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SOME FIXED POINT THEOREMS IN S-METRIC SPACES WITH NEW CONTRACTIVE MAPPINGS

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Abstract: The purpose of this paper is to introduce new contractive mapping in S-metric space using new class of function. We establish some fixed point theorems in context of these new contractive mapping in S-metric spaces.

Keywords: S-metric space; fixed point; contractive mappings.

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1. INTRODUCTION

In mathematics, Banach [1] in 1922, introduced a well- known theorem which is called Banach fixed point theorem (or Banach contraction Principle). It is a beautiful mixture of analysis, topology and geometry. It guarantees the existence and uniqueness of fixed points of certain self - maps of metric spaces and provides a constructive method to find those fixed points.

Let X be a complete metric space. A mapping $T: X \to X$ is called a contraction mapping on X if there exists $q \epsilon [0, 1)$ such that $d(Tx, Ty) \leq q\, d(x, y)$, for all $x, y \epsilon X$. Then T has a unique fixed point. Because of its simplicity and usefulness, it has become very powerful tool in solving

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various problem such as nonlinear analysis, integral and differential equations, inclusions, dynamical system theory and mathematical economics. Later, in 1968, Kannan [9], proved the following fixed point theorem which is independent of the Banach contraction principle. Let (X, d) be a metric space and T be a self-map on X, if there exists $c \epsilon \left(0, \frac{1}{2}\right]$ such that $d(Tx, Ty) \leq c\, d(x, Tx) + c\, d(y, Ty)$, for all $x, y \epsilon X$. Then T has a unique fixed point.

Using the concept of Hausdorff metric, Nadler [11] proved the fixed point theorem for multi-valued contraction maps, which is the generalization of Banach contraction principle [1]. Boyd and Wong [2] extended Banach contraction Principle to the non - linear contractive mappings.

Later on, this theorem was generalized by many authors in different metric spaces and also in generalized metric spaces. To study these generalizations, we can refer to ([3], [4], [5], [6], [7], [8], [10], [12]). Sedghi et al. [15] introduce the concept of S-metric space and proved that this concept is the generalization of G-metric space [10] and D* metric space [17]. Further in [15] authors proved that the notion of S-metric space is not the generalization of G–metric space and both metric spaces are independent of each other. More details regarding this space can be found in ([12], [14], [16]).

In this paper, we define a new class of function. Furthermore, we define some new contractive mappings which combine with the terms $d(x, x, y), d(x, x, Tx), d(y, y, Ty), d(x, x, Ty), d(y, y, Tx)$ and $d(Tx, Tx, y)$ by means of the member of newly defined class and prove some fixed point theorem using this new class of functions We begin by recalling some basic definitions and results for S- metric spaces that will be needed in the sequel.

2. PRELIMINARIES

Definition 2.1 [15]: Let X be a non-empty set. An S-metric on X is a mapping $S : X^3 \rightarrow \mathbb{R}_+$ that satisfies the following condition:

(S_1) $S(x, y, z) = 0 \; if \; and \; only \; if \; x = y = z = 0;$

(S_2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ $for\ all\ x,y,z,a \in X$

The pair (X,S) is called an S-metric space.

Example 2.2: Let X = $\mathbb{R}$. $S(x,y,z) = |x-z| + |y-z|$. Then $S(x,y,z)$ is an S-metric on $\mathbb{R}$, which is known as usual S-metric space on X .

Lemma 2.3[15]: If S is an S-metric on a non-empty set X, then $S(x,x,y) = S(y,y,x)$, for all $x,y \in X$.

Definition 2.4[13]: Let (X,S) be a S-metric space. For $r > 0$ and $x \in X$, we define the open ball with center in x and radius r, the set

$B_s(x,r) = \{ y \in X : S(x,x,y) < r \}$.

The topology induced by the S-metric is the topology determined by the base of all open balls in X.

Definition 2.5[12]: A sequence $\{x_n\}$ in (X,S) is said to be convergent to x, denoted by $\lim_{n\to\infty} x_n = x$ if $x_n \to x,\ or\ S(x_n, x_n, x) \to 0\ as\ n \to \infty$.

Definition 2.6 [12]: A sequence $\{x_n\}$ in (X,S) is said to be Cauchy sequence if $S(x_n, x_n, x_m) \to 0\ as\ n,m \to \infty$.

Definition 2.7 [12]: An S-metric space(X,S) is said to be complete if every Cauchy sequence in X is convergent.

Example 2.8: Let (X,S) be S-metric space, then the usual metric space on Xdefined in Example 2.2 is complete.

Lemma 2.9([13, 14]): Let (X,S) be S-metric space. If $x_n \to x\ and\ y_n \to y$ then $S(x_n, x_n, y_n) \to S(x,x,y)$.

Lemma 2.10[16]: Let (X,S) be an S-metric space and $x_n \to x$. Then $\lim_{n\to\infty} x_n$ is unique.

