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FIXED POINT RESULTS FOR VARIOUS CONTRACTIVE CONDITIONS IN b-MULTIPLICATIVE METRIC SPACE

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Abstract: The aim of this paper is to prove several fixed point results by using various contractive conditions in b-multiplicative metric space.

Keywords: fixed point; b-multiplicative metric space; contraction mapping.

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1. INTRODUCTION

The idea of b-metric was first given by Bakhtin [5] in 1989. Forward, the b-metric was formally defined by Czerwik [8] in 1993. In 2013, Kir et al. [11] defined some well-known fixed point theorems in b-metric space. Kamran et al. [10] introduced the concept of extended b-metric space and worked on fixed point theorems for self-mappings. Then, Suzuki [14] proved some fixed point theorems using basic inequalities on a b-metric space. Bashirov et al. [6] developed the concept of multiplicative metric space. After that, Ozavsar and Cevikel [15] developed

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multiplicative metric space accompanying their topological properties and proved some fixed point theorems for contraction mappings of multiplicative metric space. Then, in 2015 Abodayeh et al. [1] worked on the relation between metric and multiplicative metric spaces and proved some fixed point theorems in multiplicative metric space. In 2016, Shukla [13] showed that multiplicative metric is a particular case of metric space. Dosenovic et al. [9] worked on various fixed point results for various multiplicative contractive mappings. Recently in 2017, the notion of b-multiplicative metric space was introduced by Ali et al. [4] and proved some fixed point results. As an application, they established an existence theorem for solution of system of Fredholm multiplicative integral equation. Currently, in 2020, Shoaib and Saleemullah [12] extended the result of Ali et al. by applying contractive condition on closed ball in b-multiplicative metric space. In this paper, we will obtain the fixed point theorems in b-multiplicative metric space by using different contraction conditions.

2. PRELIMINARIES

Definition 2.1[2] Let X be a non-empty set. A mapping $d^*: X \times X \to \mathbb{R}^*$ is multiplicative metric if d^* satisfies the following conditions:

1. $d^*(x,y) \geq 1$ for all $x, y \in X$;

2. $d^*(x,y) = 1$ iff $x = y$ for all $x, y \in X$;

3. $d^*(x,y) = d^*(y,x)$ for all $x, y \in X$;

4. $d^*(x,z) \leq d^*(x,y) \cdot d^*(y,z)$ for all $x, y, z \in X$.

The pair (X, d^*) is called a multiplicative metric space.

Definition 2.2[3] Let X be a non-empty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \to \mathbb{R}^+$ is called a b-metric if the following conditions hold:

1. $d(x,y) \geq 0$ for all $x, y \in X$;

2. $d(x,y) = 0$ iff $x = y$ for all $x, y \in X$;

3. $d(x,y)=d(y,x)$ for all $x,y\in X$;

4. $d(x,z)\leq s\{d(x,y)+d(y,z)\}$ for all $x,y,z\in X$.

The triplet (X,d,s) is called b-metric space.

Definition 2.3[4] Let X be a non-empty set and $s\geq 1$ be a given real number. A mapping $d^*: X\times X\to\mathbb{R}^*$ is called a b-multiplicative metric if the following conditions hold:

1. $d^*(x,y)\geq 1$ for all $x,y\in X$;

2. $d^*(x,y)=1$ iff $x=y$ for all $x,y\in X$;

3. $d^*(x,y)=d^*(y,x)$ for all $x,y\in X$;

4. $d^*(x,z)\leq\{d^*(x,y)\cdot d^*(y,z)\}^s$ for all $x,y,z\in X$.

The triplet (X,d^*,s) is called a b-multiplicative metric space.

Definition 2.4[4] Let (X,d^*,s) be a b-multiplicative metric space.

1. A sequence $\{x_n\}$ is convergent iff there exists $x\in X$ such that

$$d^*(x_n,x)\to 1 \quad \text{as} \quad n\to+\infty \ .$$

2. A sequence $\{x_n\}$ is called b-multiplicative Cauchy sequence iff

$$d^*(x_n,x_m)\to 1 \quad \text{as} \quad n\to+\infty \ .$$

3. A b-multiplicative metric space (X,d^*,s) is said to be complete if every b-multiplicative Cauchy sequence in X is convergent in X.

Definition 2.5[7] Let (X,d^*) be a multiplicative metric space. A mapping $T:X\to X$ is called the multiplicative contraction if there exists $\lambda\in[0,1)$ such that for all $x,y\in X$,

$$d^*(Tx,Ty)\leq\{d^*(x,y)\}^{\lambda} \ .$$

Definition 2.6 Let (X,d^*,s) be a b-multiplicative metric space. A mapping $T:X\to X$ is called the b-multiplicative contraction if there exists $\lambda \in [0,1)$ such that for all $x, y \in X$,

$$d^*(Tx,Ty) \le \left\{d^*(x,y)\right\}^{\lambda}.$$

3. MAIN RESULTS

This section contains some fixed point theorems in b-multiplicative metric space.

Theorem 3.1. Let (X, d^*, s) be a complete b-multiplicative metric space and $T : X \to X$ be a mapping such that

$$d^*(Tx,Ty) \le \left\{d^*(x,y)\right\}^{\lambda} \quad \forall\, x, y \in X \tag{3.1}$$

where $0 \le \lambda < 1$, $s\lambda < 1$ and $s \ge 1$. Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be sequence in X defined by the recursion

$$x_n = Tx_{n-1} = ... = T^n x_0 \,, n = 1, 2, 3, ... \tag{3.2}$$

By (3.1) and (3.2), we obtain that

$$d^*(x_{n+1}, x_n) = d^*(Tx_n, Tx_{n-1}) \le \{d^*(x_n, x_{n-1})\}^{\lambda}, \quad \textit{where } \lambda < 1$$

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \le \{d^*(x_{n-1}, x_{n-2})\}^{\lambda^2}$$

Continuing this process, we get

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \le \{d^*(x_1, x_0)\}^{\lambda^n} = \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.3}$$

Therefore, T is a b-multiplicative contraction mapping.

