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MUTUAL PROXIMINALITY OF TWO SETS IN $\mathbf{L}^p(\mu, X)$

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Abstract: In this paper, we study mutual best approximation of two sets in the p–Bochner function spaces $\mathbf{L}^p(\mu,X)$, $1 \leq p < \infty$. We give some sufficient conditions on M and K as subsets of the Banach space X to guarantee that $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal subsets in $\mathbf{L}^p(\mu,X)$. The main issue in this work is that we state and prove a distance formula for mutual best approximation between the two sets $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ in $\mathbf{L}^p(\mu,X)$, for $1 \leq p < \infty$. Then we use this formula to obtain some other results related to our study.

Keywords: best approximation; mutual proximinality; distance formula; p–Bochner spaces.

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1. INTRODUCTION AND SOME PRELIMINARIES

Mutual Proximinality is a branch of best approximation theory which is concerned about the existence of mutually nearest points between two subsets in the same space. Several authors studied the existence of mutually nearest points between two sets either in Metric spaces or in general Banach spaces. See for example [1], [2] and [3]. We will study this topic for two sets in the spaces of Bochner p-integrable functions $\mathbf{L}^p(\mu,X)$, $1 \leq p < \infty$.

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In this paper, we will try to investigate some conditions that can be imposed on two given nonempty disjoint subsets M and K of a Banach space X, such that a mutual proximinal pair between $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ exists. Other than the introduction, the paper includes two sections containing the results obtained through our study. In the first section, we prove a distance formula related to mutual best approximation between the two sets $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ as subsets in $\mathbf{L}^p(\mu, X)$, $1 \leq p < \infty$, where M and K are two nonempty disjoint subsets in X. The second section contains some results in which we give partial answers to the following question:

What conditions can be imposed on the sets M and K as subsets of the Banach space X in order that $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal in $\mathbf{L}^p(\mu, X)$?

Throughout this paper, we will consider the vector space $\mathbf{L}^p(\mu, X)$ of all (equivalence classes of) strongly measurable, p-integrable functions, on a finite measure space $\boldsymbol{T}$, with values in the Banach space X, equipped with the p-norm, $1 \leq p < \infty$:

$$\| f \|_p = \Big(\int_T \| f(t) \|^p \, dt \Big)^{1/p}.$$

It is well known that $\mathbf{L}^p(\mu, X)$ is complete under the p-norm, where $1 \leq p < \infty$, whenever X is a Banach space. If Y is nonempty subset in X then the set $\mathbf{L}^p(\mu, Y)$ is nonempty subset of $\mathbf{L}^p(\mu, X)$. This can be easily seen since the set $\{y\mathbf{I}_{\boldsymbol{T}} : y \in Y\}$ is a subset of $\mathbf{L}^p(\mu, Y)$, for each p: $1 \leq p < \infty$, where $\mathbf{I}_{\boldsymbol{T}}$ is the characteristic function on $\boldsymbol{T}$.

The following properties are sometimes needed when dealing with best approximation in $\mathbf{L}^p(\mu, X)$, see for example [4] and [5].

I) If X is reflexive Banach space then $\mathbf{L}^p(\mu, X)$ is so, for $1 < p < \infty$.

II) $\mathbf{L}^p(\mu, X)$ are separable whenever X is, for $1 \leq p < \infty$, since the set of simple functions is dense in $\mathbf{L}^p(\mu, X)$.

III) If X is uniformly convex then $\mathbf{L}^p(\mu, X)$ is so, for $1 < p < \infty$.

Let $\{E_1, E_2, \ldots, E_n, \ldots\}$ be a countable collection of mutually disjoint measurable subsets of T such that $\bigcup_{i=1}^{\infty} E_i = T$. A countably-valued function, $\Psi(t)$ is represented as

$$\Psi(t) = \sum_{n=1}^{\infty} x_n \mathbf{I}_{E_n}(t),$$

where the sequence $(x_n)_{n\in N}$ in X and I_{E_n} is the characteristic function on E_n, for each n.

Denote by $\sum_p(\mu, X)$, the linear subspace in $\mathbf{L}^p(\mu, X)$ consisting of all countably-valued functions in $\mathbf{L}^p(\mu, X)$. Then one can easily see that $\sum_p(\mu, X)$ is dense in $\mathbf{L}^p(\mu, X)$, whenever X is.

