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MULTI-CRITERIA DECISION-MAKING IN COMPLEX FERMATEAN FUZZY ENVIRONMENT

V. CHINNADURAI*, S. THAYALAN, A. BOBIN

Department of Mathematics, Annamalai University, Chidambaram, India.

Abstract. In this article, we present the notion of complex Fermatean fuzzy set. We introduce various operators namely, complex Fermatean Fuzzy weighted average, complex Fermatean Fuzzy weighted geometric operator, complex Fermatean Fuzzy weighted power average and complex Fermatean Fuzzy weighted power geometric operators and discuss some of their properties. Finally, we present a MCDM problem and an algorithm to solve it with supporting case studies using these operators. We have compared the results of these operators to show the reliability of the proposed method.

Keywords: Fermatean fuzzy set; complex Fermatean fuzzy set; aggregation operator; decision-making.

2010 AMS Subject Classification: 47S40.

1. INTRODUCTION

The concept of fuzzy set (FS) introduced by Zadeh [24] is one of the important events in mathematics. Atanassov [2] proposed the notion of Intuitionistic fuzzy set (IFS) which is a generalization of FS. Wei [14] investigated geometric aggregation operators (AOs) for IFS. Wei and Lui [15] presented the concept of AOs using Einstein operations on IFS and discussed some properties. Xu and Yager [16] introduced AOs in IFS and established some of their properties. Xu [17] proposed Choquet integrals to solve multi-criteria decision-making (MCDM) problems

*Corresponding author
E-mail address: chinnaduraiau@gmail.com
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in IFS. Xu [18] established AOs in IFS and studied some of their properties. Zhao et al. [25] developed some generalized AOs for IFS. Yager [22] investigated the AOs and used these operators to solve MCDM problems. Yager [23] introduced Pythagorean fuzzy subsets and the idea of Pythagorean membership values. Yager [21] presented AOs in IFS and studied some of their properties. Yager [20] focussed on MCDM problems where the parameters are expressed in Pythagorean FS (PFS). Zhang and Xu [26] used TOPSIS method to solve MCDM problems in PFS. Zeng et al. [27] developed AOs for PFS and studied some of there properties. Zhang [29] introduced similarity measures and applied them in solving group decision-making problems.

Ramot et al. [10] extended the concept of FS to complex fuzzy set (CFS) in which the membership values are in the form of complex numbers. Alkouri and Salleh [1] generalized the concept of CFS and presented the idea of complex IFS (CIFS) by adding non-membership value to it. Chinnadurai et al. [4] defined the concept of complex cubic (IFS) and discussed some of its properties. The same authors [5] proved some properties of complex interval-valued PFS. The same authors [6] presented the idea of complex cubic set and their applications to MCDM. Zhou et al. [28] proposed complex cubic fuzzy AOs to solve MCDM problems. Yager and Abbasov [19] showed the relationship between PFS and complex numbers. Ullah et al. [13] defined some similarity measures between complex PFS (CPFS) and applied them to pattern recognition problems. Akram et al.[3] investigated AOs using Yager t-norm and s-norm in CPFS. Garg and Rani [7] defined AOs on complex IFS and discussed some properties. Garg and Rani [8] introduced the concept of AOs using T-Norm on complex IFS and presented some of there properties. Hu et al. [9] investigated AOs in CFS. Senapathi and Yager [12] proposed the concept of Fermatean FS (FFS) and compared FFS with PFS and IFS. The same authors [11] investigated AOs for FFS.

In this paper, we extend the concept of FFS to complex FFS. The main idea behind the study is that complex Fermatean fuzzy number (CFFN) is larger class than complex intuitionistic fuzzy number (CIFN) and complex Pythagorean fuzzy number (CPFN). This paper consists of six sections. Sections 1 and 2 deal with the introduction and basic definitions required for this study. Section 3 introduces the concept of CFFS. Section 4 investigates the properties of AOs. Section 5 proposes the method to solve MCDM problems. Section 6 ends with the conclusion.

2. PRELIMINARIES

In this section, we present the basic concepts of CIFS, CPFS, FFS. Through out the discussion U represent the universe.