Let $m, n \in \mathbb{N}$ where $m > n$. Then,

$$d^*(x_n, x_m) \le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2}, x_m)\}^{s^2}$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2}, x_{n+3})\}^{s^3} ... \{d^*(x_{m-1}, x_m)\}^{s^{m-n}}$$

Using (3.3), we get

$$
\begin{aligned}
d^*(x_n,x_m) &\le \{d^*(x_0,x_1)\}^{s\lambda^n}\{d^*(x_0,x_1)\}^{s^2\lambda^{n+1}}\{d^*(x_0,x_1)\}^{s^3\lambda^{n+2}}\ldots\{d^*(x_0,x_1)\}^{s^{m-n}\lambda^{m-1}}\\
&\le \{d^*(x_0,x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}\\
&\le \{d^*(x_0,x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}
\end{aligned}
$$

This shows that $d^*(x_n,x_m)\to 1$ as $m,n\to\infty$. Thus, the sequence $\{x_n\}$ is b-multiplicative Cauchy sequence in X. But (X,d^*,s) is given to be a complete metric space. Therefore, there exists a point $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$.

Now, we show that x^* is fixed point of T. For this, we have

$$
\begin{aligned}
d^*(x^*,Tx^*) &\le \{d^*(x^*,Tx_n)\,d^*(Tx_n,Tx^*)\}^s\\
&\le \{d^*(x^*,Tx_n)\}^s\{d^*(Tx_n,Tx^*)\}^s\\
&\le \{d^*(x^*,x_{n+1})\}^s\{d^*(x_n,x^*)\}^{s\lambda}
\end{aligned}
$$

$$\Rightarrow \qquad d^*(x^*,Tx^*)\to 1 \text{ as } n\to\infty.$$

This shows that x^* is fixed point of T.

Now, we prove the uniqueness of fixed point of T. For this, let x' be another fixed point of T i.e., $T(x')=x'$. Also,

$$d^*(x^*,x')=d^*(Tx^*,Tx')\le\{d^*(x^*,x')\}^{\lambda}$$

Thus, $d^*(x^*,x')=1$ and hence $x^*=x'$.

This shows that x^* is the unique fixed point of T.

Theorem 3.2. Let (X,d^*,s) be a complete b-multiplicative metric space and $T:X\to X$ be a mapping such that

$$d^*(Tx,Ty)\le\left\{d^*(Tx,x)\,d^*(Ty,y)\right\}^p \quad \forall\ x,y\in X \tag{3.4}$$

where $0<p<\frac{1}{2}$, $p(1+s)<1$ and $s\ge 1$. Then, T has a unique fixed point.

Proof. Consider $x_0\in X$. Define a sequence of points $\{x_n\}$ in X such that

$$x_n=Tx_{n-1}=\ldots=T^nx_0\ ,\ n=1,2,3,\ldots \tag{3.5}$$

From (3.4) and (3.5), we obtain that

$$\begin{aligned} d^*(x_{n+1}, x_n) &= d^*(Tx_n, Tx_{n-1}) \\ &\leq \{d^*(Tx_n, x_n)\, d^*(Tx_{n-1}, x_{n-1})\}^p \\ &= \{d^*(x_{n+1}, x_n)\, d^*(x_n, x_{n-1})\}^p \\ &= \{d^*(x_{n+1}, x_n)\}^p \{d^*(x_n, x_{n-1})\}^p \end{aligned}$$

$$\Rightarrow \quad \{d^*(x_{n+1}, x_n)\}^{1-p} \leq d^*(x_n, x_{n-1})^p$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\frac{p}{1-p}}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\lambda}, \quad \lambda = \frac{p}{1-p} < 1$$

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \leq \{d^*(x_{n-1}, x_{n-2})\}^{\lambda^2}$$

Proceeding in the same manner, we get

$$d^*(Tx_n, Tx_{n-1}) \leq \{d^*(x_1, x_0)\}^{\lambda^n} = \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.6}$$

So, T is b-multiplicative contraction mapping.

Suppose $m, n \in \mathbb{N}$ where $m > n$. Then,

$$\begin{aligned} d^*(x_n, x_m) &\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s \\ &\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_m)\}^{s^2} \\ &\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_{n+3})\}^{s^3} \ldots \{d^*(x_{m-1,} x_m)\}^{s^{m-n}} \\ &\leq \{d^*(x_0, x_1)\}^{s\lambda^n} \{d^*(x_0, x_1)\}^{s^2\lambda^{n+1}} \{d^*(x_{0,} x_1)\}^{s^3\lambda^{n+2}} \ldots \{d^*(x_{0,} x_1)\}^{s^{m-n}\lambda^{m-1}} \quad \text{(Using 3.6)} \\ &\leq \{d^*(x_0, x_1)\}^{s\lambda^n (1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})} \\ &\leq \{d^*(x_0, x_1)\}^{\frac{s\lambda^n}{1-s\lambda}} \end{aligned}$$

This signifies that $d^*(x_n, x_m) \to 1$ as $m, n \to \infty$ .Therefore, the sequence $\{x_n\}$ is b-multiplicative Cauchy sequence in X. But (X, d^*, s) is a complete metric space. So, there exists a point $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Further, we show that x^* is fixed point of T. For this, we have

$$\begin{aligned} d^*(Tx^*, x^*) &\le \{d^*(Tx^*, Tx_n) d^*(Tx_n, x^*)\}^s \\ &\le \{d^*(Tx^*, x^*)\, d^*(Tx_n, x_n)\}^{sp} \{d^*(Tx_n, x^*)\}^s \end{aligned}$$

$$\Rightarrow \quad \{d^*(Tx^*, x^*)\}^{1-sp} \le \{d^*(Tx_n, x_n)\}^{sp} \{d^*(Tx_n, x^*)\}^s$$

$$\Rightarrow \quad d^*(Tx^*, x^*) \le \{d^*(x_{n+1}, x_n)\}^{\frac{sp}{1-sp}} \{d^*(x_{n+1}, x^*)\}^{\frac{s}{1-sp}}$$

$$\Rightarrow \quad d^*(Tx^*, x^*) \to 1 \text{ as } n \to \infty.$$

This shows that T has a fixed point.

Further, let x' be another fixed point of T i.e., $T(x') = x'$.

Then,

$$d^*(x^*, x') = d^*(Tx^*, Tx') \le \{d^*(Tx^*, x^*) d^*(Tx', x')\}^p$$

$$\Rightarrow \quad d^*(x^*, x') = d^*(Tx^*, Tx') \le \{d^*(x^*, x^*) d^*(x', x')\}^p = 1.$$

Thus, $d^*(x^*, x') = 1$ and hence $x^* = x'$.