Let $(X, \| . \|)$ be a Banach Space and let M be a closed subset in X. Let x in X, $m_0 \in M$ is called best approximation point for $x \in X$ from M if

$$\| x - m_0 \| \leq \| x - m \|, \ \forall m \in M.$$

If for each $x \in X$, $\exists$ a point in M which is best approximation to x from M, then M is called Proximinal Set in X, [6].

Usually, the distance between x and the set M is defined as

$$d(x, M) = \inf\{\|x-m\|, m \in M\}.$$

Hence, $m_0 \in M$ is a best approximation point to x in X if $d(x, M) = \| x - m_0 \|$.

It is well-known that if M is compact in $\boldsymbol{X}$ then M is proximinal in $\boldsymbol{X}$. On the other hand, any finite-dimensional or reflexive subspace of $\boldsymbol{X}$ is also proximinal in $\boldsymbol{X}$.

The following theorems concern proximinality in $\mathbf{L}^p(\mu, \boldsymbol{X})$, $1 \leq p < \infty$, see [7], [8] and [9].

Theorem 1.1

Let $\boldsymbol{X}$ be a Banach space and Y is subspace of X. If Y is reflexive (or finite dimensional) in $\boldsymbol{X}$ then, for $1 \leq p < \infty$, $\mathbf{L}^p(\mu, Y)$ is proximinal in $\mathbf{L}^p(\mu, \boldsymbol{X})$.

Theorem 1.2

Let Y be closed separable in $\boldsymbol{X}$, then $\mathbf{L}^p(\mu, Y)$ is proximinal in $\mathbf{L}^p(\mu, \boldsymbol{X})$ if and only if Y is proximinal in $\boldsymbol{X}$ for $1 \leq p < \infty$.

Furthermore, many results in the theory of best approximation in function spaces and in particular for $\mathbf{L}^p(\mu, \boldsymbol{X})$ have been proved using certain distance formulas. Light opened this field by presenting the following distance formula in 1989, see [10].

Theorem 1.3

Let $\boldsymbol{X}$ be a Banach space and Y a subspace of $\boldsymbol{X}$. For $f \in \mathbf{L}^p(\mu, \boldsymbol{X})$,

$$d(f, \mathbf{L}^p(\mu, Y)) = \left\| d(f(.), Y) \right\|_p, \text{ for } 1 \leq p < \infty.$$

Where for each f in $\mathbf{L}^p(\mu, \boldsymbol{X})$, $d(f(.),Y)$: $T \to \boldsymbol{R}$ (sometimes denoted by $d_f^Y(.)$) is a real-valued function defined by

$$d(f(t),Y) = \inf \{ \|f(t) - y\|, y \text{ in } Y\}, \text{ for } a.e\ t \text{ in } T.$$

Then it is clear that $d_f^Y(.) \in \boldsymbol{L}^p(\mu)$, $1 \leq p < \infty$.

Corollary 1.4

Let X be a Banach space and let Y be a closed subspace in X, assume $(T, \sum, \mu)$ a finite measure space and $1 \leq p < \infty$. For f and g in $\mathbf{L}^p(\mu, X)$, $g(t)$ is a best approximation in Y to $f(t)$ for almost all $t \in T$ if and only if $g \in \mathbf{L}^p(\mu, Y)$ is a best approximation to f in $\mathbf{L}^p(\mu, X)$.

On the other hand, in [11], the author proved a similar distance formula as above but for certain subsets in $\mathbf{L}^p(\mu, X)$. (See: Theorem 3.11 in [11]).

In the second part of this section, we present some Definitions and Theorems concerning the subject of Mutual Proximinality, see [6], [1] and [3].

Definition 1.5

Let X be a Banach space. Let M and K be non-empty subsets in X, then the distance between M and K defined by :

$$d(M, K) = \inf \{\| m - k \| : m \in M, k \in K \}.$$

If there exists $(m_0, k_0) \in M \times K$ such that $\|m_0 - k_0\| = d(M, K)$, then (m_0, k_0) is called best Mutual proximinal pair between M and K. Hence, M and K are said to be mutually proximinal in X.