Lemma 2.11([13], [15]): Every sequence$\{x_n\}$ of elements from S-metric space (X,S), having the property that there exists $\lambda \in [0, 1)$ such that $S(x_n, x_n, x_{n+1}) \leq \lambda\, S(x_{n-1}, x_{n-1}, x_n)$ for every $n \in \mathbb{N}$, is a Cauchy.

3. MAIN RESULTS

In this section, we define some new class of functions and then by defining new contractive mapping in S-metric spaces we are going to prove some fixed point theorems which are as follows:

Definition 3.1: For any $m \in \mathbb{N}$, we define Ξ_m to be the set of all functions $\eta: [0, +\infty)^m \to [0, +\infty)$ such that

(η_1) $\eta(t_1, t_2, \dots, t_m) < \max\{t_1, t_2, \dots, t_m\}$ if $(t_1, t_2, \dots, t_m) \neq (0,0, \dots, 0)$;

(η_2) $\left\{t_i^{(n)}\right\}_{n \in \mathbb{N}}$, $1 \le i \le m$, are m sequences in $[0, +\infty)$ such that $\lim_{n\to\infty} t_i^{(n)} = t_i < +\infty$ for all i = 1, 2, . . . , m, then $\lim_{n\to\infty} \inf \eta\left(t_1^{(n)}, t_2^{(n)}, \dots, t_m^{(n)}\right) \le \eta(t_1, t_2, \dots, t_m)$.

Definition 3.2: Let (X, S) be an S-metric space and $T: X \to X$ is said to be an η-contractive mapping of Type –I if there exists $\eta \in \Xi_4$ and

$$S(Tx, Tx, Ty) \le q\, \eta\left(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x,x,Ty) + S(Tx,Tx,y)}{3}\right), \quad (3.1)$$

for all $x, y \in X$ and $q \in (0, 1)$.

Theorem 3.3: Let (X, S) be a complete S-metric space and $T : X \to X$ be an η-contractive mapping of Type – I. Then T has a unique fixed point.

Proof: Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_n = Tx_{n-1}$ for all $n \ge 1$. Assume that any two consecutive terms of the sequence $\{x_n\}$ are distinct; otherwise, T has a fixed point.

Consider

$$S(x_n, x_n, x_{n+1}) \le q\eta\left(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1},x_{n-1},x_{n+1})}{3}\right) \quad (3.2)$$

$$< q \max\left\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1},x_{n-1},x_{n+1})}{3}\right\}$$

$$= q \max\left\{S(x_{n-1}, x_{n-1}, x_n), \frac{S(x_{n-1} x_{n-1},x_{n+1})}{3}\right\}$$

$$\le q \max\left\{S(x_{n-1}, x_{n-1}, x_n), \frac{2\, S(x_{n-1},x_{n-1},x_n) + S(x_n,x_n,x_{n+1})}{3}\right\} \quad (3.3)$$

Case 1: If $S(x_n, x_n, x_{n+1}) \le q\, S(x_{n-1}, x_{n-1}, x_n)$ then by Lemma 2.11 in view of (3.3), $\{x_n\}$ is a Cauchy sequence for $q \in (0, 1)$.

Case 2: If $S(x_n, x_n, x_{n+1}) \le q \left\{\frac{2\, S(x_{n-1},x_{n-1},x_n) + S(x_n,x_n,x_{n+1})}{3}\right\}$

Then,

$3S(x_n, x_n, x_{n+1}) \leq \; q\left(2\,S(x_{n-1}, x_{n-1}, x_n) + \; S(x_n, x_n, x_{n+1})\right)$, that is,

$(3-q)S(x_n, x_n, x_{n+1}) \leq \; 2\,qS(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq \frac{2q}{3-q} S(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq$ t$S(x_{n-1}, x_{n-1}, x_n)$, where $t = \frac{2q}{3-q}$ which lies between 0 to 1 for all q.

Therefore, we get

$S(x_n, x_n, x_{n+1}) \leq \; t\;\; (t\,S(x_{n-2}, x_{n-2}, x_{n-1})$

$= \; t^2 S(x_{n-2}, x_{n-2}, x_{n-1})$, that is,

$S(x_n, x_n, x_{n+1}) \leq t^n\, S(x_0, x_0, x_1)$.

By Lemma 2.11 and (S_2) it follows that $\{x_n\}$ is a Cauchy sequence in X.

As (X, S) is a complete S-metric space, it follows that sequence $\{x_n\}$ *is convergent* to some $x \in X$.Thus, $x_n \to x$ as $n \to \infty$.