Definition 2.1[1] A CIFS $\mathfrak{F}$ in U represented as $\mathfrak{F} = \{(\breve{z}, \mu_{\mathfrak{F}}(\breve{z}), \nu_{\mathfrak{F}}(\breve{z})) : \breve{z} \in U\}$, where $\mu_{\mathfrak{F}} : U \to \{\hat{a} : \hat{a} \in C, |\hat{a}| \leq 1\}$ and $\nu_{\mathfrak{F}} : U \to \{\hat{a} : \hat{a} \in C, |\hat{a}| \leq 1\}$ are complex valued membership and non-membership functions respectively given by $\mu_{\mathfrak{F}}(\breve{z}) = r_{\mathfrak{F}}(\breve{z}) e^{i2\pi\theta_{r_{\mathfrak{F}}}(\breve{z})}$ and $\nu_{\mathfrak{F}}(\breve{z}) = q_{\mathfrak{F}}(\breve{z}) e^{i2\pi\theta_{q_{\mathfrak{F}}}(\breve{z})}$. Here $r_{\mathfrak{F}}(\breve{z}), q_{\mathfrak{F}}(\breve{z}) \in [0,1]$ such that $0 \leq r_{\mathfrak{F}}(\breve{z}) + q_{\mathfrak{F}}(\breve{z}) \leq 1$. Also $\theta_{r_{\mathfrak{F}}}(\breve{z})$ and $\theta_{q_{\mathfrak{F}}}(\breve{z})$ are real-valued, where $i = \sqrt{-1}$ for all $\breve{z} \in U$.

Definition 2.2[3] A CPFS $\mathscr{F}_t$ represented as $\mathscr{F}_t = \{(\breve{z}, P_{\mathscr{F}_t}(\breve{z}), Q_{\mathscr{F}_t}(\breve{z})) | x \in U\}$, where $P_{\mathscr{F}_t} : U \to \{\hat{a} : \hat{a} \in C : |\hat{a}| \leq 1\}$, $Q_{\mathscr{F}_t} : U \to \{\hat{a} : \hat{a} \in C : |\hat{a}| \leq 1\}$ provided that $P_{\mathscr{F}_t}(\breve{z}) = \gamma_t(\breve{z}).e^{i2\pi\theta_{\gamma_t}(\breve{z})}$ and $Q_{\mathscr{F}_t}(\breve{z}) = \kappa_t(\breve{z}).e^{i2\pi\theta_{\kappa_t}(\breve{z})}$, satisfying the conditions $0 \leq \gamma_t^2(\breve{z}) + \kappa_t^2(\breve{z}) \leq 1$ and $0 \leq \theta_{\gamma_t}^2(\breve{z}) + \theta_{\kappa_t}^2(\breve{z}) \leq 1$. The degree of hesitancy functions $H_t = \eta_t(\breve{z}).e^{i2\pi\theta_{\eta_t}(\breve{z})}$, in such that
$\eta_t(\breve{z}) = \sqrt{1 - \gamma_t^2(\breve{z}) - \kappa_t^2(\breve{z})}$ and $\theta_{\eta_t}(\breve{z}) = \sqrt{1 - \theta_{\gamma_t}^2(\breve{z}) - \theta_{\kappa_t}^2(\breve{z})}$. Then $\mathscr{F}_t = (\gamma_t.e^{i2\pi\theta_{\gamma_t}}, \kappa_t.e^{i2\pi\theta_{\kappa_t}})$ is called a CPyFN.

Definition 2.3[11] A FFS $\mathfrak{C}$ in U in this structure $\mathfrak{C} = \{\breve{z}, \langle \chi_{\mathfrak{C}}(\breve{z}), \varphi_{\mathfrak{C}}(\breve{z}) \rangle : \breve{z} \in U\}$, where $\chi_{\mathfrak{C}}(\breve{z}) : U \longrightarrow [0,1]$ and $\varphi_{\mathfrak{C}}(\breve{z}) : U \longrightarrow [0,1]$ which satisfy the conditions $0 \leq (\chi_{\mathfrak{C}}(\breve{z}))^3 + (\varphi_{\mathfrak{C}}(\breve{z}))^3 \leq 1$ for all $\breve{z} \in U$, is such that the indeterminacy degree of $\breve{z}$ to $\mathfrak{C}$ is $\xi_{\mathfrak{C}}(\breve{z}) = \sqrt[3]{1 - (\chi_{\mathfrak{C}}(\breve{z}))^3 - (\varphi_{\mathfrak{C}}(\breve{z}))^3}$ for all $\breve{z} \in U$.