This proves that T has a unique fixed point.

Theorem 3.3. Let (X, d^*, s) be a complete b-multiplicative metric space and $T: X \to X$ be a mapping such that

$$d^*(Tx, Ty) \le \left\{d^*(Tx, y)\, d^*(Ty, x)\right\}^p \quad \forall \ x, y \in X \tag{3.7}$$

where $ps < \frac{1}{2}$, $ps(1+s) < 1$ and $s \ge 1$. Then, T has a unique fixed point.

Proof. Consider $x_0 \in X$. Define a sequence of points $\{x_n\}$ in X such that

$$x_n = Tx_{n-1} = \ldots = T^n x_0, \quad n = 1, 2, 3, \ldots \tag{3.8}$$

From (3.7) and (3.8), we obtain that

$$\begin{aligned} d^*(x_{n+1}, x_n) &= d^*(Tx_n, Tx_{n-1}) \\ &\le \{d^*(Tx_n, x_{n-1})\, d^*(Tx_{n-1}, x_n)\}^p \\ &= \{d^*(x_{n+1}, x_{n-1})\, d^*(x_n, x_n)\}^p \\ &= \left\{d^*(x_{n+1}, x_n)\, d^*(x_n, x_{n-1})\right\}^{sp} \end{aligned}$$

$$\Rightarrow \quad \{d^*(x_{n+1}, x_n)\}^{1-sp} \leq \{d^*(x_n, x_{n-1})\}^{sp}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\frac{sp}{1-sp}}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\lambda}, \quad \lambda = \frac{sp}{1-sp} < 1$$

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \leq \{d^*(x_{n-1}, x_{n-2})\}^{\lambda^2}$$

Continuing this process, we get

$$d^*(Tx_n, Tx_{n-1}) \leq \{d^*(x_1, x_0)\}^{\lambda^n} = \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.9}$$

So, T is b-multiplicative contraction mapping.

Let $m, n \in \mathbb{N}$ such that $m > n$, we have

$$d^*(x_n, x_m) \leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s$$

$$\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_m)\}^{s^2}$$

$$\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2}, x_{n+3})\}^{s^3} \ldots \{d^*(x_{m-1}, x_m)\}^{s^{m-n}}$$

$$\leq \{d^*(x_0, x_1)\}^{s\lambda^n} \{d^*(x_0, x_1)\}^{s^2\lambda^{n+1}} \{d^*(x_{0,} x_1)\}^{s^3\lambda^{n+2}} \ldots \{d^*(x_{0,} x_1)\}^{s^{m-n}\lambda^{m-1}} \text{ (Using3.9)}$$

$$\leq \{d^*(x_0, x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}$$

$$\leq \{d^*(x_0, x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}$$

This proves that $d^*(x_n, x_m) \to 1$ as $m, n \to \infty$. Hence, the sequence $\{x_n\}$ is b-multiplicative Cauchy sequence in X. But (X, d^*, s) is given to be a complete metric space. Therefore, there exists a point $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, we show that x^* is fixed point of T.

$$d^*(x^*, Tx^*) \leq \{d^*(Tx^*, Tx_n) d^*(Tx_n, x^*)\}^s$$

$$\leq \{d^*(Tx^*, x_n) d^*(Tx_n, x^*)\}^{ps} \{d^*(Tx_n, x^*)\}^s$$

$$\leq \{d^*(Tx^*, x^*) d^*(x^*, x_n)\}^{ps^2} \{d^*(Tx_n, x^*)\}^{ps} \{d^*(Tx_n, x^*)\}^s$$

$$\Rightarrow \quad \{d^*(x^*, Tx^*)\}^{1-s^2p} \leq \{d^*(x^*, x_n)\}^{ps^2} \{d^*(x_{n+1}, x^*)\}^{ps+s}$$

$$\Rightarrow \quad \{d^*(x^*,Tx^*)\} \le \{d^*(x^*,x_n)\}^{\frac{ps^2}{1-s^2p}}\{d^*(x_{n+1},x^*)\}^{\frac{ps+s}{1-s^2p}}$$

$$\Rightarrow \quad d^*(x^*,Tx^*) \to 1 \text{ as } n \to \infty.$$

This signifies that x^* is fixed point of T.

Now, we prove the uniqueness of fixed point of T. For this, let x' be another fixed point of T i.e., $T(x') = x'$.

$$d^*(x^*,x') = d^*(Tx^*,Tx') \le \{d^*(Tx^*,x')d^*(Tx',x^*)\}^p$$

$$\Rightarrow \quad d^*(x^*,x') = d^*(Tx^*,Tx') \le \{d^*(x^*,x')d^*(x',x^*)\}^p$$

$$\Rightarrow \quad d^*(x^*,x') = d^*(Tx^*,Tx') \le \{d^*(x^*,x')\}^{2p}$$

$$\Rightarrow \quad \{d^*(x^*,x')\}^{1-2p} \le 1.$$

Thus, $d^*(x^*,x') = 1$ and hence $x^* = x'$.

This signifies that x^* is the unique fixed point of T.

Theorem 3.4. Let (X,d^*,s) be a complete b-multiplicative metric space. If the function $T: X \to X$ meets the following condition:

$$d^*(Tx,Ty) \le \left\{d^*(x,y)\right\}^p \left\{d^*(Tx,x)\, d^*(Ty,y)\right\}^q \quad \forall\ x,y \in X \tag{3.10}$$

where $p+2q<1, 0<q<1,\ s(p+q)+q<1$ and $s \ge 1$, then we get a unique fixed point of function T.