A sequence $\{(m_n, k_n)\}$ in $M \times K$ is called Minimizing sequence for M and K in $\boldsymbol{X}$ if

$$\lim_{n \to \infty} \|m_n - k_n\| = d(M, K).$$

Lemma 1.6

Suppose that M and K are closed subsets of the Banach space X. If some minimizing sequence $\{(m_n, k_n)\} \subseteq M \times K$ has a weak cluster point (m, k) in $M \times K$, then $d(M, K) = \|m - k\|$.

Theorem 1.7

Let X be a normed space. If M is non empty weakly sequentially compact in X and K is non empty convex and proximinal set for M in X, then a mutual proximinal pair between M and K

exists.

2. DISTANCE FORMULA BETWEEN TWO SETS IN $L^p(\mu, X)$

In this section, we prove a distance formula for the two sets $L^p(\mu, M)$ and $L^p(\mu, K)$ in $L^p(\mu, X)$, $1 \le p < \infty$. First, let us consider the following notation: Let $d_p(\cdot, M, K)$ denote the distance function from $\mathbf{T}$ into $[0, \infty)$, where M and K are two nonempty closed subsets in the Banach space X, as follows

$$d_p(\cdot, M, K)\colon \mathbf{T} \to [0, \infty),$$

such that at any value t in $\mathbf{T}$, $d_p(t, M, K)$ is defined as follows,

$$d_p(t, M, K) = \inf\{\| h(t) - g(t) \|,\ \forall h \in L^p(\mu, M) \text{ and } \forall g \in L^p(\mu, K)\}.$$

Our first task is to prove that $d_p(\cdot, M, K) \in L^p(\mu)$, as in the following Lemma.

Lemma 2.1

With the definitions above, we have

$$d_p(\cdot, M, K) \in L^p(\mu),\ 1 \le p < \infty.$$

Proof

From the definition of $d_p(\cdot, M, K)$ as,

$d_p(t, M, K) = \inf\{\| h(t) - g(t) \|,\ \forall h \in L^p(\mu, M) \text{ and } \forall g \in L^p(\mu, K)\}$, we have $d_p(\cdot, M, K)$ is a real-valued measurable function, for each p: $1 \le p < \infty$.

But, $\| h(t) - g(t) \| \le d_h^K(t) + d_g^M(t)$, a.e $t \in T$, $\forall h \in L^p(\mu, M)$ and $\forall g \in L^p(\mu, K)$, where $d_h^K(t) = d_X(h(t), K)$ and $d_g^M(t) = d_X(g(t), M)$, a.e $t \in T$. Hence, taking the inf over all $h \in L^p(\mu, M)$ and all $g \in L^p(\mu, K)$, then

$$\inf \| h(t) - g(t) \| \le d_h^K(t) + d_g^M(t).$$

Now, from **Theorem 1.3**, we have both $d_h^K(\cdot) \in L^p(\mu)$ and $d_g^M(\cdot) \in L^p(\mu)$. This implies that

$$\boldsymbol{d}_h^K(\cdot)+\boldsymbol{d}_g^M(\cdot) \in \boldsymbol{L}^p(\mu).$$

Hence, $\boldsymbol{d}_p(\cdot, M, K) \in \boldsymbol{L}^p(\mu)$. ■

Our Main Theorem in this section is the following,

Theorem 2.2 (Distance Formula)

Let M and K be two nonempty closed subsets in the Banach space $\boldsymbol{X}$, then the distance between $\boldsymbol{L}^p(\mu, M)$ and $\boldsymbol{L}^p(\mu, K)$ in $\boldsymbol{L}^p(\mu, \boldsymbol{X})$, $1 \leq p < \infty$, is given by the following formula

$$\mathrm{d}(\boldsymbol{L}^p(\mu, M), \boldsymbol{L}^p(\mu, K)) = \| \boldsymbol{d}_p(\cdot, M, K) \|_p$$

Proof

First, from the definition of $\boldsymbol{d}_p(\cdot, M, K)$ above, we have

$$\| h(t) - g(t) \| \geq \boldsymbol{d}_p(t, M, K) \text{, a.e } t \in \mathrm{T}.$$

For all h in $\mathbf{L}^p(\mu, M)$ and all g in $\mathbf{L}^p(\mu, K)$. Moving to the p-norm, we obtain,

$$\| h - g \|_p \geq \| \boldsymbol{d}_p(\cdot, M, K) \|_p$$

Then by taking the infimum over all h in $\mathbf{L}^p(\mu, M)$ and all g in $\mathbf{L}^p(\mu, K)$, we get

$$\mathrm{d}(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)) \geq \| \boldsymbol{d}_p(\cdot, M, K) \|_p \quad \text{...............(1)}$$

To prove the other direction, we proceed as follows.