3. COMPLEX FERMATEAN FUZZY SET (CFFS)

In this section, we define a new concept CFFS and discuss some of its properties.

Definition 3.1 Let U be the universal set. A Complex Fermatean Fuzzy Set (CFFS) represents as $\mathfrak{A} = \{\dot{z}, F(\dot{z}), G(\dot{z}) | \dot{z} \in U\}$ where $F(\dot{z}) : U \to \{\dot{a} : \dot{a} \in \mathfrak{A}, |\dot{a}| \leq 1\}$, $G(\dot{z}) : U \to \{\dot{a} : \dot{a} \in \mathfrak{A}, |\dot{a}| \leq 1\}$, such that $F(\dot{z}) = \dot{a}_1 = x + iy$ and $G(\dot{z}) = \dot{a}_2 = x + iy$ provided $0 \leq |a_1|^3 + |a_2|^3 \leq 1$ or $F(\dot{z}) = \gamma(\dot{z}).e^{i2\pi\theta_\gamma(\dot{z})}$ and $G(\dot{z}) = \vartheta(\dot{z}).e^{i2\pi\theta_\vartheta(\dot{z})}$. Satisfying the conditions $0 \leq (\gamma(\dot{z}))^3 + (\vartheta(\dot{z}))^3 \leq 1$ and $0 \leq (\theta_\gamma(\dot{z}))^3 + (\theta_\vartheta(\dot{z}))^3 \leq 1$. Moreover, the term $H = \varphi(\dot{z}).e^{i2\pi\theta_\varphi(\dot{z})}$, such that $\varphi(\dot{z}) = \sqrt[3]{1 - (\gamma(\dot{z}))^3 - (\vartheta(\dot{z}))^3}$ and $\theta_\varphi(\dot{z}) = \sqrt[3]{1 - (\theta_\gamma(\dot{z}))^3 - (\theta_\vartheta(\dot{z}))^3}$ define the hesitancy degree of $\dot{z}$. Furthermore $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$ is called complex Fermatean fuzzy number.

Theorem 3.1 The set of CFMGs is a larger class then the set of CPMGs and CIMGs.

Proof: Any point $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$ which is a CIMG is also a CPMG and CFMG. For any two numbers $\gamma, \vartheta \in [0,1]$ and $\theta_\gamma, \theta_\vartheta \in [0,1]$, we have $\gamma + \vartheta \leq 1 \Rightarrow (\gamma)^2 + (\vartheta)^2 \leq 1 \Rightarrow (\gamma)^3 + (\vartheta)^3 \leq 1$and $\theta_\gamma + \theta_\vartheta \leq 1 \Rightarrow (\theta_\gamma)^2 + (\theta_\vartheta)^2 \leq 1 \Rightarrow (\theta_\gamma)^3 + (\theta_\vartheta)^3 \leq 1$. There are CFMGs that are not CPMGs and CIMGs. Consider the point $(0.8.e^{i2\pi(0.6)}, 0.7.e^{i2\pi(0.9)})$. We see that $(0.8)^3 + (0.7)^3 \leq 1, (0.6)^3 + (0.9)^3 \leq 1$, hence this is a CFMG, and so $(0.8)^2 + (0.7)^2 \nleq 1, (0.6)^2 + (0.9)^2 \nleq 1$ and $(0.8.e^{i2\pi(0.6)}, 0.7.e^{i2\pi(0.9)})$ is neither a CPMGs nor CIMGs.

