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OPERATOR ALGEBRAS OF COUNTABLE RANGE MULTIPLICITY

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Abstract. In this paper, the spectrum of operators commuting with operator algebras of countable range multiplicity is studied. It is shown that if the commutant of a set which does not contain any scalar operator has countable range multiplicity then it has a non trivial invariant subspace. If the range multiplicity of an operator algebra is one then it is shown that the strong and uniform topologies coincide on the commutant of the algebra and also each collection of mutually orthogonal projections in the commutant is finite. In addition, if the operator algebra is self adjoint also then it is shown that the underline Hilbert space has a finite orthogonal decomposition such that each of its components reduces the algebra.

Keywords: strictly cyclic; range cyclic; algebra of countable range multiplicity; algebra of finite range multiplicity; algebra of finite strict multiplicity; orthogonal decomposition; resolution of identity; point evaluation.

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1. INTRODUCTION

By an operator range in a Hilbert space, we mean the range of a bounded linear operator on the space. Every closed linear space is always an operator range but not conversely. Also every operator range is a linear manifold but not conversely. Fillmore and Williams [3] have done a detailed study of operator ranges.

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In 1972, Foias [4] initiated the study of invariant operator ranges. The study was further continued by Jafarian and Radjavi [8] and others. In 1982, Harrison, Rosenthal and Longstaff [5] introduced the notion of countable range multiplicity, finite range multiplicity and range cyclicity. These notions are generalizations of strict cyclicity introduced by Lambert [9] and finite strict multiplicity introduced by Herrero [6]. The purpose of this paper is to continue the study of countable range multiplicity and in particular range cyclicity of operator algebras. The motivation of the present work comes from the investigations made by Herrero [6, 7] and Embry [1, 2] for strictly cyclic and cyclic algebras.

2. Preliminaries

Let H be a separable Hilbert space and $B(H)$, the set of all bounded linear operators on H. By an algebra, we mean a strongly closed subalgebra of $B(H)$ containing identity I.

Definition 2.1. Algebra $\mathscr{A}$ is said to be of countable (finite) range multiplicity [5] if there exists a countable (finite) subset Γ of H such that the only operator range invariant under $\mathscr{A}$ containing Γ is H itself. $\mathscr{A}$ is said to be range cyclic if there exists such a subset Γ consisting of a single element e, say. In that case e is said to be range cyclic vector for $\mathscr{A}$.

Definition 2.2. Algebra $\mathscr{A}$ is said to be cyclic (strictly cyclic) [9] if there exists a vector x_0 in H such that the only invariant subspace (linear manifold) containing x_0 is H itself. In this case vector x_0 is called cyclic (strictly cyclic) vector for $\mathscr{A}$.

In the same way algebras of finite multiplicity (finite strict multiplicity) [6] are defined. Every strictly cyclic vector for $\mathscr{A}$ is range cyclic for $\mathscr{A}$ and every range cyclic vector for $\mathscr{A}$ is cyclic for $\mathscr{A}$ [11]. However the converse is not true.

For a uniformly closed abelian unital (containing identity I) algebra, range cyclic vectors are strictly cyclic also.Thus the study of range cyclic operators coincides with the study of strictly cyclic operators.

Definition 2.3. For any $\mathscr{B} \subseteq B(H)$, the commutant of $\mathscr{B}$ is

$$\mathscr{B}^{'} = \{T \in B(H) : TB = BT \ for\, B\, in\, \mathscr{B}\}.$$

The double commutant $\mathscr{B}^{''}$ of $\mathscr{B}$ is the commutant of $\mathscr{B}^{'}$.

Definition 2.4. A closed linear subspace M of H reduces $\mathscr{B}$ if the projection of H onto M is in $\mathscr{B}^{'}$. Reducing subspace M of $\mathscr{B}$ is said to be minimal reducing subspace of $\mathscr{B}$ if $M \neq \{0\}$ and $\{0\}$ is the only reducing subspace of $\mathscr{B}$ properly contained in M.