Let h in $\mathbf{L}^p(\mu, M)$ and g in $\mathbf{L}^p(\mu, K)$. For $\varepsilon > 0$, define h_ε and g_ε to be strongly measurable M-countably, resp. K- countably valued functions, as follows:

$$g_\varepsilon(t) = \sum_{n=1}^{\infty} y_{n,\varepsilon} \, \mathbf{I}_{E_n}(t), \text{ such that} \| g - g_\varepsilon \| < \frac{\varepsilon}{3}$$

and

$$h_\varepsilon(t) = \sum_{n=1}^{\infty} z_{n,\varepsilon} \, \mathbf{I}_{E_n}(t), \text{ such that} \| h - h_\varepsilon \| < \frac{\varepsilon}{3}.$$

Where $(E_n)_{n \in \mathbf{N}}$ is a sequence of pairwise disjoint measurable subsets of T satisfying that $\bigcup_{n=1}^{\infty} E_n = T$ and $\mathbf{I}_{E_n}$ the characteristic function on each E_n. Moreover, the sequences $\{y_{n,\varepsilon}\} \in K$ and $\{z_{n,\varepsilon}\} \in \mathrm{M}$, so satisfying that $\| y_{n,\varepsilon} - z_{n,\varepsilon} \| < \mathrm{d}(M, K) + \varepsilon_n$, where the sequence

$\{\varepsilon_n\}$ can be taken such that for each n, $0 < \varepsilon_n < \frac{\varepsilon}{3(2^n)}$.

Finally, we define the scalar-valued measurable function Q_ε as follows:

$$Q_\varepsilon(t) = \sum_{n=1}^{\infty} \| y_{n,\varepsilon} - z_{n,\varepsilon} \| \, \mathbf{I}_{E_n}(t).$$

Then it is clear that

$$0 \le Q_\varepsilon(t) < \sum_{n=1}^{\infty} \left(d(M,K) + \frac{\varepsilon}{3(2^n)}\right) \mathbf{I}_{E_n}(t)$$
$$\le \sum_{n=1}^{\infty} d(M,K) \, \mathbf{I}_{E_n}(t) + \sum_{n=1}^{\infty} \frac{\varepsilon}{3(2)^n} \mathbf{I}_{E_n}(t).$$

But since the set $\{(h(t), g(t)), \forall h \in \mathbf{L}^p(\mu, M)$ and $\forall g \in \mathbf{L}^p(\mu, K)\}$ is a subset of $M \times K$, so

$$\inf\{\| h(t) - g(t)\|, h \in \boldsymbol{L}^p(\mu, M), g \in \boldsymbol{L}^p(\mu, K)\} \ge \inf\{\|m - k\|, m \in M, k \in K\}.$$

Which implies:

$$d(M, K) \le \boldsymbol{d}_p(t, M, K), \text{ for all } t \text{ in T}.$$

Again since the measure space is finite, we can write

$$\left\| \sum_{n=1}^{\infty} \frac{\mathbf{I}_{E_n}(t)}{(2)^n} \right\| \le m(T) < \infty.$$

We may choose that $m(T)=1$, hence $\left\| \sum_{n=1}^{\infty} \frac{\mathbf{I}_{E_n}(t)}{(2)^n} \right\| \le 1$.