Definition 3.2 Let $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$, $\mathfrak{A}_1 = (\gamma_1.e^{i2\pi\theta_{\gamma_1}}, \vartheta_1.e^{i2\pi\theta_{\vartheta_1}})$ and $\mathfrak{A}_2 = (\gamma_2.e^{i2\pi\theta_{\gamma_2}}, \vartheta_2.e^{i2\pi\theta_{\vartheta_2}})$ be three CFFNs, then the three operations are defined as follows

$(i)\mathfrak{A}_1 \cap \mathfrak{A}_2 = \left(min(\gamma_1,\gamma_2).e^{i2\pi min(\theta_{\gamma_1},\theta_{\gamma_2})}, max(\vartheta_1,\vartheta_2).e^{i2\pi max(\theta_{\vartheta_1},\theta_{\vartheta_2})}\right)$

$(ii)\mathfrak{A}_1 \cup \mathfrak{A}_2 = \left(max(\gamma_1,\gamma_2).e^{i2\pi max(\theta_{\gamma_1},\theta_{\gamma_2})}, min(\vartheta_1,\vartheta_2).e^{i2\pi min(\theta_{\vartheta_1},\theta_{\vartheta_2})}\right)$

$(iii)\mathfrak{A}^c = (\vartheta.e^{i2\pi\theta_\vartheta}, \gamma.e^{i2\pi\theta_\gamma})$

Definition 3.3 Let $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$, $\mathfrak{A}_1 = (\gamma_1.e^{i2\pi\theta_{\gamma_1}}, \vartheta_1.e^{i2\pi\theta_{\vartheta_1}})$ and $\mathfrak{A}_2 = (\gamma_2.e^{i2\pi\theta_{\gamma_2}}, \vartheta_2.e^{i2\pi\theta_{\vartheta_2}})$ be three CFFNs and $\lambda > 0$, then the following operators hold.

$(i)\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \left(\sqrt[3]{\gamma_1^3+\gamma_2^3-\gamma_1^3\gamma_2^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_1}^3+\theta_{\gamma_2}^3-\theta_{\gamma_1}^3\theta_{\gamma_2}^3}}, (\vartheta_1\vartheta_2).e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})}\right)$

$(ii)\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \left((\gamma_1\gamma_2).e^{i2\pi(\theta_{\gamma_1}\theta_{\gamma_2})}, \sqrt[3]{\vartheta_1^3+\vartheta_2^3-\vartheta_1^3\vartheta_2^3}.e^{i2\pi\sqrt[3]{\theta_{\vartheta_1}^3+\theta_{\vartheta_2}^3-\theta_{\vartheta_1}^3\theta_{\vartheta_2}^3}}\right)$

$(iii)\lambda\mathfrak{A} = \left(\sqrt[3]{1-(1-\gamma^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_\gamma^3)^\lambda}}, \vartheta^\lambda.e^{i2\pi(\theta_\vartheta^\lambda)}\right)$

$(iv)\mathfrak{A}^\lambda = \left(\gamma^\lambda.e^{i2\pi(\theta_\gamma^\lambda)}, \sqrt[3]{1-(1-\vartheta^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^\lambda}}\right)$

Theorem 3.2 $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and $\mathfrak{A}_3$, and $\lambda, \lambda_1, \lambda_2 > 0$, the following hold

$(i)\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \mathfrak{A}_2 \oplus \mathfrak{A}_1$ $(ii)\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \mathfrak{A}_2 \otimes \mathfrak{A}_1$ $(iii)\lambda(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \lambda\mathfrak{A}_1 \oplus \lambda\mathfrak{A}_2$,

$(iv)(\lambda_1+\lambda_2)\mathfrak{A} = \lambda_1\mathfrak{A} \oplus \lambda_2\mathfrak{A}$ $(v)(\mathfrak{A}_1 \otimes \mathfrak{A}_2)^\lambda = \mathfrak{A}_1^\lambda \otimes \mathfrak{A}_2^\lambda$ $(vi)\mathfrak{A}^{\lambda_1} \otimes \mathfrak{A}^{\lambda_2} = \mathfrak{A}^{\lambda_1+\lambda_2}$

Proof: For three CFFNs $\mathfrak{A}, \mathfrak{A}_1$ and $\mathfrak{A}_2$, and $\lambda, \lambda_1, \lambda_2 > 0$, according to Definition 3.2, we obtain

$(i)\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \left(\sqrt[3]{\gamma_1^3+\gamma_2^3-\gamma_1^3\gamma_2^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_1}^3+\theta_{\gamma_2}^3-\theta_{\gamma_1}^3\theta_{\gamma_2}^3}}, (\vartheta_1\vartheta_2).e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})}\right)$