Now, Consider

$$S(Tx_n, Tx_n, Tx) \leq q\,\eta\left(S(x_n, x_n, x),\; S(x_n, x_n, x_{n+1}),\; S(x, x, Tx),\; \frac{S(x_n, x_n, Tx) + S(x, x, Tx_n)}{3}\right).$$

Taking limit as $n \to \infty$, we get

$$S(x, x, Tx) \leq q\,\eta\left(S(x, x, x), S(x, x, x), S(x, x, Tx), \frac{S(x, x, Tx) + S(x, x, Tx)}{3}\right)$$

$$< q \max\left\{0,\; 0,\; S(x, x, Tx),\; \frac{2\,S(x, x, Tx)}{3}\right\}$$

$$= \quad q\,S(x, x, Tx).$$

Thus, we get, $S(x, x, Tx) < \; q\,S(x, x, Tx)$, where $q \in$ (0, 1), which is contradiction.

Therefore, $Tx = x$.

Now let $Ty = y$ for some $y \in X$ and suppose that $x \neq y$, then consider

$S(x, x, y) = S(Tx, Tx, Ty)$

$$\leq q\,\eta\left(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, Ty) + S(y, y, Tx)}{3}\right)$$

$$< q \max \left\{S(x,x,y), S(x,x,x), S(y,y,y), \frac{S(x,x,y)+S(y,y,x)}{3}\right\}$$

$$= q \max \left\{S(x,x,y),\ 0,\ 0,\ \frac{2\,S(x,x,y)}{3}\right\}$$

$$= qS(x,x,y),$$

a contradiction. Therefore, $x = y$. ■

Theorem 3.4: Let (X, S)be a complete S-metric space and $T : X \to X$ be an η-contractive mapping where $\eta \,\epsilon \Xi_4$ given by

$$S(Tx,Tx,Ty) \le \eta\left(S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{S(x,x,Ty)+S(Tx,Tx,y)}{3}\right), \tag{3.4}$$

for all $x, y \,\epsilon\, X$. Then T has a unique fixed point.

Proof: Let $x_0 \,\epsilon\, X$. Define a sequence $\{x_n\}$ in X as $x_n = Tx_{n-1}$ for all $n \ge 1$. Assume that any two consecutive terms of the sequence $\{x_n\}$ are distinct; otherwise, T has a fixed point.

Consider

$$S(x_n,x_n,x_{n+1}) \le \eta\begin{pmatrix} S(x_{n-1},x_{n-1},x_n), S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \\ \frac{S(x_{n-1},x_{n-1},x_{n+1})}{3} \end{pmatrix} \tag{3.5}$$

$$< \max \left\{S(x_{n-1},x_{n-1},x_n), S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \frac{S(x_{n-1},x_{n-1},x_{n+1})}{3}\right\}$$

$$= \max \left\{S(x_{n-1},x_{n-1},x_n), \frac{S(x_{n-1}x_{n-1},x_{n+1})}{3}\right\}$$

$$\le \max\left\{S(x_{n-1},x_{n-1},x_n), \frac{2\,S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})}{3}\right\} \tag{3.6}$$

If $\max\left\{S(x_{n-1},x_{n-1},x_n),\ \frac{2\,S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})}{3}\right\} = \frac{2\,S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})}{3}$

then $S(x_n,x_n,x_{n+1}) \le \left\{\frac{2\,S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})}{3}\right\}$.

Therefore, we have

$$S(x_n,x_n,x_{n+1}) \le S(x_{n-1},x_{n-1},x_n) \tag{3.7}$$

From (3.7), we see that $\{S(x_n,x_n,x_{n+1})\}$ is monotonically decreasing sequence and bounded below. Therefore, $S(x_n,x_n,x_{n+1}) \to k, k \ge 0$.

Now suppose that $k > 0$, taking $\lim\inf \to +\infty$ in (3.5), we have $k \le \eta(k,k,k,k')$, where

$$k' = \lim_{n\to+\infty} \sup \frac{S(x_{n-1},x_{n-1},x_{n+1})}{3}$$

$$\leq \lim_{n\to+\infty} \sup \frac{2\, S(x_{n-1},x_{n-1},x_n)+S(x_n,x_n,x_{n+1})}{3}$$

$$= \frac{2k+k}{3} = k.$$

Therefore, $k \leq \eta(k,k,k,k') < \max\{k,k,k,k'\} = k,$ a contradiction.

Therefore, we get

$$\lim_{n\to+\infty} S(x_n,x_n,x_{n+1}) = 0 \tag{3.8}$$

Now suppose $\{x_n\}$ is not a Cauchy sequence then there exist $\varepsilon > 0$ such that for any $c \, \epsilon \, N$ *there exist* $m_c > n_c \geq c$ *such that*

$$S(x_{m_c},x_{m_c},x_{n_c}) \geq \varepsilon \tag{3.9}$$

Suppose m_c is smallest natural number greater then n_c such that (3.9) holds. Then,

$$\epsilon \leq S(x_{m_c},x_{m_c},x_{n_c})$$

$$\leq 2\, S(x_{m_c},x_{m_c},x_{m_{c-1}}) + S(x_{n_c},x_{n_c},x_{m_{c-1}})$$

$$< 2\, S(x_{m_c},x_{m_c},x_{m_{c-1}}) + \epsilon.$$

Taking $\lim c \to +\infty$, we get

$$\lim_{c\to+\infty} S(x_{m_c},x_{m_c},x_{n_c}) = \varepsilon \tag{3.10}$$

Now consider,

$$S(x_{m_c},x_{m_c},x_{n_c}) \leq 2\, S(x_{m_c},x_{m_c},x_{m_{c+1}}) + S(x_{n_c},x_{n_c},x_{m_{c+1}})$$

