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RESULTS ON UNICITY OF MEROMORPHIC FUNCTION WITH ITS SHIFT AND q-DIFFERENCE

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Abstract. This paper is insisted to studying the problems on sharing value for the derivative of meromorphic function with its shift and q-difference. The results in this paper improve and generalize the recent results to C. Meng and G. Liu (2020).

Keywords: Nevanlinna theory; uniqueness; value sharing; meromorphic functions.

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1. Introduction

In what follows, we assume that the reader is familiar with standard notations and main results of Nevanlinna theory [37]. As usual the abbrevation CM means "Counting Multiplicity", while IM stands for "Ignoring Multiplicity".

Let f and g are two non constant meromorphic functions. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point of f with multiplicity k is counted k times in the set. If these zeros are only once counted, then we denote the set by $\overline{E}(a,f)$. If $E(a,f) = E(a,g)$, so that f and g share the value a CM, and f and g share a IM if $\overline{E}(a,f) = \overline{E}(a,g)$.

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For a complex number $a \in \mathbb{C} \cup \infty$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ for a complex number $a \in \mathbb{C} \cup \infty$ we say that f and g share the value a with weight k ([16], page 195).

The definition implies that if f and g share a value a with weight k , then z_0 is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f-a$ with multiplicity $m(> k)$ if and only if it is a zero of $g-a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f and g share (a,k) to mean that f and g share the value a with weight k. Clearly if f and g share (a,k) then f and g share (a, p) for all integer p, $0 \leq p \leq k$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.

We denotes $E_{k)}(a, f)$ the set of all a points of f with multiplicities not exceeding k, where an a point is counted accordingly and the set of distinct a points of f with multiplicities not greater than k is $\overline{E}_{k)}(a, f)$.

And $N_{k)}(r, \frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ the corresponding one for which multiplicities is not counted.

Meromorphic functions sharing values with their derivatives has become a subject of great interest in uniqueness theory. The paper by Rubel and Yang is the starting point of this topic, along with the following.

2. Preliminaries and Lemmas

Theorem 2.1. *([33], page 101) Let f be a non-constant entire function. If f and f' share two distinct finite values CM, then $f = f'$.*

The function $f = e^{e^z} \int\limits_0^z e^{-e^t}(1-e^t)dt$ from [4] shows clearly that f and f' share 1 CM but $f \neq f'$. In a special case, we recall a well-known conjecture by Brück:

Conjecture 2.1. *([4], page 22) Let f be a non-constant entire function such that hyper-order $\rho_2(f) := limsup_{r\to\infty}\frac{loglogT(r,f)}{logr}$ is not a positive integer or infinity. If f and f' share the finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is nonzero constant.*

The conjecture has been verified in the special cases when $a = 0$ [4], or when f is of finite order [12], or when $\rho_2(f) < \frac{1}{2}$ [7]. Many results have been obtained for this and related topics (See [1, 5, 11, 17, 18],[23]-[28],[34, 35, 38, 39, 41, 43],[45]-[48], and the references therein). Heittokangas et al. considered analogues of Br*ü*ck's conjecture for meromorphic functions concerning their shifts, and proved the following theorem.

Theorem 2.2. *([15], Theorem 1, page 353) Let f be a meromorphic function of order*

$$\rho(f) := \lim_{r\to\infty} sup\frac{logT(r,f)}{logr} < 2$$

and let $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share the values $a \in \mathbb{C}$ and ∞ CM, then

$$\frac{f(z+c)-a}{f(z)-a} = \tau,$$

Since then, many mathematicians considered this topic (See [6, 8, 10, 19, 22, 30, 42] and the references therein). In 2018, Qi, Li and Yang considered the value sharing problem related to $f'(z)$ and $f(z+c)$, where c is a complex number. They obtained the following result.

Theorem 2.3. *([29], Theorem 1.5, page 570) Let f be a non-constant meromorphic function of finite order and $n \geq 9$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

It is natural to ask whether the f' can be extended to $f^{(k)}$ in Theorem 2.3. Here f^n denotes the n^{th} power of the function f and $f^{(k)}$ stands for the k^{th} derivative of f, where k is a non-negative integer. Considering this question, C. Meng and G. Liu proved the following results.

