi)$ in (3.5), we have

$$
\begin{aligned}
(3.7)\qquad & l_{S,T}(x,y,\varphi) \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n-1},Sx_{2n-1})+\varphi(x_{2n-1})+\varphi(Sx_{2n-1}), \\
& d(x_{2n},Tx_{2n})+\varphi(x_{2n})+\varphi(Tx_{2n})\} \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}), \\
& d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\} \\
= \; & \max\{d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\}.
\end{aligned}
$$

It follows from (3.3) that

$$
\begin{aligned}
(3.8)\qquad & \psi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}))) \\
= \; & \psi(F(d(Tx_{2n-1},Sx_{2n})+\varphi(Tx_{2n-1})+\varphi(Sx_{2n}))) \\
\leq \; & \psi(F(m_{S,T}(x_{2n-1},x_{2n},\varphi)))-\phi(F(l_{S,T}(x_{2n-1},x_{2n},\varphi))).
\end{aligned}
$$

If $d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n})<d(x_{2n},x_{2n+})+\varphi(x_{2n})+\varphi(x_{2n+1})$ for some positive integer n, then

$$
\begin{aligned}
& F(d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n})) \\
< \; & F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}))
\end{aligned}
$$

and by (3.8), we get

$$\begin{aligned}&\psi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})))\\ \leq\ &\psi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})))\\ &-\phi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}))),\end{aligned}$$

which implies

$$\phi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})))=0,$$

so

$$F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1}))=0,$$

and

$$d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})=0.$$

Hence, $x_{2n}=x_{2n+1}$ and $\varphi(x_{2n})=\varphi(x_{2n+1})=0$ which is contraction.
Thus, we have

$$\begin{aligned}&d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})\\ \leq\ &d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}),\end{aligned} \tag{3.9}$$

for all $n=1,2,3,...$ and by (3.6) and (3.7), we obtain

$$m_{S,T}(x_{2n-1},x_{2n},\varphi)=d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}) \tag{3.10}$$

and

$$l_{S,T}(x_{2n-1},x_{2n},\varphi)=d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}), \tag{3.11}$$

for all $n=1,2,3,....$
It follows from (3.8) that

$$\begin{aligned}&\psi(F(d(x_{2n},x_{2n+1})+\varphi(x_{2n})+\varphi(x_{2n+1})))\\ \leq\ &\psi(F(d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n})))\\ &-\phi(F(d(x_{2n-1},x_{2n})+\varphi(x_{2n-1})+\varphi(x_{2n}))).\end{aligned} \tag{3.12}$$

It follows from (3.9) that the sequence $\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$ is a monotone nonincreasing. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty}[d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})] = r. \tag{3.13}$$

We claim that $r=0$. Otherwise, $r>0$. By (3.12) we have

$$\begin{aligned} & \psi(F(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))) \\ \leq\ & \psi(F(d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n))) \\ & -\phi(F(d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n))), \end{aligned} \tag{3.14}$$

which implies that

$$\begin{aligned} & \sup_{i\geq n}(\psi(F(d(x_i,x_{i+1})+\varphi(x_i)+\varphi(x_{i+1})))) \\ \leq\ & \sup_{i\geq n}(\psi(F(d(x_{i-1},x_i)+\varphi(x_{i-1})+\varphi(x_i)))) \\ & -\inf_{i\geq n}(\phi(F(d(x_{i-1},x_i)+\varphi(x_{i-1})+\varphi(x_i)))). \end{aligned} \tag{3.15}$$

Then taking limit as $n\to\infty$ on (3.15), by the continuity of ψ and the lower semicontinuity of ϕ it follows that

$$0 < \limsup_{t\to r}\psi(F(t)) \leq \limsup_{t\to r}\psi(F(t)) - \liminf_{t\to r}\phi(F(t)),$$

which implies that

$$0 < \psi(F(r)) \leq \psi(F(r)) - \phi(F(r)).$$

Since $r>0$, we have $\phi(r)>0$. Hence

$$\psi(F(r)) \leq \psi(F(r)) - \phi(F(r)) < \psi(F(r)),$$

a contradiction. Hence $\lim_{n\to\infty}[d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})]=0$, which implies that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \tag{3.16}$$

and

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{3.17}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not Cauchy, then by Lemma 2.1 there exist $\varepsilon > 0$ and subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that (2.1) and (2.2) hold.
From (3.4), we have

$$\begin{aligned}(3.18)\qquad & m_{S,T}(x_{n(k)},x_{m(k)},\varphi)\\ = \;& \max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{n(k)},Sx_{n(k)})+\varphi(x_{n(k)})\\ & +\varphi(Sx_{n(k)}),d(x_{m(k)},Tx_{m(k)})+\varphi(x_{m(k)})+\varphi(Tx_{m(k)})+\frac{1}{2}[d(x_{n(k)},Tx_{m(k)})\\ & +\varphi(x_{n(k)})+\varphi(Tx_{m(k)})+d(x_{m(k)},Sx_{n(k)})+\varphi(x_{m(k)})+\varphi(Sx_{n(k)})]\}\\ = \;& \max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})\\ & +\varphi(x_{n(k)+1}),d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1})+\frac{1}{2}[d(x_{n(k)},x_{m(k)+1})\\ & +\varphi(x_{n(k)})+\varphi(x_{m(k)+1})+d(x_{m(k)},x_{n(k)+1})+\varphi(x_{m(k)})+\varphi(x_{n(k)+1})]\}.\end{aligned}$$