Proof. Let x_0 be any point in X. Define a sequence of points $\{x_n\}$ in X such that

$$x_n = Tx_{n-1} = \ldots = T^n x_0 \quad , \ n = 1,2,3,\ldots$$

We have

$$\begin{aligned} d^*(x_{n+1},x_n) &= d^*(Tx_n,Tx_{n-1}) \\ &\le \{d^*(x_n,x_{n-1})\}^p \left\{d^*(Tx_n,x_n)\, d^*(Tx_{n-1},x_{n-1})\right\}^q \\ &= \{d^*(x_n,x_{n-1})\}^p \left\{d^*(x_{n+1},x_n)\, d^*(x_n,x_{n-1})\right\}^q \\ &= \left\{d^*(x_n,x_{n-1})\right\}^{p+q} \left\{d^*(x_{n+1},x_n)\right\}^q \end{aligned}$$

$$\Rightarrow \quad \{d^*(x_{n+1}, x_n)\}^{1-q} \le \{d^*(x_n, x_{n-1})\}^{p+q}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \le \{d^*(x_n, x_{n-1})\}^{\frac{p+q}{1-q}}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \le \{d^*(x_n, x_{n-1})\}^{\lambda}, \quad \lambda = \frac{p+q}{1-q} < 1$$

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \le \{d^*(x_{n-1}, x_{n-2})\}^{\lambda^2}$$

$$\le \{d^*(x_{n-2}, x_{n-3})\}^{\lambda^3}$$

$$\vdots$$

$$\le \{d^*(x_1, x_0)\}^{\lambda^n} = \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.11}$$

Now for $m > n$, we have

$$d^*(x_n, x_m) \le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_m)\}^{s^2}$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_{n+3})\}^{s^3} \ldots \{d^*(x_{m-1,} x_m)\}^{s^{m-n}}$$

$$\le \{d^*(x_0, x_1)\}^{s\lambda^n} \{d^*(x_0, x_1)\}^{s^2\lambda^{n+1}} \{d^*(x_{0,} x_1)\}^{s^3\lambda^{n+2}} \ldots \{d^*(x_{0,} x_1)\}^{s^{m-n}\lambda^{m-1}} \qquad \text{(Using 3.11)}$$

$$\le \{d^*(x_0, x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}$$

$$\le \{d^*(x_0, x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}$$

This signifies that $d^*(x_n, x_m) \to 1$ as $m, n \to \infty$. Thus, the sequence $\{x_n\}$ is b-multiplicative Cauchy sequence in X. Being the completeness of (X, d^*, s), there exists a point $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now,

$$d^*(x^*, Tx^*) \le \{d^*(Tx^*, Tx_n) d^*(Tx_n, x^*)\}^s$$

$$\le \{d^*(Tx_n, Tx^*)\}^s \{d^*(Tx_n, x^*)\}^s$$

$$\le [\{d^*(x^*, x_n)\}^p \{d^*(Tx_n, x_n) d^*(Tx^*, x^*)\}^q]^s \{d^*(Tx_n, x^*)\}^s$$

$$\le \{d^*(x^*, x_n)\}^{ps} \{d^*(x_{n+1}, x_n)\}^{qs} \{d^*(Tx^*, x^*)\}^{qs} \{d^*(x_{n+1}, x^*)\}^s$$

$$\Rightarrow \quad \{d*(x*,Tx*)\}^{1-qs} \le \{d*(x*,x_n)\}^{ps}\{d*(x_{n+1},x_n)\}^{qs}\{d*(x_{n+1},x*)\}^{s}$$

$$\Rightarrow \quad \{d*(x*,Tx*)\} \le \{d*(x*,x_n)\}^{\frac{ps}{1-qs}}\{d*(x_{n+1},x_n)\}^{\frac{qs}{1-qs}}\{d*(x_{n+1},x*)\}^{\frac{s}{1-qs}}$$

$$\Rightarrow \quad d*(x*,Tx*) \to 1 \text{ as } n \to \infty.$$

This signifies that $Tx* = x*$.

So, we can say that $x*$ is fixed point of the mapping T.

Now, if x' be another fixed point of mapping T, then we have

$$d*(x*,x') = d*(Tx*,Tx') \le \{d*(x*,x')\}^{p}\{d*(Tx*,x*)d*(Tx',x')\}^{q}$$

$$\Rightarrow \quad d*(x*,x') = d*(Tx*,Tx') \le \{d*(x*,x')\}^{p}\{d*(x*,x*)d*(x',x')\}^{q}$$

$$\Rightarrow \quad d*(x*,x') = d*(Tx*,Tx') \le \{d*(x*,x')\}^{p}$$

$$\Rightarrow \quad \{d*(x*,x')\}^{1-p} \le 1.$$

Thus, $d*(x*,x') = 1$. So, $x* = x'$.

Hence, we obtain a unique fixed point of T.

Theorem 3.5. Let $(X,d*,s)$ be a complete b-multiplicative metric space. If the function $T: X \to X$ meets the following condition:

$$d*(Tx,Ty) \le \left\{d*(Tx,x)\,d*(Ty,y)\right\}^{p}\left\{d*(Tx,y)\,d*(Ty,x)\right\}^{q} \quad \forall\, x,y \in X \tag{3.12}$$

where $p + sq < \frac{1}{2}$, $p(1+s) + sq(1+s) < 1$ and $s \ge 1$. Then, we get a unique fixed point $x* \in X$ such that $Tx* = x*$.

Proof. Let x_0 be any point in X. Define a sequence of points $\{x_n\}$ in X such that

$$x_n = Tx_{n-1} = \ldots = T^n x_0, \quad n = 1,2,3,\ldots$$

We have

$$\begin{aligned} d*(x_{n+1},x_n) &= d*(Tx_n,Tx_{n-1}) \\ &\le \{d*(Tx_n,x_n)\,d*(Tx_{n-1},x_{n-1})\}^{p}\left\{d*(Tx_n,x_{n-1})\,d*(Tx_{n-1},x_n)\right\}^{q} \\ &= \{d*(x_{n+1},x_n)\,d*(x_n,x_{n-1})\}^{p}\left\{d*(x_{n+1},x_{n-1})\,d*(x_n,x_n)\right\}^{q} \\ &= \left\{d*(x_{n+1},x_n)\,d*(x_n,x_{n-1})\right\}^{p}\left\{d*(x_{n+1},x_n)\,d*(x_n,x_{n-1})\right\}^{sq} \end{aligned}$$

$$\Rightarrow \quad \{d^*(x_{n+1}, x_n)\}^{1-p-sq} \leq \{d^*(x_n, x_{n-1})\}^{p+sq}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\frac{p+sq}{1-p-sq}}$$

$$\Rightarrow \quad d^*(x_{n+1}, x_n) \leq \{d^*(x_n, x_{n-1})\}^{\lambda}, \quad \lambda = \frac{p+sq}{1-p-sq} < 1$$

$$\Rightarrow \quad d^*(Tx_n, Tx_{n-1}) \leq \{d^*(x_{n-1}, x_{n-2})\}^{\lambda^2}$$

$$\leq \{d^*(x_{n-2}, x_{n-3})\}^{\lambda^3}$$

$$\vdots$$

$$\leq \{d^*(x_1, x_0)\}^{\lambda^n} = \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.13}$$