Theorem 2.4. *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ \ [f^{(k)}(z)]^n \ \ and \ \ f^n(z+c) \ \ share \ \ (1,2),(\infty,0) \ \ and \ \ n \geq 2k+8;$$

$$(II)\ \ [f^{(k)}(z)]^n \ \ and \ \ f^n(z+c) \ \ share \ \ (1,2),(\infty,\infty) \ \ and \ \ n \geq 2k+7;$$

$$(III)\ \ [f^{(k)}(z)]^n \ \ and \ \ f^n(z+c) \ \ share \ \ (1,0),(\infty,0) \ \ and \ \ n \geq 3k+14;$$

then $f^{(k)}(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

If they consider entire function instead of meromorphic function, the counting functions related to the poles of $[f^{(k)}(z)]^n$ and $f^n(z+c)$ can be neglected. Arguing similarly as in Theorem 2.4, one can see that k is not related to the coefficient of $N_{k+1}\left(r, \frac{1}{f}\right)$. So obtained the following corollary.

Corollary 2.1. *Let* f *be a non-constant entire function of finite order and* $n \geq 5$ *be an integer. If* $[f^{(k)}(z)]^n$ *and* $f^n(z+c)$ *share (1,2), then* $f^{(k)}(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

If the shifts $f(z+c)$ in Theorem 2.3 and 2.4 are replaced by q-difference $f(qz)$, where $q \in \mathbb{C}\backslash\{0\}$, they obtained:

Theorem 2.5. *Let* f *be a non-constant meromorphic function of zero order and* n *be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ \ [f^{(k)}(z)]^n \ \ and\ \ f^n(qz)\ \ share\ \ (1,2),(\infty,0)\ \ and\ \ n \geq 2k+8;$$

$$(II)\ \ [f^{(k)}(z)]^n \ \ and\ \ f^n(qz)\ \ share\ \ (1,2),(\infty,\infty)\ \ and\ \ n \geq 2k+7;$$

$$(III)\ \ [f^{(k)}(z)]^n \ \ and\ \ f^n(qz)\ \ share\ \ (1,0),(\infty,0)\ \ and\ \ n \geq 3k+14;$$

then $f^{(k)}(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

Corollary 2.2. *Let* f *be a non-constant entire function of zero order and* $n \geq 5$ *be an integer. If* $[f^{(k)}(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$*, then* $f^{(k)}(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

We present some **lemmas** which will be needed later on. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right)\left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

where F and G are non-constant meromorphic functions. From above, it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different,

(iv) zeros of F' which are not the zeros of $F(F-1)$ and zeros of G' which are not the zeros of $G(G-1)$. And we define the following notations which are used in the proof.

$$N_2\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right),$$

where a simple zero point of f is counted once and a multiple zero point of f is counted twice. Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. We denote by $N_E^{1)}\left(r,\frac{1}{f-1}\right)$ the counting function of those 1-points of f where $p=q=1$; by $N_L\left(r,\frac{1}{f-1}\right)$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g; each point in these counting functions is counted only once. We are ignoring g in the notations above only because the reader can interpret from the context with which function g we are comparing the function f.

Lemma 2.1. . *(*[2]*, Lemma 2.13, page 13) Let F, G be two non-constant meromorphic functions. If F, G share (1,2) and* (∞,k)*, where* $0\leq k\leq\infty$*, and* $H\not\equiv 0$*, then*

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G),$$

where $\overline{N}_*(r,\infty;F,G)$ *denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.*

Lemma 2.2. *(*[36]*, Lemma 2, page 108) Let f be a non-constant meromorphic function, and let* $a_1,a_2,...,a_n$ *be finite complex numbers,* $a_n\neq 0$*. Then*

$$T(r,a_nf^n+...+a_2f^2+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2.3. *(*[19]*, Theorem 2.1, page 109) Let f be a meromorphic function of finite order* $\rho(f)$*, and let c be a nonzero constant. Then*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\rho(f)-1+\varepsilon})+O(logr),$$

for an arbitrary $\varepsilon>0$*.*

We mention that Lemma 2.3 holds also for $c=0$ as in the case $T(r,f(z+c))=T(r,f(z))$.

