Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 6910-6922
https://doi.org/10.28919/jmcs/6477
ISSN: 1927-5307

NUMERICAL METHODS FOR FINDING PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH STRONG NONLINEARITY

AMER DABABNEH[1], AMJED ZRAIQAT[1], ABDULKARIM FARAH[2], HASSAN AL-ZOUBI[1],
MA'MON ABU HAMMAD[1,*]

[1]Department of Mathematics, Faculty of Science & IT, Al-Zaytoonah University of Jordan, Amman, Jordan

[2]Department of Mathematics, Faculty of Science & IT, Isra University, Amman, Jordan

Abstract: The paper introduces an ordinary differential equation consisting of a positively homogeneous main part of an order of m > 1 and a periodic perturbation. Approximate method of finding periodic solutions, which attempts to find zeros of explicitly defined finite-dimensional mappings, is presented.

Keywords: differential equations; periodic solutions; difference scheme.

2010 AMS Subject Classification: 65L05.

1. INTRODUCTION

It is known that finding the exact solutions of a system of nonlinear differential equations is only possible in some particular cases. This problem becomes even more complicated when considering the issue of finding periodic solutions. It justifies the need for resorting in order to

*Corresponding author
E-mail address mabuhammad@gmail.com
Received July 09, 2021

approximate methods for finding periodic solutions, which in return, can provide important information around the phenomenon under study.

A considerable number of published articles is devoted to the problem of determining the conditions for finding the existence of periodic solutions that are related to differential equations and numerical methods. The existing conditions for periodic solutions are mainly related to the existence of a fixed point of some mappings, which are induced by a given equation, or by the application of variational methods in which the periodic solution is considered an extremal of some functions. In general, despite the fact that the proven fact of the periodic solution existence does not allow one to determine the points through which this solution passes, the problem of constructing numerical methods for finding a periodic solution is extremely significant.

It is known that ω-periodicity in t of the right-hand side of the differential equation $x' = f(t,x),\ x \in R^N$ does not imply the existence of a ω-periodic solution. For example, a system of linear differential equations

$$x' = Ax + f(t),\ \ x \in R^N,\ f(t) \in C^1(R),\ f(t+\omega) = f(t)$$

may even not contain ω-periodic solutions, such as the linear Hamiltonian system, which is derived from [2]. A simpler example of a similar system can be provided as follows:

It is obvious that this system does not have 2π-periodic solutions, in spite of the fact that 2π exists in the equation $f(t+2\pi) = f(t)$.

Thus, the problem of finding periodic solutions of a differential equation by numerical methods must be accompanied by an evidence for the existence of such solutions.

One of the numerical methods for finding periodic solutions is the Collocation method. The description of this method and its applications can be found in [2]. In some cases, periodic solutions can be found for a system of quasi-linear equations based on the use of the method that is proposed by A.M. Samoilenko [12]. A description of this method and references can be found in [3].

The purpose of this paper is to explore the periodic solutions of a differential equation with a positively homogeneous main part of an order of $m > 1$ and with a periodic perturbation as:

$$x' = P(\mathrm{x}) + p(\mathrm{t},\mathrm{x}) \tag{1}$$

Where,

$$x \in R^N, t \in R, P: R^N \to R^N, \quad p: R \times R^N \to R^N$$

$$P(\lambda x) = \lambda^\alpha P(x), \ \lambda \geq 0, \ m > 1, \tag{2}$$

$$p(t+\omega, x) = p(t, x) \tag{3}$$

$$p(\mathrm{t}, \mathrm{x}) / |x|^m \to 0, \ |x| \to \infty \quad \text{is uniformed in } t. \tag{4}$$

Such system as (1) is widely used in physics and in mathematical models that are applied in many different fields as engineering, chemistry, economy, finance, etc. [4].

System (1) was researched in a number of different previous studies. The existence of periodic solutions for system (1) was studied in [5].