Now for $m > n$, we have

$$d^*(x_n, x_m) \leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s$$

$$\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_m)\}^{s^2}$$

$$\leq \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_{n+3})\}^{s^3} \ldots \{d^*(x_{m-1,} x_m)\}^{s^{m-n}}$$

$$\leq \{d^*(x_0, x_1)\}^{s\lambda^n} \{d^*(x_0, x_1)\}^{s^2\lambda^{n+1}} \{d^*(x_{0,} x_1)\}^{s^3\lambda^{n+2}} \ldots \{d^*(x_{0,} x_1)\}^{s^{m-n}\lambda^{m-1}} \text{ (Using 3.13)}$$

$$\leq \{d^*(x_0, x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}$$

$$\leq \{d^*(x_0, x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}$$

This shows that $d^*(x_n, x_m) \to 1$ as $m, n \to \infty$ .So, the sequence $\{x_n\}$ is b- multiplicative Cauchy sequence in X. Being the completeness of (X, d^*, s), there exist a point $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, we show that x^* is fixed point of T.

$$d^*(x^*, Tx^*) \leq \{d^*(Tx^*, Tx_n) d^*(Tx_n, x^*)\}^s$$

$$\Rightarrow \quad d^*(x^*, Tx^*) \leq \{d^*(Tx_n, Tx^*)\}^s \{d^*(Tx_n, x^*)\}^s$$

$$\Rightarrow \quad \{d^*(x^*, Tx^*)\} \leq \{d^*(Tx_n, x_n) d^*(Tx^*, x^*)\}^{sp} \{d^*(Tx_n, x^*) d^*(Tx^*, x_n)\}^{sq} \{d^*(Tx_n, x^*)\}^s$$

$$\Rightarrow \{d*(x*,Tx*)\} \le \{d*(x_{n+1},x_n)\}^{sp}\{d*(Tx*,x*)\}^{sp}\{d*(x_{n+1},x*)\}^{sq}\{d*(Tx*,x_n)\}^{sq}\{d*(x_{n+1},x*)\}^{s}$$

$$\Rightarrow \{d*(x*,Tx*)\}^{1-sp} \le \{d*(x_{n+1},x_n)\}^{sp}\{d*(x*,x_{n+1})\}^{sq+s}\{d*(Tx*,x_n)\}^{sq}$$

$$\Rightarrow d*(x*,Tx*) \le \{d*(x_{n+1},x_n)\}^{\frac{sp}{1-sp}}\{d*(x*,x_{n+1})\}^{\frac{sq+s}{1-sp}}\{d*(Tx*,x_n)\}^{\frac{sq}{1-sp}}$$

$$\Rightarrow d*(x*,Tx*) \to 1 \text{ as } n \to \infty.$$

This signifies that $Tx* = x*$ and hence $x*$ is fixed point of function T.

Again if x' is some other fixed point of function T, then we have

$$d*(x*,x') = d*(Tx*,Tx') \le \{d*(Tx*,x*)d*(Tx',x')\}^{p}\{d*(Tx*,x')d*(Tx',x*)\}^{q}$$

$$\Rightarrow d*(x*,x') = d*(Tx*,Tx') \le \{d*(x*,x')\}^{2q}\{d*(x*,x*)d*(x',x')\}^{p}$$

$$\Rightarrow d*(x*,x') = d*(Tx*,Tx') \le \{d*(x*,x')\}^{2q}$$

$$\Rightarrow \{d*(x*,x')\}^{1-2q} \le 1$$

Therefore, $d*(x*,x') = 1$ and thus $x* = x'$.

Hence, we get a unique fixed point of T.

Theorem 3.6. Let $(X,d*,s)$ be a complete b-multiplicative metric space. If the function $T: X \to X$ meets the following condition:

$$d*(Tx,Ty) \le \left\{d*(x,y)\right\}^{p}\left\{d*(Tx,x)\,d*(Ty,y)\right\}^{q}\left\{d*(Tx,y)\,d*(Ty,x)\right\}^{r} \quad \forall \ x,y \in X$$

where $p+2q+2sr<1$, $sp+q(1+s)+sr(1+s)<1$ and $s \ge 1$, then we get a unique fixed point $x* \in X$ such that $Tx* = x*$.

Proof. The proof follows from Theorem 3.4 and Theorem 3.5.

Theorem 3.7. Let $(X,d*,s)$ be a complete b-multiplicative metric space. If the function $T: X \to X$ meets the following condition:

$$d*(Tx,Ty) \le [\max\{d(x,Tx),d(y,Ty),d(x,y)\}]^p\{d(x,Ty)d(y,Tx)\}^q \quad \forall x,y \in X \tag{3.14}$$

where $p,q > 0$ and $p + 2sq < 1$. Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X defined by recursion

$$x_n = Tx_{n-1} = ... = T^n x_0,\ n = 1,2,3,... \tag{3.15}$$

By (3.14) and (3.15), we obtain

$$d*(Tx_{n-1},Tx_n) \le [\max\{d*(x_{n-1},Tx_{n-1}),d*(x_n,Tx_n),d*(x_{n-1},x_n)\}]^p[d*(x_{n-1},Tx_n)\,d*(x_n,Tx_{n-1})]^q$$

$$\Rightarrow \quad d*(Tx_{n-1},Tx_n) \le [\max\{d*(x_{n-1},x_n),d*(x_n,x_{n+1}),d*(x_{n-1},x_n)\}]^p[d*(x_{n-1},x_{n+1})\,d*(x_n,x_n)]^q$$

$$\Rightarrow \quad d*(Tx_{n-1},Tx_n) \le [\max\{d*(x_{n-1},x_n),d*(x_n,x_{n+1})\}]^p[d*(x_{n-1},x_{n+1})]^q$$

$$\Rightarrow \quad d*(Tx_{n-1},Tx_n) \le M_1{}^p\{d*(x_{n-1},x_n)\,d*(x_n,x_{n+1})\}^{sq}$$

where $M_1 = \max\{d*(x_{n-1},x_n),d*(x_n,x_{n+1})\}$.