Definition 2.5. A collection $\{M_j\}$ of closed linear subspaces of H is an orthogonal decomposition of H if and only if $M_j^{\prime}s$ are pairwise orthogonal and span H.

Definition 2.6. A collection $\{P_j\}_{j=1}^{n}$ of projections is a resolution of identity if the collection $\{P_j(H)\}_{j=1}^{n}$ of ranges of P_j is an orthogonal decomposition of H.

3. Main Results

Theorem 3.1. *If $\mathscr{A}$ is a range cyclic algebra, then the strong and the uniform topologies coincide on $\mathscr{A}^{'}$, the commutant of $\mathscr{A}$.*

Proof. Let e be a range cyclic vector for $\mathscr{A}$. The point evaluation E_e mapping $B(H)$ to H defined by $E_e(B) = Be$ is bounded below on $\mathscr{A}^{'}$ [10]. Thus there exists a constant $K > 0$ such that

$$\|E_eB\| \geq K\|B\|$$

for all B in $\mathscr{A}^{'}$. Let $\{A_\lambda\}$ be a set in $\mathscr{A}^{'}$ converging strongly to A in $\mathscr{A}^{'}$. Then

$$\|A_\lambda - A\| \leq \frac{1}{K}\|(A_\lambda - A)e\| \to 0$$

showing that $\{A_\lambda\}$ converges to A uniformly.

Applying Theorem 3.1 to $\mathscr{A}^{'}$, we obtain the following.

Corollary 3.2. *If $\mathscr{A}$ is a subalgebra of $B(H)$ such that $\mathscr{A}^{'}$ is range cyclic, then the strong and the uniform topologies coincide on $\mathscr{A}$.*

Theorem 3.3. *Let $\mathscr{A}$ be an algebra of countable range multiplicity on H. Let $E \in \mathscr{A}^{'}$ and $z \in \sigma(E)$, the spectrum of E. Then either $Ker(zI - E) \neq 0$ or $\overline{R(zI - E)} \neq H$.*

Proof. Let Γ be countable subset of H such that the only operator range invariant under $\mathscr{A}$ containing Γ is H itself. Let $Ker(zI - E) = \{0\}$. Then $zI - E$ is one -one and so it is not onto. As $R(zI - E)$ is an operator range invariant under $\mathscr{A}$, the set Γ cannot be contained in $R(zI - E)$. Thus $R(zI - E) \neq H$. By [[10], Corollary 1], $\overline{R(zI - E)} \neq H$.

Corollary 3.4. *If A is not a scalar multiple of I and $\{A\}^{'}$ has countable range multiplicity, then A has a non- trivial hyper - invariant subspace.*

Proof. Let $z \in \sigma(A)$. Applying Theorem 3.3 to $\{A\}^{'}$, we get that either $Ker(zI-A) \neq 0$ or $\overline{R(zI-A)} \neq H$ and each is invariant under $\{A\}^{'}$.

Corollary 3.5. *If $\mathscr{B}$ is a subset of $B(H)$ not consisting of scalar operators and $\mathscr{B}^{'}$ has countable range multiplicity then $\mathscr{B}^{'}$ has a non - trivial invariant subspace.*

Proof. There exists A in $\mathscr{B}$ such that A is not a scalar multiple of identity. As $\mathscr{B}^{'} \subseteq \{A\}^{'}$, we get that $\{A\}^{'}$ has countable range multiplicity. By Corollary 3.4, $\{A\}^{'}$ has a non - trivial invariant subspace and that is invariant under $\mathscr{B}^{'}$ also.

Corollary 3.6. *If $\mathscr{A}$ is an algebra of countable range multiplicity on H and $E \in \mathscr{A}^{'}$, then*

(i) $\sigma(E) = \sigma_p(E) \cup \overline{\sigma_p(E^*)}$

(ii) $\partial\sigma(E) \subset \overline{\sigma_p(E^*)}$,

where $\overline{\sigma_p(E^*)} = \left\{\bar{\lambda} : \lambda \in \sigma_p(E^*)\right\}$.