Conditions for the existence of aprioristic estimate of periodic solutions and sufficient conditions for the existence of periodic solutions are all presented in [6–8]. There has been a few numbers of studies that are related to the problem of creating approximate methods for finding periodic solutions. For instance, the application of the collocation method does not always yield to satisfactory results for system (1). The reason behind this is that: If the main part of system (1) is positively homogeneous, and its order is $m > 1$, one can deal with strong nonlinearity where not all solutions of system (1) are non-locally extended.

Next we assume that the functions are continuously differentiable, and the following condition is met:

$$P(x) = 0 \Leftrightarrow x = 0 \tag{5}$$

2. MAIN RESULTS

M Replace system (1) by a different equation, and study this discrete analogue of Equation (1).

Divide the interval into n equal parts.

Let $h = \omega / n, \ t_k = kh; \quad k = 0,1,\ldots,\mathrm{n}; \ x_k = x(t_k) \in \mathrm{R}^N$.

Approximate a derivative as follows:

$$x'(t_k) \approx \frac{x(t_{k+1}) - x(t_{k-1})}{2h}$$

Construct a difference scheme for finding periodic solutions. The periodicity condition for the solution implies that the following condition is met: $x_0 = x_n$.

Considering the periodicity condition, the following discrete problem can be obtained:

$$\begin{cases} x_0 = x_2 - 2hF_1 \\ x_1 = x_3 - 2hF_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-2} = x_0 - 2hF_{n-1} \\ x_{n-1} = x_1 - 2hF_0 \end{cases}, \qquad (6)$$

where,

$$F_k = P(x_k) + p(t_k, x_k).$$

Problem (6) is a nonlinear system of equations containing $N \cdot n$ unknowns.

Let

$$X^T = \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \end{pmatrix}; \qquad F^T(X) = \begin{pmatrix} F_0 & F_1 & \ldots & F_{n-1} \end{pmatrix};$$

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 \ldots & 0 & 1 \\ 1 & 0 & 0 & 0 \ldots & 0 & 0 \end{pmatrix}; \qquad (7)$$

Let $\| X \| = \max_k | x_k |$, where $| x_k |$ is the norm in R^N.

Remark: Given a matrix

$$C = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0,n-1} \\ c_{10} & c_{11} & \cdots & c_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1,0} & c_{n-1,1} & \cdots & c_{n-1,n-1} \end{pmatrix},$$

then, multiplying C by X gives:

$$(CX)^T = \left(\sum_{j=0}^{n-1} c_{0j} x_j \quad \sum_{j=0}^{n-1} c_{1j} x_j \quad \ldots \quad \sum_{j=0}^{n-1} c_{n-1,j} x_j \right), \qquad x_j \in R^N$$

and hence, equation (2) becomes:

$$X = QX - 2hBF(X) \tag{8}$$

It is normal to assume that the solution of equation (8) is an approximate value of the periodic solution of system (1). Possible methods for estimating the rate of convergence of solutions of equation (8) to the solution of system (1) can be found in [11]. The existence of solutions of equation (8) requires the fulfilment of other additional conditions. Equation (8) could also be obtained as follows:

Assume that system (1) is integrated over $[t_{i-1};t_{i+1}]$ interval. The following equation is obtained:

$$x(t_{i+1}) - x(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} F(t, x(t))dt, \quad F(t,x) = P(x) + p(t,x)$$

By applying various formulas for approximate calculations of a definite integral, and by taking into consideration the condition of periodicity solution, various types of discrete equations for the approximate determination of periodic solutions can be obtained. In particular, knowing that

$$\int_{t_{i-1}}^{t_{i+1}} F(t, x(t))dt \approx 2hF(t_i, x(t_i)) = 2hF_i$$

Then, Problem (2) can be obtained.