Lemma 2.4. *([48], Lemma 2.1, page 4) Let f be a non-constant meromorphic function, p, k be positive integers, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f),$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.

We point out that in Lemma 2.4 one obviously has that $\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right)$

Lemma 2.5. *([13], Theorem 2.1, page 465) Let f be a non-constant meromorphic function of finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^{\delta}}\right) = S(r,f).$$

Lemma 2.6. *([43], Lemma 3.3, page 349) Suppose that two non-constant meromorphic functions F and G share 1 and ∞ IM. Let H be given as above. If $H \not\equiv 0$, then*

$$\begin{aligned}T(r,F)+T(r,G) \leq & 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\ & +2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)\\ & +S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.7. *([44], Theorem 1.1, page 538) Let f be a zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$T(r,f(qz)) = (1+o(1))T(r,f(z))$$

and

$$N(r,f(qz)) = (1+o(1))N(r,f(z))$$

on a set of lower logarithmic density 1.

Lemma 2.8. *([3], Theorem 1.1, page 457) Let f be a zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r,\frac{f(qz)}{f(z)}\right) = S(r,f)$$

on a set of logarithmic density 1.

3. MAIN RESULTS

In this paper, by considering the difference operator $\Delta_c F$ in Theorem 2.4 and 2.5, we obtain analogous results which are more general.

Theorem 3.1. *Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ \ [f^{(k)}(z)]^n \ \ and\ \ \Delta_c F\ \ share\ \ (1,2),(\infty,0)\ \ and\ \ n\geq k+6;$$

$$(II)\ \ [f^{(k)}(z)]^n \ \ and\ \ \Delta_c F\ \ share\ \ (1,2),(\infty,\infty)\ \ and\ \ n\geq k+5;$$

$$(III)\ \ [f^{(k)}(z)]^n \ \ and\ \ \Delta_c F\ \ share\ \ (1,0),(\infty,0)\ \ and\ \ n\geq 2k+12;$$

where $\Delta_c F = f^n(z+n)-f^{(n)}(z)$ then $f^{(k)}(z)=tf(z+c)$, for a constant t that satisfies $t^n=\frac{1}{2}$.

proof. Let

$$F=\Delta_c F = f^n(z+n)-f^n(z) \tag{1}$$

(I). Suppose $[f^{(k)}(z)]^n$ and $\Delta_c F$ share (1,2), $(\infty,0)$ and $n\geq 2k+8$. Then it follows directly from the assumption of the theorem that F and G share (1,2) and $(\infty,0)$. Let H be defined as above. Suppose that $H\neq 0$. It follows from Lemma 2.1 that

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \tag{2}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F)=nT(r,f(z+\eta))+nT(r,f(z))+S(r,f)=2nT(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \tag{3}$$

It is obvious that

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) =&2\overline{N}\left(r,\frac{1}{f^n(z+\eta)-f^n(z)}\right)\\ \leq&2\overline{N}\left(r,\frac{1}{f(z+\eta)}\right)+2\overline{N}\left(r,\frac{1}{f(z)}\right)\\ \leq&4T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \end{aligned} \tag{4}$$

$$\begin{aligned} \overline{N}(r,F) =&\overline{N}(r,f^n(z+\eta)-f^n(z))\\ \leq&2T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \end{aligned} \tag{5}$$

$$\overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,F) \leq 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \tag{6}$$

Since $\overline{E}(\infty, f^{(k)}) = \overline{E}(\infty, f)$, we have

$$\overline{N}(r,G) = \overline{N}(r,[f^{(k)}(z)]^n) = \overline{N}(r,f^{(k)}(z)) = \overline{N}(r,f) \leq T(r,f). \tag{7}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) =& 2\overline{N}\left(r,\frac{1}{f^{(k)}}\right) \leq 2N_{k+1}\left(r,\frac{1}{f}\right) + 2k\overline{N}(r,f) + S(r,f) \\ \leq& (2+2k)T(r,f) + S(r,f). \end{aligned} \tag{8}$$