Proof.

(i) Let $\lambda \in \sigma(E)$. By Theorem 3.3, either

$$Ker(\lambda I - E) \neq 0$$

or

$$Ker(\bar{\lambda} I - E^*) = \overline{R(\lambda I - E)}^{\perp} \neq \{0\}$$

This implies that either $\lambda \in \sigma_p(E)$ or $\lambda \in \overline{\sigma_p(E^*)}$. Thus

$$\sigma(E) = \sigma_p(E) \cup \overline{\sigma_p(E^*)}$$

(ii) Let $\lambda \in \sigma(E)/\ \overline{\sigma_p(E^*)}$. Then λ is in spectrum of E and $Ker(\bar{\lambda} I - E^*) = \{0\}$. This implies that $\overline{R(\lambda I - E)} = H$. As $R(\lambda I - E)$ is invariant under $\mathscr{A}$, by [[10], Corollary 1], $R(\lambda I - E) = H$. This cannot happen if λ is a boundary point of $\sigma(E)$. Thus $\partial\sigma(E) \subset \overline{\sigma_p(E^*)}$.

Theorem 3.7. *Let $\mathscr{A}$ be a range cyclic algebra on H. Each collection of mutually orthogonal projections in $\mathscr{A}^{'}$ is finite.*

Proof. Let $\{P_j\}$ be a collection of mutually orthogonal projections in $\mathscr{A}'$. Without any loss of generality, we may assume $\{P_j\}$ to be countable. Let

$$Q_n = \sum_{j=1}^{n} P_j \quad \text{and} \quad Q = \sum_{j=1}^{\infty} P_j$$

We see that $Q_n \to Q$ strongly. By Theorem 3.1, $Q_n \to Q$ uniformly. However $Q - Q_n$ is a projection and hence has norm zero or one. Thus for sufficiently large n, $Q_n = Q$. This means that $\{P_j\} = \{P_j\}_{j=1}^{n}$ is finite.

Corollary 3.8. *Let $\mathscr{A}$ be range cyclic operator algebra on H. Then each normal element of $\mathscr{A}'$ has finite spectrum.*

Proof. Let $E \in \mathscr{A}'$ be normal operator. Then E has no residual spectrum. Also by Theorem 3.3, E has no continuous spectrum. Thus spectrum of E consists entirely of point spectrum. By Theorem 3.7, E has only a finite number of distinct eigenspaces. Thus spectrum of E is finite.

Theorem 3.9. *If $\mathscr{A}$ is a self - adjoint range cyclic operator algebra on H, then there exists a finite orthogonal decomposition $\{M_k\}$ of H such that each M_k reduces $\mathscr{A}$ and $\mathscr{A}/M_k$ is strongly dense in $B(M_k)$.*

Proof. If $\mathscr{A}$ has only trivial reducing subspaces then by [[5], Corollary 1] $\mathscr{A}$ is strongly dense in $B(H)$. Thus the trivial decomposition $\{H\}$ of H satisfies the requirements of the theorem.

Let $\{M_k\}_{k=1}^{p}$ be a colletion of mutually orthogonal subspaces of H such that each M_k reduces $\mathscr{A}$ and $\mathscr{A}/M_k$ is strongly dense in $B(M_k)$. If $M_k's$ span H, the conclusion of the theorem is satisfied. Otherwise consider

$$\mathscr{A}_1 = \mathscr{A}|_{\{M_1, M_2, \ldots, M_p\}^{\perp}}.$$

By [[10], Theorem 7], $\mathscr{A}_1$ is range cyclic on $\{M_1, M_2, \ldots, M_p\}^{\perp}$. If $\mathscr{A}_1$ has only trivial reducing subspaces then again by [5], $\mathscr{A}_1$ is strongly dense in $B(\{M_1, M_2, \ldots, M_p\}^{\perp})$ and the construction is complete. Otherwise $\mathscr{A}_1$ has a non - trivial reducing subspace M_{p+1} and by [5], $\mathscr{A}_1|_{M_{p+1}}$ is strongly dense in $B(M_{p+1})$.