$$\leq 2\, S(x_{m_c},x_{m_c},x_{m_{c+1}}) + \eta\begin{pmatrix} S(x_{m_c},x_{m_c},x_{n_{c-1}}), S(x_{m_c},x_{m_c},x_{m_{c+1}}), S(x_{n_{c-1}},x_{n_{c-1}},x_{n_c}), \\ \frac{S(x_{m_c},x_{m_c},x_{m_{c+1}})+S(x_{n_c},x_{n_c},x_{m_{c+1}})}{3} \end{pmatrix}.$$

Taking limit as $c \to +\infty$, on both sides and using (3.8) and (3.10) we get,

$\varepsilon \leq 2.0 + \eta(\epsilon,0,0,\epsilon'),$ where

$$\epsilon' = \lim_{c\to+\infty} \sup \frac{S(x_{m_c},x_{m_c},x_{m_{c+1}}) + S(x_{n_c},x_{n_c},x_{m_{c+1}})}{3}$$

$$\leq \lim_{c\to+\infty} \sup \frac{S(x_{m_c},x_{m_c},x_{m_{c+1}})+2S(x_{n_c},x_{n_c},x_{m_c})+S(x_{m_c},x_{m_c},x_{m_{c+1}})}{3}$$

$= \frac{0+2\varepsilon+0}{3} = \frac{2\varepsilon}{3}$.

Therefore, $\varepsilon \leq 0 + \eta\left(\varepsilon, 0, 0, \frac{2\varepsilon}{3}\right) < \max\left\{\varepsilon, 0, 0, \frac{2\varepsilon}{3}\right\} = \varepsilon$,

which is contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. As (X, S) is a complete S-metric space, it follows that sequence $\{x_n\}$ is convergent to some $x \epsilon X$.

Thus, $x_n \to x$ as $n \to \infty$.

Now, consider

$$S\ (Tx_n, Tx_n, Tx) \leq \eta\left(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \frac{S(x_n, x_n, Tx)+S(x,x,Tx_n)}{3}\right)$$

Taking limit as n→ +∞ we get

$$S(x,x,Tx) \leq \eta\left(S(x,x,x), S(x,x,x), S(x,x,Tx), \frac{S(x,x,Tx)+S(x,x,Tx)}{3}\right)$$

$$< \max\left\{0,\ 0,\ S(x,x,Tx),\ \frac{2\,S(x,x,Tx)}{3}\right\}$$

$$= S(x,x,Tx).$$

Thus, we get, $S(x,x,Tx) < S(x,x,Tx)$, which is contradiction.

Therefore, $Tx = x$.

Now let $Ty = y$ for some $y \epsilon X$ and suppose that $x \neq y$, then consider

$$S(x,x,y) = S(Tx,Tx,Ty)$$

$$\leq \eta\left(S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{S(x,x,Ty)+S(y,y,Tx)}{3}\right)$$

$$< \max\left\{S(x,x,y), S(x,x,x), S(y,y,y), \frac{S(x,x,y)+S(y,y,x)}{3}\right\}$$

$$= \max\left\{S(x,x,y),\ 0,\ 0,\ \frac{2\,S(x,x,y)}{3}\right\}$$

$$= S(x,x,y),$$

which is contradiction. Therefore, $x = y$. ■

Definition 3.5: Let (X,S) be an S-metric space. The mapping $T: X \to X$ is said to be an η-contractive mapping of Type –II if there exists $\eta \epsilon \Xi_5$ and

$$S(Tx,Tx,Ty) \leq q\,\eta\left(S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{S(x,x,Ty)}{3},\ S(Tx,Tx,y)\right), \tag{3.11}$$

for all $x, y \epsilon X$ and $q \epsilon\ (0, 1)$.

Theorem 3.6: Let (X, S) be a complete S-metric space and $T : X \to X$ be an η-contractive mapping of Type – II. Then T has a unique fixed point.

Proof: Let $x_0 \epsilon X$. Define a sequence $\{x_n\}$ *in* X as $x_n = Tx_{n-1}$ for all $n \geq 1$. Assume that any two consecutive terms of the sequence $\{x_n\}$are distinct; otherwise, T has a fixed point.

Consider

$$S(x_n, x_n, x_{n+1}) \leq \text{q } \eta \begin{pmatrix} S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \\ \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3}, S(x_n, x_n, x_{n+1}) \end{pmatrix} \tag{3.12}$$

$$< q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \right\}$$

$$= q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{S(x_{n-1} x_{n-1}, x_{n+1})}{3} \right\}$$

$$\leq q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{2\, S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3} \right\} \tag{3.13}$$

Case 1: If $S(x_n, x_n, x_{n+1}) \leq q\, S(x_{n-1}, x_{n-1}, x_n)$ then by Lemma 2.11 in view of (3.13), $\{x_n\}$ is a Cauchy sequence for all $q \epsilon (0, 1)$.