Letting $k\to\infty$ in (3.18) and using Lemma 2.1, (3.16) and (3.17), it follows that

$$m_{S,T}(x_{n(k)},x_{m(k)},\varphi)=\max\{\varepsilon,0,\frac{1}{2}[0+0]\}=\varepsilon. \tag{3.19}$$

It follows from (3.5) that

$$\begin{aligned}& l_{S,T}(x_{n(k)},x_{m(k)},\varphi)\\ = \;& \max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{n(k)},Sx_{n(k)})+\varphi(x_{n(k)})\\ & +\varphi(Sx_{n(k)}),d(x_{m(k)},Tx_{m(k)})+\varphi(x_{m(k)})+\varphi(Tx_{m(k)})\}\\ = \;& \max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})\\ & +\varphi(x_{n(k)+1}),d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1})\}.\end{aligned}$$

Hence

$$l_{S,T}(x_{n(k)},x_{m(k)},\varphi)=\max\{\varepsilon,0,0\}=\varepsilon. \tag{3.20}$$

From (3.3) we have

$$\begin{aligned}(3.21)\qquad & \psi(F(d(Sx_{n(k)},Tx_{m(k)})+\varphi(Sx_{n(k)})+\varphi(Tx_{m(k)})))\\ \leq\; & \psi(F(m_{S,T}(x_{n(k)},x_{m(k)},\varphi)))-\phi(F(l_{S,T}(x_{n(k)},x_{m(k)},\varphi))).\end{aligned}$$

Letting $k \to \infty$ in (3.22), by Lemma 2.1, the continuity of ψ, the lower semicontinuity of ϕ, and (3.19), (3.20) and (3.22), we have

$$\psi(F(\varepsilon)) \leq \psi(F(\varepsilon)) - \phi(F(\varepsilon)) < \psi(F(\varepsilon)),$$

which is contradiction because $\phi(F(\varepsilon)) > 0$.

Hence, the sequence $\{x_n\}$ is Cauchy sequence and hence it is convergent by the completeness of X. Denote $\lim_{n\to\infty} x_n = q$. Since φ is lower semicontinuos,

$$\varphi(q) \leq \liminf_{n\to\infty} \varphi(x_n) \leq \lim_{n\to\infty} \varphi(x_n) = 0,$$

which implies that

$$\varphi(q) = 0. \tag{3.22}$$

Finally, we prove that q is a unique common fixed point of S and T.

It follows from (3.4) that

$$\begin{aligned} & m_{S,T}(x_n, q, \varphi) \\ = \;& \max\{d(x_n, q) + \varphi(x_n) + \varphi(q), d(x_n, Sx_n) + \varphi(x_n) \\ & +\varphi(Sx_n), d(q, Tq) + \varphi(q) + \varphi(Tq) + \frac{1}{2}[d(x_n, Tq) \\ & +\varphi(x_n) + \varphi(Tq) + d(q, Sx_n) + \varphi(q) + \varphi(Sx_n)]\}. \end{aligned} \tag{3.23}$$

So, we have

$$\begin{aligned} \lim_{n\to\infty} m_{S,T}(x_n, q, \varphi) &= d(q, Tq) + \varphi(q) + \varphi(Tq) \\ &= d(q, Tq) + \varphi(Tq). \end{aligned} \tag{3.24}$$

Also, we have

$$\begin{aligned} & l_{S,T}(x_n, q, \varphi) \\ = \;& \max\{d(x_n, q) + \varphi(x_n) + \varphi(q), d(x_n, Sx_n) + \varphi(x_n) + \varphi(Sx_n), \\ & d(q, Tq) + \varphi(q) + \varphi(Tq)\}, \end{aligned} \tag{3.25}$$

so

$$\lim_{n\to\infty} l_{S,T}(x_n,q,\varphi) = d(q,Tq)+\varphi(q)+\varphi(Tq) = d(q,Tq)+\varphi(Tq). \tag{3.26}$$

It follows from (3.3) that

$$\begin{aligned} &\psi(F(d(x_{n+1},Tq)+\varphi(x_{n+1})+\varphi(Tq))) \\ = &\ \psi(F(d(Sx_n,Tq)+\varphi(Sx_n)+\varphi(Tq))) \\ \leq &\ \psi(F(m_{S,T}(x_n,q,\varphi)))-\phi(F(l_{S,T}(x_n,q,\varphi))). \end{aligned} \tag{3.27}$$

By taking the limit as $n\to\infty$ in (3.27) and using the continuity of ψ, the lower semicontinuity of ϕ, (3.24) and (3.26), we have

$$\begin{aligned} &\psi(F(d(q,Tq)+\varphi(Tq))) \\ \leq &\ \psi(F(d(q,Tq)+\varphi(Tq)))-\phi(F(d(q,Tq)+\varphi(Tq))). \end{aligned} \tag{3.28}$$

So, we have

$$\phi(F(d(q,Tq)+\varphi(Tq)))=0,$$

which implies that

$$F(d(q,Tq)+\varphi(Tq))=0,$$

or

$$d(q,Tq)+\varphi(Tq)=0.$$

Hence, we get $d(q,Tq)=0$ implies that $q=Tq$ and $\varphi(Tq)=0$.