Thus $M_1, M_2, \ldots, M_{p+1}$ are pairwise orthogonal reducing subspaces for$\mathscr{A}$ and $\mathscr{A}|_{M_k}$ is strongly dense in $B(M_k)$ for $k = 1, 2, \ldots, p+1$. By Theorem 3.7, the construction must terminate with a finite number of pairwise othogonal reducing subspaces.

Theorem 3.7, Theorem 3.9 above and the following results are due to Embry [2] in the case of strictly cyclic operator algebras. Embry's techniques, together with Theorem 3.9 give us the following.

Theorem 3.10. *Let $\mathscr{A}$ be a self - adjoint range cyclic operator algebra. Let $\{M_k\}_{k=1}^n$ be a decomposition of H as described in the statement of Theorem* 3.9. *Let P_k be the orthogonal projection of H onto M_k. Then $\mathscr{A}' = \sum_{j,k=1}^n P_j \mathscr{A}' P_k$ and for each value of j and k, $P_j \mathscr{A}' P_k$ is of dimension one or zero. In particular $\mathscr{A}'$ is finite dimensional.*

Corollary 3.11. *If $\mathscr{A}$ is a self - adjoint range cyclic operator algebra with an abelian commutant, then*

$$\mathscr{A}' = \left\{ \sum_{j=1}^n \lambda_j P_j : \lambda_j \text{ is complex} \right\}$$

where $\{P_j\}$ is a resolution of identity as described in the statement of Theorem 3.10. *In particular, $\mathscr{A}'$ consists of normal operators with finite spectra.*

Corollary 3.12. *Let N be a normal operator with a range cyclic commutant $\{N\}'$. Then there exist orthogonal projections $P_1, P_2, \ldots P_n$ such that*

$$\{N\}'' = \left\{ \sum \lambda_j P_j : \lambda_j \text{ complex} \right\}.$$

Corollary 3.13. *The decomposition $\{M_k\}_{k=1}^n$ as described in the statement of Theorem* 3.9 *is unique if and only if $\mathscr{A}'$ is abelian.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] M.R. Embry, Strictly cyclic operator algebras on a Banach space, Pac. J. Math. 45 (1973), 443-452.

[2] M.R. Embry, Self adjoint strictly cyclic operator algebras, Pac. J. Math. 52 (1974), 53-57.

[3] P. A. Fillmore and J.P. Williams, On operator ranges, Adv. Math. 7 (1971), 254-281.

[4] C. Foias, Invariant paraclosed subspaces, Indiana Univ. Math. J. 21 (1972), 887-906.

[5] K. J. Harrison, W.E. Longstaff and P.Rosenthal, Some tractable non self - adjoint operator algebras, J. Lon. Math. Soc. s2-26 (1982), 325-330.

[6] D.A. Herrero, Operator algebras of finite strict multiplicity, Indiana Univ. Math. J. 22 (1972), 13–24.

[7] D.A. Herrero, Operator algebras of finite strict multiplicity II, Indiana Univ. Math. J. 27 (1978), 9-18.

[8] A.A. Jafarian and H. Radjavi, Compact operator ranges and reductive algebras, Acta Sci. Math. 40 (1978), 73-79.

[9] A. Lambert, Strictly cyclic operator algebras, Pac. J. Math. 39 (1971), 717-726.

[10] W.E. Longstaff and P.Rosenthal, On operator algebras and operator ranges, Integral Equ. Oper. Theory. 9 (1986), 820-830.

[11] W.E. Longstaff, Cyclic vectors for operator algebras, Bull. Greek Math. Soc. 27 (1986), 73-83.