Based on other approximation formulas, it is possible to obtain other more exact discrete equations. For example, if it is assumed that

$$\int_{t_{i-1}}^{t_{i+1}} F(t,x(t))dt \approx h\left(\frac{1}{2}F(t_{i-1},x(t_{i-1})) + F(t_i,x(t_i)) + \frac{1}{2}F(t_{i+1},x(t_{i+1}))\right) =$$

$$= \frac{h}{2}\left(F_{i-1} + 2F_i + F_{i+1}\right)$$

then the following equation can be obtained:

$$\begin{cases} x_0 = x_2 - \frac{h}{2}\left(F_0 + 2F_1 + F_2\right) \\ x_1 = x_3 - \frac{h}{2}\left(F_1 + 2F_2 + F_3\right) \\ \dots\dots\dots\dots\dots\dots\dots \\ x_{n-2} = x_0 - \frac{h}{2}\left(F_{n-2} + 2F_{n-1} + F_0\right) \\ x_{n-1} = x_1 - \frac{h}{2}\left(F_{n-1} + 2F_0 + F_1\right) \end{cases}$$

which can be rewritten as follows:

$$X = QX - \frac{h}{2}B_1F(\mathrm{X}),$$

Where,

$$B_1 = \begin{pmatrix} 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & 2 \\ 2 & 1 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad (9)$$

By applying Simpson’s rule

$$\int_{t_{i-1}}^{t_{i+1}} F(t,x(t))dt \approx \frac{h}{3}\left(F(t_{i-1},x(t_{i-1}))+4F(t_i,x(t_i))+F(t_{i+1},x(t_{i+1}))\right)=$$

$$=\frac{h}{3}\left(F_{i-1}+4F_i+F_{i+1}\right),$$

other discrete problems can be derived as follows:

$$X = QX - \frac{h}{3}B_2 F(\mathrm{X})$$

where,

$$B_2 = \begin{pmatrix} 1 & 4 & 1\ldots & 0 & 0 & 0 \\ 0 & 1 & 4\ \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0\ldots & 4 & 1 & 0 \\ 0 & 0 & 0\ \ldots & 1 & 4 & 1 \\ 1 & 0 & 0\ldots & 0 & 1 & 4 \\ 4 & 1 & 0\ \ldots & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Further, assume that B is a matrix whose form is similar to (7), (9) or (10).

Theorem 1. Assume that the conditions (2)-(5) are met. Then, the solutions of equation (8) allow aprioristic assessment.

Proof. Assume that there is a sequence X_k, such that

$$r_k = \| X_k \|, \;\; r_k \to \infty, k \to \infty .$$

Let $U_k = X_k / r_k, \quad \| \mathrm{U}_k \| = 1.$

Without loss of generality, it is possible to consider that

$U_k \to U_*, k \to \infty, \; \| U_* \| = 1.$

Now, assume that

$$r_k U_k = r_k Q U_k - 2hBF(r_k U_k).$$

Since

$$F_k(t_k, r_k u_k) = P(r_k u_k) + p(t_k, r_k u_k) = r_k^m \left(P(u_k) + \frac{1}{r_k^m} p(t_k, r_k u_k) \right),$$

and hence,

$$\frac{1}{r_k^{m-1}} U_k = \frac{1}{r_k^{m-1}} QU_k - 2hB\left(P(U_k) + \frac{1}{r_k^m} p(U_k) \right).$$

Passing to a limit at $k \to \infty$, $BP(U_*) = 0$ will be obtained. Nonetheless, in this case, $P(U_*) = 0$.

We have arrived at a contradiction.

It follows from Theorem 1 that on the sphere of sufficiently large radius, the mappings $\Psi = E - Q + 2hBF$ and $\Psi_0 = BF$ are homotopic.

Therefore, we can say that if $\underset{x=0}{ind}\, P(x) \neq 0$, then, Equation (8) has a solution.