From (3.23) and (3.25), we get

$$\begin{aligned} &m_{S,T}(q,q,\varphi) \\ = &\ \max\{d(q,q)+\varphi(q)+\varphi(q), d(q,Sq)+\varphi(q)+\varphi(Sq), d(q,Tq)+\varphi(q)+\varphi(Tq), \\ &\ \frac{1}{2}[d(q,Tq)+\varphi(q)+\varphi(Tq)+d(q,Sq)+\varphi(q)+\varphi(Sq)]\} \\ = &\ d(q,Sq)+\varphi(q)+\varphi(Sq) \end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
(3.30)\qquad & l_{S,T}(q,q,\varphi) \\
= \ & \max\{d(q,q)+\varphi(q)+\varphi(q), d(q,Sq)+\varphi(q)+\varphi(Sq), d(q,Tq)+\varphi(q)+\varphi(Tq)\} \\
= \ & d(q,Sq)+\varphi(q)+\varphi(Sq).
\end{aligned}$$

Suppose that $Sq = q$. By (3.21), (3.29) and (3.30), we obtain

$$\begin{aligned}
(3.31)\qquad 0 &< \psi(F(d(Sq,q)+\varphi(Sq)+\varphi(q))) \\
&= \psi(F(d(Sq,Tq)+\varphi(Sq)+\varphi(Tq))) \\
&\leq \psi(F(m_{S,T}(q,q,\varphi))) - \phi(F(l_{S,T}(q,q,\varphi))) \\
&= \psi(F(d(q,Sq)+\varphi(q)+\varphi(Sq))) - \phi(F(d(q,Sq)+\varphi(q)+\varphi(Sq))) \\
&< \psi(F(d(q,Sq)+\varphi(q)+\varphi(Sq))),
\end{aligned}$$

which is contradiction. This $q = Sq = Tq$. For uniqueness, we assume that there exists another point $p \in X$ such that $Tp = Sp = p \neq q = Sq = Tq$. So $d(p,q) \neq 0$. Observe that

$$\begin{aligned}
0 &< \psi(F(d(p,q)+\varphi(p)+\varphi(q))) \\
&= \psi(F(d(Sp,Tq)+\varphi(Sp)+\varphi(Tq))) \\
&\leq \psi(F(m_{S,T}(p,q,\varphi))) - \phi(F(l_{S,T}(p,q,\varphi))) \\
&= \psi(F(d(p,q)+\varphi(p)+\varphi(q))) - \phi(F(d(p,q)+\varphi(p)+\varphi(q))) \\
&< \psi(F(d(p,q)+\varphi(p)+\varphi(q))),
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &< \psi(F(d(p,q))) \\
&\leq \psi(F(d(p,q))) - \phi(F(d(p,q))) \\
&< \psi(F(d(p,q))).
\end{aligned}$$

This is contradiction. Hence $p = q$. The proof is completed. □

Corollary 3.1. *Let (X,d) be a complete metric space and Suppose that S,T satisfies the following condition:*

$$\begin{aligned}&\psi(F(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty)))\\ \leq\ &\psi(F(m_{S,T}(x,y,\varphi)))-\phi(F(m_{S,T}(x,y,\varphi))),\end{aligned}$$

for all $x,y\in X$, $F\in\mathscr{F}$, $\psi\in\Psi$ and $\varphi\in\Phi$. Then there exists a unique common fixed point of S and T.

Corollary 3.2. *Let (X,d) be a complete metric space and Suppose that S,T satisfies the following condition:*

$$\begin{aligned}&\psi(F(d(Sx,Ty)+\varphi(Sx)+\varphi(Ty)))\\ \leq\ &\psi(F(d(x,y)+\varphi(x)+\varphi(y)))-\phi(F(d(x,y)+\varphi(x)+\varphi(y))),\end{aligned}$$

for all $x,y\in X$, $F\in\mathscr{F}$, $\psi\in\Psi$ and $\varphi\in\Phi$. Then there exists a unique common fixed point of S and T.

Corollary 3.3. *Let (X,d) be a complete metric space and Suppose that S,T satisfies the following condition:*

$$\begin{aligned}&\psi(F(d(S^kx,T^ky)+\varphi(S^kx)+\varphi(T^ky)))\\ \leq\ &\psi(F(m_{S^k,T^k}(x,y,\varphi)))-\phi(F(l_{S^k,T^k}(x,y,\varphi))),\end{aligned}$$

for all $x,y\in X$, $F\in\mathscr{F}$, $\psi\in\Psi$, $\varphi\in\Phi$ and k is a positive integer. Then there exists a unique common fixed point of S and T.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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