Now, two cases arise:

CASE 1. If suppose $M_1 = d*(x_n,x_{n+1})$, then

$$d*(x_n,x_{n+1}) = d*(Tx_{n-1},Tx_n) \le \{d*(x_n,x_{n+1})\}^p\{d*(x_{n-1},x_n)\}^{sq}\{d*(x_n,x_{n+1})\}^{sq}$$

$$\Rightarrow \quad \{d*(x_n,x_{n+1})\}^{1-p-sq} \le \{(d*(x_{n-1},x_n)\}^{sq}$$

$$\Rightarrow \quad d*(x_n,x_{n+1}) \le \{d*(x_{n-1},x_n)\}^{\frac{sq}{1-p-sq}}$$

$$\Rightarrow \quad d*(x_n,x_{n+1}) \le \{d*(x_{n-1},x_n)\}^{\lambda}, \qquad \lambda = \frac{sq}{1-p-sq} < 1$$

$$\Rightarrow \quad d*(x_n,x_{n+1}) \le \{d*(x_{n-2},x_{n-1})\}^{\lambda^2}$$

Continuing this process, we get

$$d*(x_n,x_{n+1}) \le \{d*(x_0,x_1)\}^{\lambda^n}$$

$$\Rightarrow \quad d*(Tx_{n-1},Tx_n) \le \{d*(x_0,x_1)\}^{\lambda^n}$$

Thus, T is b-multiplicative contractive mapping.

CASE 2. Now, assume $M_1 = d*(x_{n-1},x_n)$, then

$$d^*(x_n, x_{n+1}) \le \{d^*(x_{n-1}, x_n)\}^p \{d^*(x_{n-1}, x_n)\}^{sq} \{d^*(x_n, x_{n+1})\}^{sq}$$

$$\Rightarrow \quad \{d^*(x_n, x_{n+1})\}^{1-sq} \le \{d^*(x_{n-1}, x_n)\}^{p+sq}$$

$$\Rightarrow \quad d^*(x_n, x_{n+1}) \le \{d^*(x_{n-1}, x_n)\}^{\frac{p+sq}{1-sq}}$$

$$\Rightarrow \quad d^*(x_n, x_{n+1}) \le \{d^*(x_{n-1}, x_n)\}^{\lambda}, \quad \lambda = \frac{p+sq}{1-sq} < 1$$

$$\Rightarrow \quad d^*(x_n, x_{n+1}) \le \{d^*(x_{n-2}, x_{n-1})\}^{\lambda^2}$$

Continuing like this, we obtain

$$d^*(x_n, x_{n+1}) \le \{d^*(x_0, x_1)\}^{\lambda^n} \tag{3.16}$$

$$\Rightarrow \quad d^*(Tx_{n-1}, Tx_n) \le \{d^*(x_0, x_1)\}^{\lambda^n}$$

Hence, T is multiplicative contraction mapping.

Now, we show that $\{x_n\}$ is a multiplicative Cauchy sequence in X.

Now for $m > n$, we have

$$d^*(x_n, x_m) \le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_m)\}^s$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_m)\}^{s^2}$$

$$\le \{d^*(x_n, x_{n+1})\}^s \{d^*(x_{n+1}, x_{n+2})\}^{s^2} \{d^*(x_{n+2,} x_{n+3})\}^{s^3} \ldots \{d^*(x_{m-1,} x_m)\}^{s^{m-n}}$$

$$\le \{d^*(x_0, x_1)\}^{s\lambda^n} \{d^*(x_0, x_1)\}^{s^2\lambda^{n+1}} \{d^*(x_{0,} x_1)\}^{s^3\lambda^{n+2}} \ldots \{d^*(x_{0,} x_1)\}^{s^{m-n}\lambda^{m-1}} \qquad \text{(Using 3.16)}$$

$$\le \{d^*(x_0, x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}$$

$$\le \{d^*(x_0, x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}$$

This shows that $d^*(x_n, x_m) \to 1$ as $m, n \to \infty$.

Hence, $\{x_n\}$ is a b-multiplicative Cauchy sequence in X. Since X is complete, therefore $\{x_n\}$ converges to $x^* \in X$.

Now, we show that x^* is the fixed point of T.

$$d^*(x^*, Tx^*) \le \{d^*(x^*, x_{n+1})\}^s \{d^*(x_{n+1}, Tx^*)\}^s$$

$$\Rightarrow \quad d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^s \{d^*(Tx_n,Tx^*)\}^s$$

$$\Rightarrow d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^s [\max\{d^*(x_n,Tx_n) d^*(x^*,Tx^*) d^*(x_n,x^*)\}]^{sp} \{d^*(x_n,Tx^*) d^*(x^*,Tx_n)\}^{sq}$$

$$\Rightarrow d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^s [\max\{d^*(x_n,x_{n+1}), d^*(x^*,Tx^*), d^*(x_n,x^*)\}]^{sp}$$
$$\{d^*(x_n,x^*)\}^{s^2q} \{d^*(x^*,Tx^*)\}^{s^2q} \{d^*(x^*,x_{n+1})\}^{sq}$$

$$\Rightarrow \{d^*(x^*,Tx^*)\}^{1-s^2q} \le \{d^*(x^*,x_{n+1})\}^{s(1+q)} (M_2)^{sp} \{d^*(x_n,x)\}^{s^2q}$$

$$\text{where } M_2 = \max\{d^*(x_n,x_{n+1}), d^*(x^*,Tx^*), d^*(x_n,x^*)\} \tag{3.17}$$

Suppose $M_2 = d^*(x_n,x_{n+1})$, then from (3.17), we have

$$\{d^*(x^*,Tx^*)\}^{1-s^2q} \le \{d^*(x^*,x_{n+1})\}^{s(1+q)} \{d^*(x_n,x_{n+1})\}^{sp} \{d^*(x_n,x^*)\}^{s^2q}$$

$$\Rightarrow \quad \{d^*(x^*,Tx^*)\}^{1-s^2q} \le \{d^*(x^*,x_{n+1})\}^{s(1+q)} \{d^*(x_n,x^*)\}^{s^2p} \{d^*(x^*,x_{n+1})\}^{s^2p} \{d^*(x_n,x^*)\}^{s^2q}$$

$$\Rightarrow \quad \{d^*(x^*,Tx^*)\}^{1-s^2q} \le \{d^*(x^*,x_{n+1})\}^{s(1+q+ps)} \{d^*(x_n,x^*)\}^{s^2(p+q)}$$

$$\Rightarrow \quad d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^{\frac{s(1+q+ps)}{1-s^2q}} \{d^*(x_n,x^*)\}^{\frac{s^2(p+q)}{1-s^2q}} .$$

As $\lim_{n\to\infty} x_n = x^*$, thus we obtain $\lim_{n\to\infty} d^*(x^*,Tx^*) = 1$ and hence $x^* = Tx^*$.