Assume that there is a function $P(\alpha, \mathrm{x}),\ P \in C\left([0;1] \times \mathrm{R}^N, R^N\right)$ for which the following conditions are met:

$$P(\alpha, \lambda \mathrm{x}) = \lambda^m P(\alpha, \mathrm{x}),\ \lambda \geq 0, \quad P(\alpha, \mathrm{x}) = 0 \Leftrightarrow x = 0$$

Consider a family of equations:

$$X = QX - 2hBF(\alpha, \mathrm{X}), \tag{11}$$

Where,

$$F(\alpha, X) = \begin{pmatrix} F_0(\alpha) \\ F_1(\alpha) \\ \ldots \\ F_{n-1}(\alpha) \end{pmatrix} \qquad F_k(\alpha) = P(\alpha, x_k) + p(t_k, x_k)$$

From Theorem 1, the following statement is obtained:

Theorem 2. Assume that Equation (11) has the solution at $\alpha = 0$ and for any function $p(t, x)$ which meets Conditions (3) and (4) uniformly in *α*. Then, at $\alpha = 1$, the following equation is obtained:

$$X = QX - 2hBF(1, \mathrm{X}) \tag{12}$$

which also has the solution for any function $p(t,x)$ meeting Conditions (3) and (4).

Proof. Let A denote the set of numbers $\alpha \in [0;1]$ for which the equation $X = QX - 2hBF(\alpha, \mathrm{X})$ for some functions $p_\alpha(t,x)$ has no solutions. We should note that if for some functions $p_1(t,x)$ the equation $X = QX - 2hBF(1, \mathrm{X})$ has no solution, the set number A is not empty and open. On the other hand, choosing ε as a sufficiently small positive number, it is not difficult to realize that if for some α_0 the equation $X = QX - 2hBF(\alpha_0, \mathrm{X})$ does not have solutions for some functions $p_{\alpha_0}(t,x)$, then for each $\alpha \in [\alpha_0 - \varepsilon; \alpha_0 + \varepsilon]$, then such a function $p_\alpha(t,x)$ can be determined for which the equation $X = QX - 2hBF(\alpha, \mathrm{X})$ also does not have solutions. It should be noted that the choice of ε does not depend on α. The existence of such a number ε follows from the existence of a general a priori estimate for all solutions of Equation (11). Consequently, the set number A of numbers α for which the equation $X = QX - 2hBF(\alpha, \mathrm{X})$ has no solutions for some functions $p_\alpha(t,x)$ is closed. But then $\mathrm{A} = [0;1]$.

Let $E_0 = \ker(E - Q),\ E_1 = \mathrm{Im}(E - Q)$.

Further, assume that n is an odd number, then in this case, $\dim(\mathrm{E}_0) = 1$.

Assume that P_0 is a projection on E_0, and P_1 is a projection on E_1.

Let $X = X_0 + X_1$, $X_0 \in E_0$, $X_1 \in E_1$, Equation (8) can be rewritten as follows:

$$\begin{cases} X_1 = P_1 Q X_1 - 2hP_1 BF(\mathrm{X}_0 + \mathrm{X}_1) \\ P_0 Q X_1 = 2hP_0 BF(\mathrm{X}_0 + \mathrm{X}_1) \end{cases} \tag{13}$$

Then, the following equation can be obtained:

$$X_1 = -2h(E - P_1Q)^{-1} P_1 BF(X_0 + X_1) \tag{14}$$

If h is sufficiently a small number, then the operator

$$G(X_1) = -2h(E - P_1Q)^{-1} P_1 BF(X_0 + X_1)$$

is a contraction operator in E_1. Again.

Thus, the approximate finding of the periodic solution of Equation (1) can be replaced with the finding of the system solution of Equation (13).

One can find the solution from System (13) using the following algorithm:

1) Set an initial value $X_1^{(0)} \in E_1$ и $X_0^{(0)} \in E_0$,

2) Calculate $X_1^{(1)} = -2h(E - P_1Q)^{-1} P_1 BF(X_0^{(0)} + X_1^{(0)})$,

3) Find $X_0^{(1)}$ from the equation $P_0 Q X_1^{(1)} = 2hP_0 BF(X_0 + X_1^{(1)})$,

4) Calculate $X_1^{(2)} = -2h(E - P_1Q)^{-1} P_1 BF(X_0^{(1)} + X_1^{(1)})$,

5) Go to step (3),

6) Continue calculations until the required accuracy is reached.