Case 2: If $S(x_n, x_n, x_{n+1}) \leq q \left\{ \frac{2\, S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3} \right\}$.

Then,

$3S(x_n, x_n, x_{n+1}) \leq q \left(2\, S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\right)$, that is,

$(3 - q)\, S(x_n, x_n, x_{n+1}) \leq 2\, qS(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq \frac{2q}{3-q} S(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq t\, S(x_{n-1}, x_{n-1}, x_n)$, where $t = \frac{2q}{3-q}$ which lies between 0 to 1 for all q.

Therefore, $S(x_n, x_n, x_{n+1}) \leq t\ (tS(x_{n-2}, x_{n-2}, x_{n-1})$

$$= t^2\, S(x_{n-2}, x_{n-2}, x_{n-1})$$

$$\vdots$$

$$\leq t^n S(x_0, x_0, x_1).$$

By Lemma 2.11 and (S_2) it follows that $\{x_n\}$ is a Cauchy sequence in X.

As (X,S) is a complete S-metric space, it follows that sequence $\{x_n\}$ is convergent to some $x \in X$. Thus, $x_n \to x$ as $n \to \infty$.

Now consider,

$$S(Tx_n, Tx_n, Tx) \leq q\,\eta\left(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \frac{S(x_n,\ x_n, Tx)}{3}, S(x, x, Tx_n)\right)$$

Taking limit as n→ ∞, we get

$$S(x, x, Tx) \leq q\,\eta\left(S(x, x, x), S(x, x, x), S(x, x, Tx), \frac{S(x, x, Tx)}{3}, S(x, x, x)\right)$$

$$< q \max\left\{0,\ 0,\ S(x, x, Tx),\ \frac{2\,S(x,x,Tx)}{3},\ 0\right\}$$

$$= \quad q\,S(x, x, Tx).$$

Thus, we get, $S(x, x, Tx) < q\,S(x, x, Tx)$ where $q \in$ (0, 1), which is contradiction.

Therefore, $Tx = x$.

Now let $Ty = y$ for some $y \in X$ and suppose that $x \neq y$, then consider

$$S(x, x, y) = S(Tx, Tx, Ty)$$

$$\leq q\,\eta\left(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x,x,Ty)}{3}, S(y, y, Tx)\right)$$

$$< q \max\left\{S(x, x, y), S(x, x, x), S(y, y, y), \frac{S(x,x,y)}{3}, S(y, y, x)\right\}$$

$$= q \max\left\{S(x, x, y),\ 0,\ 0,\ \frac{S(x,x,y)}{3}, S(x, x, y)\right\}$$

$$= qS(x, x, y),$$

which is a contradiction. Therefore, $x = y$. ■

Theorem 3.7: Let (X,S) be a complete S-metric space. Let T be a mapping $T\colon X \to X$ such that

$$S(Tx, Tx, Ty) \leq \alpha \max\{S(x, x, Tx), S(y, y, Ty), S(x, x, y)\}$$
$$+\beta\,\{S(x, x, Ty) + S(y, y, Tx)\}, \tag{3.14}$$

$\alpha, \beta > 0$ *such that* $\alpha + 3\beta < 1$ *for all* $x, y \in X$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$. Define a sequence $\{x_n\}$ *in* X as

$$x_n = Tx_{n-1} = T^n x_0, \tag{3.15}$$

for all $n \geq 1$.

By (3.14) and (3.15) we obtain that

$$S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n)$$
$$\leq \alpha \max\{S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n), S(x_{n-1}, x_{n-1}, x_n)\} +$$
$$\beta \{S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_n, Tx_{n-1})\}$$
$$= \alpha \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, x_n)\}$$
$$+ \beta \{S(x_{n-1}, x_{n-1}, x_{n+1}) + S(x_n, x_n, x_n)\}$$
$$\leq \alpha \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}$$
$$+ \beta \{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\}, \text{ that is,}$$

$$S(x_n, x_n, x_{n+1}) \leq \alpha A_1 + \beta\{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\}, \tag{3.16}$$

where $A_1 = \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}$

Now two cases arises:

Case I: Suppose that $A_1 = S(x_{n-1}, x_{n-1}, x_n)$, then from (3.16)

$S(x_n, x_n, x_{n+1}) \leq \alpha S(x_{n-1}, x_{n-1}, x_n) + 2\beta S(x_{n-1}, x_{n-1}, x_n) + \beta S(x_n, x_n, x_{n+1})$, that is,

$(1 - \beta) S(x_n, x_n, x_{n+1}) \leq (\alpha + 2\beta) S(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq \frac{\alpha+2\beta}{(1-\beta)} S(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq \kappa\, S(x_{n-1}, x_{n-1}, x_n)$, where $\kappa = \frac{\alpha+2\beta}{(1-\beta)} < 1$.