Therefore, x^* is fixed point of T .

Now, assume $M_2 = d^*(x_n,x^*)$.

Then (from 3.17), we obtain

$$\{d^*(x^*,Tx^*)\}^{1-s^2q} \le \{d^*(x^*,x_{n+1})\}^{s(1+q)} \{d^*(x_n,x^*)\}^{sp} \{d^*(x_n,x^*)\}^{s^2q}$$

$$\Rightarrow \quad d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^{\frac{s(1+q)}{1-s^2q}} \{d^*(x_n,x^*)\}^{\frac{sp+s^2q}{1-s^2q}}$$

Proceeding limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d^*(x^*,Tx^*) = 1 \Rightarrow x^* = Tx^*.$$

Therefore, x^* is fixed point of T .

Suppose that $M_2 = d^*(x^*,Tx^*)$.

Then (from 3.17), we get

$$\{d*(x*,Tx*)\}^{1-s^2q} \le \{d*(x*,x_{n+1})\}^{s(1+q)}\{d*(x*,Tx*)\}^{sp}\{d*(x_n,x*)\}^{s^2q}$$

$$\Rightarrow \{d*(x*,Tx*)\}^{(1-s^2q-sp)} \le \{d*(x*,x_{n+1})\}^{s(1+q)}\{d*(x_n,x*)\}^{s^2q}$$

$$\Rightarrow d*(x*,Tx*) \le \{d*(x*,x_{n+1})\}^{\frac{s(1+q)}{(1-sp-s^2q)}}\{d*(x_n,x*)\}^{\frac{s^2q}{(1-sp-s^2q)}}.$$

On taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d*(x*,Tx*) = 1 \Rightarrow x* = Tx*.$$

Therefore, $x*$ is the fixed point of T.

<u>Uniqueness of Fixed point:</u>

Assume that x' is another fixed point of T, then we have

$$d*(Tx*,Tx') \le [\max\{d*(x*,Tx*), d*(x',Tx'), d*(x*,x')\}]^p\{d*(x*,Tx')d*(x',Tx*)\}^q$$

$$\Rightarrow d*(x*,x') \le [\max\{d*(x*,x*), d*(x',x'), d*(x*,x')\}]^p\{d*(x*,x')d*(x',x*)\}^q$$

$$\Rightarrow d*(x*,x') \le \{d*(x*,x')\}^p\{d*(x*,x')\}^{2q}$$

$$\Rightarrow d*(x*,x') \le \{d*(x*,x')\}^{p+2q}$$

$$\Rightarrow \{d*(x*,x')\}^{1-p-2q} \le 1$$

As $d*(x*,x') \ge 1$, thus $d*(x*,x') = 1$ and hence $x* = x'$.

Hence, we obtain that mapping T has a unique fixed point.

Theorem 3.8. Let $(X, d*, s)$ be a complete b-multiplicative metric space. If the function $T : X \to X$ meets the following condition:

$$d*(Tx,Ty) \le \left\{d*(x,Tx)\,d*(y,Ty)\right\}^{\frac{p}{2}} \left\{d*(x,Ty)\,d*(y,Tx)\right\}^{\frac{q}{2}} \left\{d*(x,y)\right\}^{r} \quad \forall\ x, y \in X,$$

where $p,q,r > 0$ and $p + sq + r < 1$. Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X defined by recursion

$$x_n = Tx_{n-1} = ... = T^n x_0,\ n = 1, 2, 3, ...$$

Now,

$$d^*(x_n,x_{n+1})=d^*(Tx_{n-1},Tx_n)\le\{d^*(x_{n-1},Tx_{n-1})d^*(x_n,Tx_n)\}^{\frac{p}{2}}\{d^*(x_{n-1},Tx_n)d^*(x_n,Tx_{n-1})\}^{\frac{q}{2}}\{d^*(x_{n-1},x_n)\}^r$$

$$d^*(Tx_{n-1},Tx_n)\le\{d^*(x_{n-1},x_n)d^*(x_n,x_{n+1})\}^{\frac{p}{2}}\{d^*(x_{n-1},x_{n+1})d^*(x_n,x_n)\}^{\frac{q}{2}}\{d^*(x_{n-1},x_n)\}^r$$

$$d^*(Tx_{n-1},Tx_n)\le\{d^*(x_{n-1},x_n)d^*(x_n,x_{n+1})\}^{\frac{p}{2}}\{d^*(x_{n-1},x_n)d^*(x_n,x_{n+!})\}^{\frac{sq}{2}}\{d^*(x_{n-1},x_n)\}^r$$

$$d^*(Tx_{n-1},Tx_n)\le\{d^*(x_{n-1},x_n)\}^{\frac{p}{2}}\{d^*(x_n,x_{n+1})\}^{\frac{p}{2}}\{d^*(x_{n-1},x_n)\}^{\frac{sq}{2}}\{d^*(x_n,x_{n+!})\}^{\frac{sq}{2}}\cdot\{d^*(x_{n-1},x_n)\}^r$$

$$\{d^*(x_n,x_{n+1})\}^{\left(1-\frac{p}{2}-\frac{sq}{2}\right)}\le\{d^*(x_{n-1},x_n)\}^{\left(\frac{p}{2}+\frac{sq}{2}+r\right)}$$

$$d^*(x_n,x_{n+1})\le\{d^*(x_{n-1},x_n)\}^{\left(\frac{\frac{p}{2}+\frac{sq}{2}+r}{1-\frac{p}{2}-\frac{sq}{2}}\right)}$$

$$d^*(x_n,x_{n+1})\le\left\{d^*(x_{n-1},x_n)\right\}^{\lambda},\quad \lambda=\frac{\frac{p}{2}+\frac{sq}{2}+r}{1-\frac{p}{2}-\frac{sq}{2}}$$