For instance, consider a system of equations:

$$\begin{cases} \dot{x}_1 = x_2\sqrt{x_1^2 + x_2^2} + x_1 \cdot \cos t + x_2 \sin t - 2\sin t - 1 \\ \dot{x}_2 = x_1\sqrt{x_1^2 + x_2^2} \end{cases} \tag{15}$$

In accordance with the above notation, we have:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad P(x) = \begin{pmatrix} x_2\sqrt{x_1^2 + x_2^2} \\ x_1\sqrt{x_1^2 + x_2^2} \end{pmatrix}, \quad p(t,x) = \begin{pmatrix} x_1 \cdot \cos t + x_2 \sin t - 2\sin t - 1 \\ 0 \end{pmatrix}$$

As a matrix B, take the matrix from Equation (7), $\omega = 2\pi$.

Take $n = 19$. Then, $h = 2\pi / n$.

Pre-calculate the auxiliary matrix $(E - P_1Q)^{-1} P_1 B$ and $P_0 B$.

After 12 iterations, the following values were obtained for x_1 and x_2:

t_i	x_1	x_2	t_i	x_1	x_2
0	0,934379	0,010138	3,30694	-0,92312	-0,13624
0,330694	0,879337	0,296918	3,637634	-0,82405	-0,41945
0,661388	0,729388	0,55658	3,968328	-0,63416	-0,66567
0,992082	0,503193	0,763644	4,299022	-0,3757	-0,84332
1,322776	0,225289	0,890692	4,629715	-0,07719	-0,92518
1,65347	-0,07575	0,917269	4,960409	0,229505	-0,89748
1,984164	-0,369	0,838737	5,291103	0,511823	-0,76498
2,314858	-0,62439	0,668435	5,621797	0,739255	-0,54914
2,645552	-0,81524	0,431628	5,952491	0,885971	-0,28056
2,976246	-0,91955	0,155476	3,30694	-0,92312	-0,13624
3,30694	-0,92312	-0,13624	3,637634	-0,82405	-0,41945

Table (1)

Table 1 illustrates the approximate values for x_1 and x_2 after 12 iterations.

The solution of the equation $P_0QX_1^{(1)} = 2hP_0BF(\mathrm{X}_0 + \mathrm{X}_1^{(1)})$ was determined with an accuracy of up to $\Delta = 0.01$.

The exact solution of System (15) has the form: $x_1(t) = \cos t;\quad x_2(t) = \sin t$.