Thus, we have

$S(x_n, x_n, x_{n+1}) \leq \kappa(\kappa(S(x_{n-2}, x_{n-2}, x_{n-1}))$.

Continue this process, we get

$S(x_n, x_n, x_{n+1}) \leq \kappa^n S(x_0, x_0, x_1)$.

Case II: Suppose that $A_1 = S(x_n, x_n, x_{n+1})$, then by (3.16), we have

$S(x_n, x_n, x_{n+1}) \leq \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_{n-1}, x_{n-1}, x_n) + \beta S(x_n, x_n, x_{n+1})$, that is,

$(1 - \alpha - \beta) S(x_n, x_n, x_{n+1}) \leq 2\beta S(x_{n-1}, x_{n-1}, x_n)$, or,

$S(x_n, x_n, x_{n+1}) \leq \frac{2\beta}{1-\alpha-\beta} S(x_{n-1}, x_{n-1}, x_n)$, that is,

$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-1}, x_{n-1}, x_n)$, where $\kappa = \frac{2\beta}{1-\alpha-\beta} < 1$.

$S(x_n, x_n, x_{n+1}) \leq \kappa(\kappa(S(x_{n-2}, x_{n-2}, x_{n-1}))$.

Continue this process, we get

$S(x_n, x_n, x_{n+1}) \leq \kappa^n S(x_0, x_0, x_1)$.

Thus, from both the cases and by Lemma 2.11 it follows that $\{x_n\}$ is a Cauchy sequence in X. As (X, S) is a complete S-metric space, it follows that sequence $\{x_n\}$ is convergent to some $z \epsilon X$. That is, $x_n \to z$ as $n \to \infty$.

Now, we show that z is the fixed point of T.

$$S(z,z,Tz) \leq 2S(z,z,x_{n+1}) + S(x_{n+1}, x_{n+1}, Tz)$$
$$= 2S(z,z,x_{n+1}) + S(Tx_n, Tx_n, Tz), \text{ that is,}$$

$$S(z,z,Tz) \leq 2S(z,z,x_{n+1})$$
$$+ \alpha \max\{S(x_n, x_n, Tx_n), S(z,z,Tz), S(x_n, x_n, z)\}$$
$$+ \beta\{S(x_n, x_n, Tz) + S(z,z,Tx_n)\}$$
$$= 2S(z,z, x_{n+1}) + \alpha \max\{S(x_n, x_n, x_{n+1}), S(z,z,Tz), S(x_n, x_n, z)\}$$
$$+ \beta\{S(x_n, x_n, Tz) + S(z,z,x_{n+1})\}$$
$$\leq 2S(z,z, x_{n+1}) + \alpha \max\{S(x_n, x_n, x_{n+1}), S(z,z,Tz), S(x_n, x_n, z)\}$$
$$+ \beta\{2S(x_n, x_n, z) + S(z,z,Tz) + S(z,z,x_{n+1})\},$$

that is,

$$(1-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha A_2, \quad (3.17)$$

where $A_2 = \max\{S(x_n, x_n, x_{n+1}), S(z,z,Tz), S(x_n, x_n, z)\}$

Case I: Suppose that $A_2 = S(x_n, x_n, x_{n+1})$, then from (3.17), we get

$$(1-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha S(x_n, x_n, x_{n+1})$$
$$\leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha\{2S(x_n, x_n, z) + S(z,z,x_{n+1})\},$$

that is,

$$(1-\beta)S(z,z,Tz) \leq (2+\beta+\alpha)S(z,z,x_{n+1}) + 2(\alpha+\beta)\, S(x_n, x_n, z), \text{ that is,}$$

$$S(z,z,Tz) \leq \frac{(2+\alpha+\beta)}{(1-\beta)}\, S(z,z,x_{n+1}) + \frac{2(\alpha+\beta)}{(1-\beta)})\, S(x_n, x_n, z).$$

Taking limit as $n \to \infty$, we get

$\lim_{n\to\infty} S(z,z,Tz) = 0 \implies Tz = z.$

Therefore, z is the fixed point of T.

Case II: Suppose that $A_2 = S(z,z,Tz)$ then from (3.17)

$(1-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n,x_n,z) + \alpha S(z,z,Tz)$, that is,

$(1-\alpha-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n,x_n,z)$, that is,

$S(z,z,Tz) \leq S\frac{(2+\beta)}{(1-\alpha-\beta)}(z,z,x_{n+1}) + \frac{2\beta}{((1-\alpha-\beta))} S(x_n,x_n,z).$

Taking limit as $n \to \infty$, we get

$\lim_{n\to\infty} S(z,z,Tz) = 0 \implies Tz = z$

Therefore, z is the fixed point of T.