Then from the argument above and moving to the p-norm, we get

$$\|Q_\varepsilon\|_p \le \|\boldsymbol{d}_p(., M, K)\|_p + \frac{\varepsilon}{3}.$$

The last step of the proof goes as follows:

For $h \in \mathbf{L}^p(\mu, M)$ and $g \in \mathbf{L}^p(\mu, K)$, we have

$$\|g - h\|_p \le \|g - g_\varepsilon\|_p + \|h - h_\varepsilon\|_p + \|h_\varepsilon - g_\varepsilon\|_p.$$

So,

$$\|g - h\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|Q_\varepsilon\|_p < \frac{2\varepsilon}{3} + \|\boldsymbol{d}_p(., M, K)\|_p + \frac{\varepsilon}{3}.$$

Finally, we get

$$\|g - h\|_p < \|\boldsymbol{d}_p(., M, K)\|_p + \varepsilon$$

Then taking the inf over all h in $\mathbf{L}^p(\mu, M)$ and g in $\mathbf{L}^p(\mu, K)$. Also, since ε arbitrary, we have

$$d(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)) \le \|\boldsymbol{d}_p(., M, K)\|_p \ldots\ldots\ldots\ldots \quad (2)$$

From (1) and (2), we deduce that

$$d(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)) = \|\boldsymbol{d}_p(., M, K)\|_p . \blacksquare$$

Corollary 2.3

Let M and K be two nonempty disjoint closed subsets of $\boldsymbol{X}$. Let (T, μ) be a finite measure space, h in $\mathbf{L}^p(\mu, M)$ and g in $\mathbf{L}^p(\mu, K)$, $1 \leq p < \infty$. The pair $(h, g) \in \mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$ is a mutual proximinal pair if and only if $(h(t), g(t)) \in M \times K$ is a mutual proximinal pair of M and K a.e $t \in T$.

3. Results on Mutual Proximinality in $\mathbf{L}^p(\mu, \boldsymbol{X})$

In this section, we will prove some results concerning when the two sets $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal in $\mathbf{L}^p(\mu, \boldsymbol{X})$, for $1 \leq p < \infty$. The first result is the following Lemma which states that the existence of a weak limit for a minimizing sequence in $\mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$ implies that $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal in $\mathbf{L}^p(\mu, \boldsymbol{X})$. We also recall that a sequence $\{(m_n, k_n)\}$ in $M \times K$ is called a minimizing sequence for M and K in $\boldsymbol{X}$, if $\lim_{n\to\infty} \|m_n - k_n\| =$ $d(M, K)$.

Lemma 3.1

Let M and K be two nonempty closed subsets of X. If $\{(h_n, g_n)\}$ is a minimizing sequence in $\mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$ and has a weak limit point (h_0, g_0), then

$$\|h_0 - g_0\|_p = d(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)).$$

Hence, $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal in $\mathbf{L}^p(\mu, X)$.

Proof

Since $\{(h_n, g_n)\}$ is a minimizing sequence in $\mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$, then

$$\lim_{n\to\infty} \|h_n - g_n\|_p = d(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)).$$

Hence, from **Corollary 2.3** on distance formula $\{(h_n(t), g_n(t))\}$ is a minimizing sequence in $M \times K$, a.e $t \in \boldsymbol{T}$. But also given that $\{(h_n, g_n)\} \xrightarrow{w} (h_0, g_0)$ in $\mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$ and this implies that $\{(h_n(t), g_n(t))\} \to (h_0(t), g_0(t))$, a.e $t \in \boldsymbol{T}$. Now, by **Lemma 1.6** we get $(h_0(t),$

$g_0(t)$) is mutual proximinal pair in $M \times K$ a.e $t \in \boldsymbol{T}$. Then by **Corollary 2.3** again (h_0, g_0) is a mutual proximinal pair in $\mathbf{L}^p(\mu, K) \times \mathbf{L}^p(\mu, M)$ so $\| h_0 - g_0 \|_p = \mathrm{d}(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K))$. ∎

The next two results present some conditions on the two sets M and K in $\boldsymbol{X}$ in order to achieve the mutual proximinality of $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ in $\mathbf{L}^p(\mu, \boldsymbol{X})$, $1 < p < \infty$.

Theorem 3.2

Let M be a nonempty subset that is weak sequentially compact in $\boldsymbol{X}$ and K nonempty convex subset and separable that is proximinal with respect to M in $\boldsymbol{X}$, then a mutual proximinal pair of $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ exists.