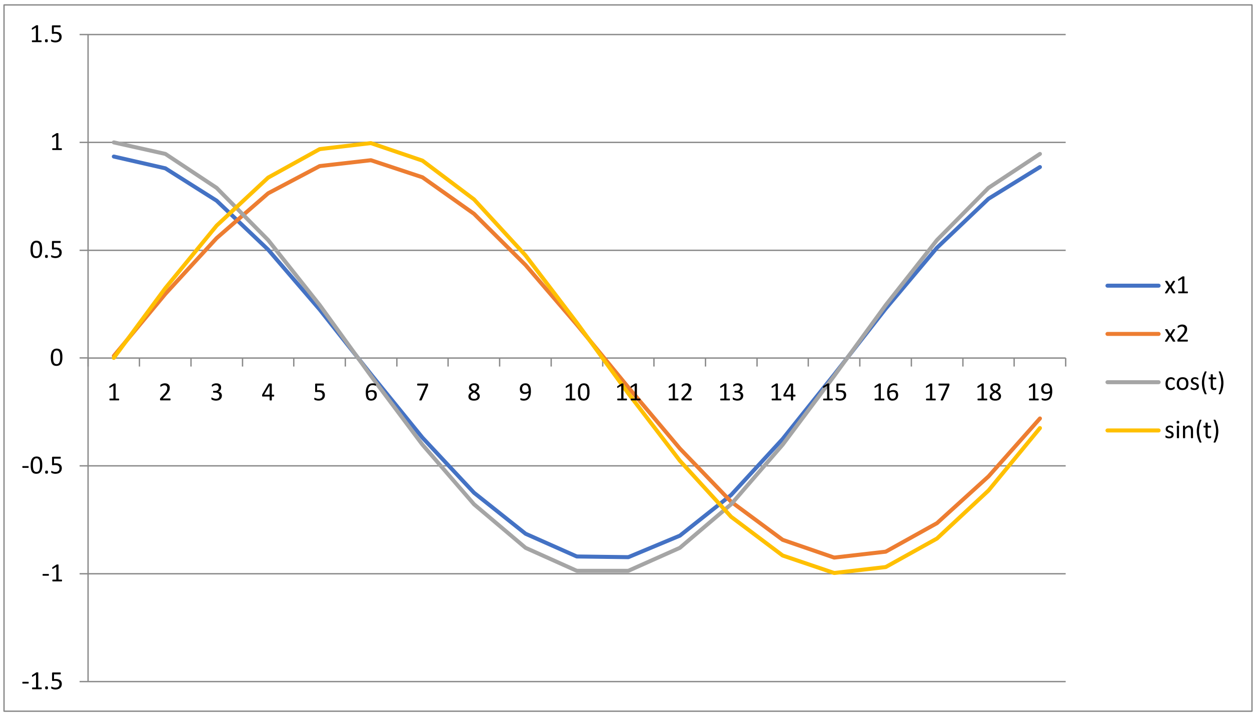

Fig. (1)

Figure 1 illustrates different graphs of exact and approximate solutions.

The problem of finding periodic solutions for Equation (1) can be similarly considered for the case of homogeneity order of $m \in (0;1)$. In this case, the problem also attempts to find the solution of System (13).

3. CONCLUSION

The paper introduced an effective algorithm for an approximate finding of periodic solutions of the equations with positively homogeneous main parts. The algorithm can be applied both to the equations with homogeneity order of $m > 1$, and to the equations with homogeneity order of $0 < m < 1$. In this paper, sufficient conditions for the aprioristic assessment of the existence of periodic solutions', and the condition for existence of periodic solutions are formulated. The statement on homotopy invariance of periodic solutions' existence for Equation (8) is announced. It will allow defining necessary and sufficient conditions for the existence of periodic solutions.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] U.M. Ascher, R.M. Mattheij and R.D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. SIAM Publications, Philadelphia, (1995).

[2] J.P. Aubin, I. Ekeland, Applied Nonlinear Analysis. Wiley, Hoboken, (1984).

[3] N.A. Bobylev, M.A. Krasnosel'sky, Principles of affinity in nonlinear problems, J. Math. Sci. 83(4) (1997), 485-521.

[4] N.A. Bobylev, S.V. Emel'yanov, S.K. Korovin, Geometrical methods in variational problems, Kluwer Academic Publishers, Dordrecht (1999).

[5] S.V.E. and S.K. Korovin, Control of complex and uncertain systems: new types of feedback, Meas. Sci. Technol. 12 (2001), 653–653.

[6] M.A. Krasnosel'skii, Approximate Solution of Operator Equations, Springer, (1972).

[7] M.A. Krasnoselskii, P.P. Zabreiko, Geometrical methods of nonlinear analysis, Springer, (1984).

[8] J.H. He, Homotopy perturbation technique, Comput. Meth. Appl. Mech. Eng. 178 (1999), 257–262.

[9] J.H. He, The Homotopy perturbation method nonlinear oscillators with discontinuities, Appl. Math. Comput. 151 (2004), 287–292.

[10] N.J. Mallon, Collocation: a method for computing periodic solutions of ordinary differential equations, (DCT rapporten; Vol. 2002.035), Technische Universiteit Eindhoven, (2002).

[11] A. Rontó, M. Rontó, Successive Approximation Techniques in Non-Linear Boundary Value Problems for Ordinary Differential Equations, in: Handbook of Differential Equations: Ordinary Differential Equations, Elsevier, 2008: pp. 441–592.

[12] A.M. Samoilenko, Certain questions of the theory of periodic and quasi-periodic systems, D. Sc. Dissertation, Kiev. (1967).