Case III: Suppose that $A_2 = S(x_n,x_n,z)$ then from (3.17)

$(1-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + 2\beta S(x_n,x_n,z) + \alpha S(x_n,x_n,z)$, that is,

$S(z,z,Tz) \leq \frac{(2+\beta)}{(1-\beta)}S(z,z,x_{n+1}) + \frac{2\beta+\alpha}{(1-\beta)}S(x_n,x_n,z).$

Taking limit as $n \to \infty$, we get

$\lim_{n\to\infty} S(z,z,Tz) = 0 \implies Tz = z.$

Therefore, z is the fixed point of T.

Uniqueness of the fixed point:

We have to show that z is the unique fixed point of T.

Let z' be another fixed point of T then we have $Tz' = z'$.

Now, $S(z,z,z') = S(Tz,Tz,Tz')$

$$\leq \alpha \max\{S(z,z,Tz), S(z',z',Tz'), S(z,z,Tz')\} + \beta\{S(z,z,Tz') + S(z',z',z)\}$$

$$\leq \alpha \max\{S(z,z,z), S(z',z',z'), S(z,z,z')\} + \beta\{S(z,z,z') + S(z',z',z)\}$$

$$= \alpha S(z,z,z') + 2\beta S(z,z,z')$$

$$= (\alpha+2\beta)S(z,z,z').$$

Thus, we get

$S(z,z,z') \leq (\alpha+2\beta)S(z,z,z')$, that is,

$S(z,z,z') < (1-\beta)\, S(z,z,z')$ where $q > 0$ with $\alpha + 3\beta < 1$, a contradiction.

Therefore $z = z'$.

Hence z is the unique fixed point. This completes the proof. ■

Example 3.8: Let $X = [0,1]$. Let us consider the usual S-metric on X defined as follows $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \epsilon X$. Let $Tx = \frac{x}{3}$ for all $x \epsilon [0,1]$. Then T is a self-mapping on the S-metric space $[0,1]$.

Here we have $S(Tx, Tx, Ty) = \frac{2}{3}|x - y|$ and

$$\alpha \max\{S(x,x,Tx), S(y,y,Ty), S(x,x,y)\} + \beta \{S(x,x,Ty) + S(y,y,Tx)\}$$
$$= \alpha \max\{2|x - Tx|, 2|y - Ty|, 2|x - y|\} + \beta\{2|x - Ty| + |y - Tx|\}$$

which for all value of $x, y \epsilon [0,1]$, condition (3.14) is satisfied with $\alpha + 3\beta < 1$.

Therefore, T has a unique fixed point $x = 0$.

Theorem 3.9: Let (X,S) be a complete S-metric space. Let T be a mapping $T: X \to X$ such that

$$S(Tx,Tx,Ty) \leq \alpha\{S(x,x,Tx) + S(y,y,Ty)\} + \beta\{S(x,x,Ty) + S(y,y,Tx)\}, \quad (3.18)$$

where $\alpha, \beta > 0$ such that $2\alpha + 3\beta < 1$ for all $x, y \epsilon X$. Then T has a unique fixed point.

Proof: Let $x_0 \epsilon X$. Define a sequence $\{x_n\}\ in\ X$ as

$$x_n = Tx_{n-1} = T^n x_0, \quad (3.19)$$

for all $n \geq 1$

By (3.18) and (3.19), we obtain that

$$S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n)$$
$$\leq \alpha\{S(x_{n-1}, x_{n-1}, Tx_{n-1}) + S(x_n, x_n, Tx_n)\} + \beta\{S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_n, Tx_{n-1})\}$$
$$= \alpha\{S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\} + \beta \left\{ \begin{matrix} S(x_{n-1}, x_{n-1}, x_{n+1}) \\ + S(x_n, x_n, x_n) \end{matrix} \right\}$$
$$\leq \alpha S(x_{n-1}, x_{n-1}, x_n) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_{n-1}, x_{n-1}, x_n)$$
$$+\beta\{S(x_n, x_n, x_{n+1}),$$ that is,

$$(1 - \alpha - \beta) S(x_n, x_n, x_{n+1}) \leq (\alpha + 2\beta) S(x_{n-1}, x_{n-1}, x_n),$$ or,

$$S(x_n, x_n, x_{n+1}) \leq \frac{(\alpha+2\beta)}{(1-\alpha-\beta)} S(x_{n-1}, x_{n-1}, x_n),$$ that is,

$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-1}, x_{n-1}, x_n)$, where $\kappa = \frac{(\alpha+2\beta)}{(1-\alpha-\beta)} < 1$.

Hence, by Lemma 2.11 and (S_2) it follows that $\{x_n\}$ is a Cauchy sequence in X.

As (X, S) is a complete S-metric space, it follows that sequence $\{x_n\}$ is convergent to some point $z \in X$.

That is, $x_n \to z$ as $n \to \infty$.