Proof

Given M a nonempty weak sequentially compact subset in $\boldsymbol{X}$, hence, $\mathbf{L}^p(\mu, M)$ is weakly closed in $\mathbf{L}^p(\mu, \boldsymbol{X})$, $1 < p < \infty$. So, for any bounded sequence $\{h_n\}$ in $\mathbf{L}^p(\mu, M)$ has a weak limit in $\mathbf{L}^p(\mu, M)$. Also given that K is proximinal with respect to M. This implies that any minimizing sequence $\{(h_n, g_n)\}$ for $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ has a weak limit in $\mathbf{L}^p(\mu, \boldsymbol{X})$, and so a mutual proximinal pair of $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ exists, by **Lemma 3.1** above. ∎

Theorem 3.3

Let $\boldsymbol{X}$ be a reflexive space and M be a nonempty, bounded and weakly closed subset in $\boldsymbol{X}$. Let K be non empty closed convex subset in $\boldsymbol{X}$, then a mutual proximinal pair of $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ exists.

Proof

Since M is bounded weakly closed in a reflexive normed space so M is sequentially a weak compact in $\boldsymbol{X}$. Also K is a closed convex set in a normed reflexive space so K is proximinal for the total space, but M is a subset in $\boldsymbol{X}$ so K is proximinal with respect to M. This implies using **Theorem 3.2** above that any minimizing sequence $\{(h_n, g_n)\}$ for $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ has a weak limit in $\mathbf{L}^p(\mu, \boldsymbol{X})$, and so a mutual proximinal pair of $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ exists. ∎

Our final result in this paper discusses the converse of the question raised at the begining of the paper, which is the statement of the following theorem.

Theorem 3.4

Let $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ be mutually proximinal sets in $\mathbf{L}^p(\mu, \boldsymbol{X})$, for M and K are two closed nonempty sets in $\boldsymbol{X}$ and $1 \leq p < \infty$. Then M and K are mutually proximinal sets in $\boldsymbol{X}$.

Proof

Let $\{(m_n , k_n)\}$ in $M \times K$ be a minimizing sequence for M and K in $\boldsymbol{X}$. Hence,

$$\lim_{n\to\infty} \|m_n - k_n\| = \mathrm{d}(M, K).$$

Now define the measurable functions h_n in $\mathbf{L}^p(\mu, M)$ and g_n in $\mathbf{L}^p(\mu, K)$ such that for each n, we have $h_n(t) = m_n \mathbf{I}_T(t)$ and $g_n(t) = k_n \mathbf{I}_T(t)$, a.e $t \in T$. Then $\{(h_n, g_n)\} \in \mathbf{L}^p(\mu, M) \times \mathbf{L}^p(\mu, K)$ and we get the following:

$$\mathrm{d}(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)) \leq \|h_n - g_n\|_p = \|m_n - k_n\| \, \mu(T) \to \mathrm{d}(M, K)\, \mu(T).$$

But,

$$\mathrm{d}(M, K)\, \mu(T) \leq \|\boldsymbol{d}_p(., M, K)\|_p = \mathrm{d}(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)).$$

Hence,

$$\|h_n - g_n\|_p \to \mathrm{d}(\mathbf{L}^p(\mu, M), \mathbf{L}^p(\mu, K)).$$

This means that $\{(h_n, g_n)\}$ is a minimizing sequence for $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ in $\mathbf{L}^p(\mu, \boldsymbol{X})$, so it converges say to (h_0, g_0) by the given that $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu,K)$ are mutually proximinal in $\mathbf{L}^p(\mu,X)$. Finally, by the definition of h_n and g_n, we can write $h_0(t) = m_0 \mathbf{I}_T(t)$ and $g_0(t) = k_0 \mathbf{I}_T(t)$, a.e $t \in T$ and for some m_0 in M and k_0 in K. Hence, (m_0 , k_0) is a mutual proximinal pair for M and K in $\boldsymbol{X}$. ■

4. CONCLUSION

In this paper, we studied mutual proximinality between two sets in $\mathbf{L}^p(\mu,X)$, $1 \leq p < \infty$, where we investigated some sufficient conditions to be imposed on M and K as subsets of a Banach space X to get that $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ are mutually proximinal in $\mathbf{L}^p(\mu,X)$. In order to approach our aim, we first stated and proved a new distance formula between the two sets $\mathbf{L}^p(\mu, M)$ and $\mathbf{L}^p(\mu, K)$ in $\mathbf{L}^p(\mu,X)$, $1 \leq p < \infty$. Then this formula was used to obtain other results related to our study.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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