$$d^*(x_n,x_{n+1})\le\{d^*(x_{n-2},x_{n-1})\}^{\lambda^2}$$

Continuing this process we get,

$$d^*(x_n,x_{n+1})\le\{d^*(x_0,x_1)\}^{\lambda^n} \tag{3.18}$$

$$d^*(Tx_{n-1},Tx_n)\le\{d^*(x_0,x_1)\}^{\lambda^n}.$$

Now for $m>n$, we have

$$d^*(x_n,x_m)\le\{d^*(x_n,x_{n+1})\}^s\{d^*(x_{n+1},x_m)\}^s$$

$$\le\{d^*(x_n,x_{n+1})\}^s\{d^*(x_{n+1},x_{n+2})\}^{s^2}\{d^*(x_{n+2,},x_m)\}^{s^2}$$

$$\le\{d^*(x_n,x_{n+1})\}^s\{d^*(x_{n+1},x_{n+2})\}^{s^2}\{d^*(x_{n+2,},x_{n+3})\}^{s^3}\ldots\{d^*(x_{m-1,},x_m)\}^{s^{m-n}}$$

$$\le\{d^*(x_0,x_1)\}^{s\lambda^n}\{d^*(x_0,x_1)\}^{s^2\lambda^{n+1}}\{d^*(x_{0,},x_1)\}^{s^3\lambda^{n+2}}\ldots\{d^*(x_{0,},x_1)\}^{s^{m-n}\lambda^{m-1}} \quad \text{(Using 3.18)}$$

$$\le\{d^*(x_0,x_1)\}^{s\lambda^n(1+s\lambda+s^2\lambda^2+\ldots+s^{m-n-1}\lambda^{m-n-1})}$$

$$\le\{d^*(x_0,x_1)\}^{\frac{s\lambda^n}{1-s\lambda}}$$

This signifies that $d^*(x_n,x_m)=1$ as $m,n\to\infty$.

So, $\{x_n\}$ is a b-multiplicative Cauchy sequence in X. Since X is a complete metric space, therefore $\{x_n\}$ converges to $x^* \in X$.

Now,

$$d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})d^*(x_{n+1},Tx^*)\}^s$$

$$d^*(x^*,Tx^*) \le \{d^*(x^*,x_{n+1})\}^s\{d^*(Tx_n,Tx^*)\}^s$$

$$d^*(x^*,Tx^*) \le d^*(x^*,x_{n+1})^s\{d^*(x_n,Tx_n)d^*(x^*,Tx^*)\}^{\frac{sp}{2}}\{d^*(x_n,Tx^*)d^*(x^*,Tx_n)\}^{\frac{sq}{2}}\cdot\{d^*(x_n,x^*)\}^{sr}$$

$$d^*(x^*,Tx^*) \le d^*(x^*,x_{n+1})^s\{d^*(x_n,x_{n+1})d^*(x^*,Tx^*)\}^{\frac{sp}{2}}\{d^*(x_n,x^*)d^*(x^*,Tx^*)\}^{\frac{s^2q}{2}}$$
$$\{d^*(x^*,x_{n+1})\}^{\frac{sq}{2}}\{d^*(x_n,x^*)\}^{sr}$$

$$d^*(x^*,Tx^*) \le d^*(x^*,x_{n+1})^s\{d^*(x_n,x^*)d^*(x^*,x_{n+1})\}^{\frac{s^2p}{2}}\{d^*(x^*,Tx^*)\}^{\frac{sp}{2}}\{d^*(x_n,x^*)d^*(x^*,Tx^*)\}^{\frac{s^2q}{2}}$$
$$\{d^*(x^*,x_{n+1})\}^{\frac{sq}{2}}\{d^*(x_n,x^*)\}^{sr}$$

$$\{d^*(x^*,Tx^*)\}^{\left(1-\frac{sp}{2}-\frac{s^2q}{2}\right)} \le \{d^*(x_n,x^*)\}^{\left(sr+\frac{s^2p}{2}+\frac{s^2q}{2}\right)}\{d^*(x^*,x_{n+1})\}^{\left(s+\frac{s^2p}{2}+\frac{sq}{2}\right)}$$

$$d^*(x^*,Tx^*) \le \{d^*(x_n,x^*)\}^{\left(\frac{sr+\frac{s^2p}{2}+\frac{s^2q}{2}}{1-\frac{sp}{2}-\frac{s^2q}{2}}\right)}\{d^*(x^*,x_{n+1})\}^{\left(\frac{s+\frac{s^2p}{2}+\frac{sq}{2}}{1-\frac{sp}{2}-\frac{s^2q}{2}}\right)}$$

As limit $n \to \infty$, thus we obtain $d(x^*,Tx^*) = 1$.Therefore, $x^* = Tx^*$.

Hence, x^* is fixed point of T.

Further, we will show that T has unique fixed point.

Assume that x' is another fixed point of T, then we have $Tx' = x'$ and $Tx^* = x^*$.

$$d(x^*,x') = d(Tx^*,Tx') \le \{d(x^*,Tx^*)d(x',Tx')\}^{\frac{p}{2}}\{d(x^*,Tx')d(x',Tx^*)\}^{\frac{q}{2}}\{d(x^*,x')\}^r$$

$$\le \{d(x^*,x^*)d(x',x')\}^{\frac{p}{2}}\{d(x^*,x')d(x',x^*)\}^{\frac{q}{2}}\{d(x^*,x')\}^r$$

$$\leq \{d^*(x^*, x')\}^{\frac{2q}{2}} \cdot \{d^*(x^*, x')\}^r$$

$\Rightarrow \{d^*(x^*, x')\}^{1-q-r} \leq 1$.

As $d^*(x^*, x') \geq 1$, thus $d^*(x^*, x') = 1$ and hence $x^* = x'$.

Hence, T has a unique fixed point.

Corollary 3.9. Let (X, d^*, s) be complete b-multiplicative metric space and $T : X \to X$ be a mapping such that

$$d^*(Tx,Ty) \leq \{d^*(x,Tx)d^*(y,Ty)\}^p \{d^*(x,Ty)d^*(y,Tx)\}^q \{d^*(x,y)\}^r \quad \forall\, x, y \in X$$

where $p, q, r > 0$ and $2p + q(2s+1) + r < 1$ .Then, T has a unique fixed point.

Proof. The proof easily follows from Theorem 3.8.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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