Now, we show that z is the fixed point of T.

$$
\begin{aligned}
S(z,z,Tz) &\leq 2S(z,z,x_{n+1}) + S(x_{n+1}, x_{n+1}, Tz) \\
&= 2S(z,z,x_{n+1}) + S(Tx_n, Tx_n, Tz) \\
&\leq 2S(z,z,x_{n+1}) + \alpha\{S(x_n, x_n, Tx_n) + S(z,z,Tz)\} \\
&\quad +\beta\{S(x_n, x_n, Tz) + S(z,z,Tx_n)\} \\
&\leq 2S(z,z,x_{n+1}) + \alpha\{S(x_n, x_n, x_{n+1}) + S(z,z,Tz)\} \\
&\quad +\beta\{S(x_n, x_n, Tz) + S(z,z,x_{n+1})\},
\end{aligned}
$$

that is,

$$
\begin{aligned}
(1-\alpha)S(z,z,Tz) &\leq (2+\beta)S(z,z,x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + \beta S(x_n, x_n, Tz) \\
&\leq (2+\beta)S(z,z,x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_n, x_n, z) + \beta S(z,z,Tz),
\end{aligned}
$$

that is,

$(1-\alpha-\beta)S(z,z,Tz) \leq (2+\beta)S(z,z,x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_n, x_n, z)$, that is,

$$
\begin{aligned}
S(z,z,Tz) \leq \frac{(2+\beta)}{(1-\alpha-\beta)} S(z,z,x_{n+1}) + \frac{\alpha}{(1-\alpha-\beta)} S(x_n, x_n, x_{n+1}) \\
+ \frac{2\beta}{(1-\alpha-\beta)} S(x_n, x_n, z)
\end{aligned}
$$

Now, taking $\lim n \to \infty$ we get

$$\lim_{n\to\infty} S(z,z,Tz) = 0 \implies Tz = z$$

Therefore, z is the fixed point of T.

Uniqueness of the fixed point:

We have to show that z is the unique fixed point of T.

Let z' be another fixed point of T then we have $Tz' = z'$.

Now, $S(z,z,z') = S(Tz,Tz,Tz')$

$$\leq \alpha\{S(z,z,Tz) + S(z',z,'Tz')\} + \beta\{S(z,z,Tz') + S(z',z',Tz)\}$$

$= \alpha\{S(z,z,z) + S(z',z,'z')\} + \beta\{S(z,z,z') + S(z',z',z)\}$, that is,

$S(z,z,z') \leq 2\beta S(z,z,z')$ with $\beta > 0$ such that $2\alpha + 3\beta < 1$, a contradiction.

Therefore $z = z'$.

Hence z is the unique fixed point. This completes the proof. ■

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Banach, Sur les operations dans les ensembles abstraits el leur application aux equations integrals, Fund. Math. 3(1992), 133–181.

[2] D. W. Boyd, S. W. Wong, On nonlinear contractions, Proc. Am. Math. Soc. 20 (1969), 458–464.

[3] M. Bukatin, R. Kopperman, S. Matthews, M. Pajoohesh, Partial metric spaces, Am. Math. Mon 116 (2009), 708–718.

[4] L. Ciric, Some recent results in metrical fixed point theory, University of Belgrade, Beograd 2003, Serbia.

[5] B. C. Dhage, Generalized metric space and mapping with fixed point, Bull. Calcutta. Math. Soc. 84 **(**1992), 329–336.

[6] B. C. Dhage, Generalized metric space and topological structure I, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Fiz. 46 (1) (2000), 3–24.

[7] T. Dosenovic, S. Radenovic, S. Sedghi, Generalized metric spaces: Survey, TWMS J. Pure Appl. Math. 9 (1) (2018), 3–17.

[8] J. Esfahani, Z. D. Mitrovic, S. Radenovic, S. Sedghi, Suzuki-type point results in S-metric type spaces, Comm. Appl. Nonlinear Anal. 25 (3) (2018), 27–36.

[9] R. Kannan, Some results on fixed point|. Am. Math. Mon. 76 (1969), 405-408.

[10] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.

[11] S. B. Naddler Jr., Multi valued contraction mapping, Pac. J. Math. 30 (1969), 475- 488.

[12] Y. N. Ozgur, N. Tas, Some fixed point theorems on S-metric spaces, Mat. Vesnik, 69 (1) (2017), 39–52.

[13] M. M. Rezaee, M. Shahraki, S. Sedghi, I. Altun, Fixed point theorems for weakly contractive mappings on S-metric spaces and a homotopy result, Appl. Math. E-Notes, 17 (2017), 1607–2510.

[14] S. Sedghi, N. V. Dung, Fixed point theorems on S-metric spaces, Mat. Vesnik, 66 (1) (2014), 113–124.

[15] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64 (2012), 258–266.

[16] S. Sedghi, N. Shobe, T. Dosenovic, Fixed point results in S-metric spaces, Nonlinear Funct. Anal. Appl. 20 (1) (2015), 55–67.

[17] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D∗-metric spaces, Fixed Point Theory Appl. 2007 (2007), Article ID 27906.