Combining (2)-(8), we deduce

$$\begin{aligned} T(r,F) \leq& N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) \\ &+ \overline{N}_*(r,\infty;F,G) + S(r,f) + S(r,g), \\ 2nT(r,f) \leq& 4T(r,f) + (2+2k)T(r,f) + 2T(r,f) + T(r,f) \\ &+ 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \\ 2nT(r,f) \leq& (2k+11)T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \end{aligned}$$

$$(2n-2k-11)T(r,f) \leq O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{9}$$

which contradicts $n \geq \frac{2k+12}{2} \geq k+6$. Therefore $H \equiv 0$, that is

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By integrating twice, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{10}$$

where $A \neq 0$ and B are constants. From (10) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF - (B+1)} \tag{11}$$

Suppose that $B \neq 0, -1$. From (11), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r,G) \tag{12}$$

From the second fundamental theorem and Lemma 2.3, we have

$$
\begin{aligned}
2nT(r,f) =& T(r,F)+S(r,f)\\
\le& \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)+S(r,f)\\
\le& \overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+\overline{N}(r,f)+S(r,f)\\
\le& 2T(r,f)+2T(r,f)+T(r,f)+S(r,f)\\
\le& 5T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f),
\end{aligned}
\tag{13}
$$

which contradicts $n \ge k+6$. Suppose that $B=-1$. From (11) we have

$$G=\frac{(A+1)F-A}{F} \tag{14}$$

If $A \ne -1$, we obtain from (14) that

$$\overline{N}\left(r,\frac{1}{F-\frac{A}{A+1}}\right)=\overline{N}\left(r,\frac{1}{G}\right) \tag{15}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$
\begin{aligned}
2nT(r,f) =& T(r,F)+S(r,f) \le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{A}{A+1}}\right)+S(r,f)\\
\le& \overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\
\le& \overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}r,f+S(r,f)\\
\le& (k+5)T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f),
\end{aligned}
\tag{16}
$$

which contradicts with $n \ge k+6$. Hence $A=-1$. From (14) we get $FG=1$, that is

$$[f^n(z+\eta)-f^n(z)]\ \ [f^{(k)}(z)]^n=1 \tag{17}$$

Since $[f^{(k)}(z)]^n$ and $[f(z+\eta)-f^n(z)]$ share $(\infty,0)$, from (17) we get

$$N(r,f^{(k)})=0, \quad T(r,f^{(k)})=T(r,f(z+\eta))+S(r,f), \tag{18}$$

and

$$[f^{(k)}(z)]^{2n}=\frac{[f^{(k)}(z)]^n}{f^n(z+\eta)-f^n(z)}=\frac{\frac{[f^{(k)}(z)]^n}{f^n(z)}}{\frac{f^n(z+\eta)-f^n(z)}{f^n(z)}}. \tag{19}$$

From Lemma 2.5 and logarithmic derivative lemma, we get

$$2nT(r,f^{(k)}) = T(r,[f^{(k)}]^{2n}) = m(r,[f^{(k)}]^{2n}) + N(r,[f^{(k)}]^{2n}) = m(r,[f^{(k)}(z)]^{2n}) = S(r,f).$$

that is

$$T(r,f^{(k)}) = S(r,f) \tag{20}$$

By (18) and (20), we know that

$$T(r,f(z+\eta)) = T(r,f^{(k)}) = S(r,f). \tag{21}$$

which is a contradiction with Lemma 2.3.

Suppose that $B=0$. From (11), we have

$$G = AF-(A-1) \tag{22}$$

If $A \neq 1$, from (22) we obtain

$$\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) = \overline{N}\left(r,\frac{1}{G}\right). \tag{23}$$

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
2nT(r,f) =& T(r,F)+S(r,f)\\
\leq& \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)+S(r,f)\\
\leq& \overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)++S(r,f)\\
\leq& \overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\\
\leq& (k+5)T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f),
\end{aligned} \tag{24}$$

which contradicts with $n \geq k+6$. Thus $A=1$. From (22) we have $F=G$, that is $f^n(z+\eta)-f^n(z) = [f^{(k)}(z)]^n$. Hence $f^{(k)}(z) = tf(z+\eta)$, for a constant t with $t^n = \frac{1}{2}$. We can get the conclusion of Theorem 3.1.

(II). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+\eta)-f^n(z)$ share $(1,2),(\infty,\infty)$ and $n \geq 2k+7$. Then it follows directly from the assumption of the theorem that F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \tag{25}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F) = nT(r,f(z+\eta)) + nT(r,f(z)) + S(r,f) = 2nT(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \tag{26}$$

It is obvious that

$$N_2\left(r,\frac{1}{F}\right) = 4T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{27}$$

$$\overline{N}(r,F) = 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{28}$$

$$\overline{N}(r,G) = \overline{N}(r,f) \le T(r,f), \tag{29}$$

$$\overline{N}_*(r,\infty;F,G) = 0 \tag{30}$$

Lemma 2.4 gives

$$N_2\left(r,\frac{1}{G}\right) = (2k+2)T(r,f) + S(r,f). \tag{31}$$

Combining (25)-(31), we deduce

$$(2n-2k-9)T(r,f) \le O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \tag{32}$$

which contradicts with $n \ge k+5$. Therefore $H \equiv 0$. Similar to the proof in (I), we can get the conclusion of Theorem 3.1.

(III). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+\eta) - f^n(z)$ share $(1,0), (\infty,0)$ and $n \ge 3k+14$. Then it follows directly from the assumption of the theorem that F and G share $(1,0)$ and $(\infty,0)$. Let

H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.6 that

(33)
$$T(r,F)+T(r,G) \leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)$$
$$+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).$$

Since

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)$$
(34)
$$\leq N\left(r,\frac{1}{G-1}\right) \leq T(r,G)+O(1),$$

we get from (33) and (34) that

$$T(r,F) \leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)$$
(35)
$$+N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).$$

According to Lemma 2.2 and Lemma 2.3, we have

(36)
$$T(r,F)=2nT(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f).$$

It is obvious that

(37)
$$\overline{N}(r,F)=2T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f),$$

(38)
$$N_2\left(r,\frac{1}{F}\right)=4T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f),$$

$$N_L\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{F}{F'}\right) \leq N\left(r,\frac{F'}{F}\right)+S(r,f)$$
$$\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)$$
$$=\overline{N}(r,\Delta_c f)+\overline{N}\left(r,\frac{1}{\Delta_c f}\right)+S(r,f)$$

(39)
$$\leq 4T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f).$$

Lemma 2.4 gives

$$N_2\left(r,\frac{1}{G}\right)\leq (2k+2)T(r,f)+S(r,f), \tag{40}$$

$$\begin{aligned} N_L\left(r,\frac{1}{G-1}\right) \leq & N\left(r,\frac{G}{G'}\right)\leq N\left(r,\frac{G'}{G}\right)+S(r,f)\\ \leq & \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ \leq & \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ \leq & \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\end{aligned}$$

$$\leq (k+2)T(r,f)+S(r,f). \tag{41}$$

Combining (35)-(41), we deduce

$$(2n-3k-22)T(r,f)\leq O(r^{\rho(f)-1+\varepsilon})+S(r,f). \tag{42}$$

which contradicts with $n\geq \frac{3k+23}{2}\geq 2k+12$. Therefore $H\equiv 0$. Similar to the proof of (I), we can get the conclusion of Theorem 3.1. □

Corollary 3.1. *Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

$$(I)\ \ [f^{(k)}(z)]^n \ \ \textit{and}\ \ \Delta_c f(qz)\ \ \textit{share}\ \ (1,2),(\infty,0)\ \ \textit{and}\ \ n\geq k+6;$$

$$(II)\ \ [f^{(k)}(z)]^n \ \ \textit{and}\ \ \Delta_c f(qz)\ \ \textit{share}\ \ (1,2),(\infty,\infty)\ \ \textit{and}\ \ n\geq k+5;$$

$$(III)\ \ [f^{(k)}(z)]^n \ \ \textit{and}\ \ \Delta_c f(qz)\ \ \textit{share}\ \ (1,0),(\infty,0)\ \ \textit{and}\ \ n\geq 2k+12;$$

where $\Delta_c F = f^n(qz+n)-f^{(n)}(qz)$ then $f^{(k)}(z)=tf(qz)$, for a constant t that satisfies $t^n=\frac{1}{2}$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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