$= \left(\sqrt[3]{\gamma_2^3+\gamma_1^3-\gamma_2^3\gamma_1^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_2}^3+\theta_{\gamma_1}^3-\theta_{\gamma_2}^3\theta_{\gamma_1}^3}}, (\vartheta_2\vartheta_1).e^{i2\pi(\theta_{\vartheta_2}\theta_{\vartheta_1})}\right)$

$= \mathfrak{A}_2 \oplus \mathfrak{A}_1$

$(ii)\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \left((\gamma_1\gamma_2).e^{i2\pi(\theta_{\gamma_1}\theta_{\gamma_2})}, \sqrt[3]{\vartheta_1^3+\vartheta_2^3-\vartheta_1^3\vartheta_2^3}.e^{i2\pi\sqrt[3]{\theta_{\vartheta_1}^3+\theta_{\vartheta_2}^3-\theta_{\vartheta_1}^3\theta_{\vartheta_2}^3}}\right)$

$= \left((\gamma_2\gamma_1).e^{i2\pi(\theta_{\gamma_2}\theta_{\gamma_1})}, \sqrt[3]{\vartheta_2^3+\vartheta_1^3-\vartheta_2^3\vartheta_1^3}.e^{i2\pi\sqrt[3]{\theta_{\vartheta_2}^3+\theta_{\vartheta_1}^3-\theta_{\vartheta_2}^3\theta_{\vartheta_1}^3}}\right)$

$= \mathfrak{A}_2 \otimes \mathfrak{A}_1$

$$(iii)\lambda(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \lambda\left(\sqrt[3]{\gamma_1^3+\gamma_2^3-\gamma_1^3\gamma_2^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_1}^3+\theta_{\gamma_2}^3-\theta_{\gamma_1}^3\theta_{\gamma_2}^3}}, (\vartheta_1\vartheta_2).e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})}\right)$$
$$= \left(\sqrt[3]{1-(1-\gamma_1^3-\gamma_2^3+\gamma_1^3\gamma_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_1}^3-\theta_{\gamma_2}^3+\theta_{\gamma_1}^3\theta_{\gamma_2}^3)^\lambda}}, (\vartheta_1\vartheta_2)^\lambda.e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})^\lambda}\right)$$
$$= \left(\sqrt[3]{1-(1-\gamma_1^3)^\lambda(1-\gamma_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_1}^3)^\lambda(1-\theta_{\gamma_2}^3)^\lambda}}, (\vartheta_1^\lambda\vartheta_2^\lambda).e^{i2\pi(\theta_{\vartheta_1}^\lambda\theta_{\vartheta_2}^\lambda)}\right)$$
$$\lambda\mathfrak{A}_1 \oplus \lambda\mathfrak{A}_2 = \left(\sqrt[3]{1-(1-\gamma_1^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_1}^3)^\lambda}}, \vartheta_1^\lambda.e^{i2\pi(\theta_{\vartheta_1}^\lambda)}\right) \oplus$$
$$\left(\sqrt[3]{1-(1-\gamma_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_2}^3)^\lambda}}, \vartheta_2^\lambda.e^{i2\pi(\theta_{\vartheta_2}^\lambda)}\right)$$
$$= \left(\sqrt[3]{1-(1-\gamma_1^3)^\lambda(1-\gamma_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_1}^3)^\lambda(1-\theta_{\gamma_2}^3)^\lambda}}, (\vartheta_1^\lambda\vartheta_2^\lambda).e^{i2\pi(\theta_{\vartheta_1}^\lambda\theta_{\vartheta_2}^\lambda)}\right)$$
$$= \lambda(\mathfrak{A}_1 \oplus \mathfrak{A}_2)$$
$$(iv)(\lambda_1+\lambda_2)\mathfrak{A} = \left(\sqrt[3]{1-(1-\gamma^3)^{\lambda+\lambda_2}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\gamma^3)^{\lambda_1+\lambda_2}}}, \vartheta^{\lambda_1+\lambda_2}.e^{i2\pi(\theta_\vartheta^{\lambda_1+\lambda_2})}\right)$$
$$= \left(\sqrt[3]{1-(1-\gamma^3)^{\lambda_1+\lambda_2}(1-\gamma^3)^{\lambda_1+\lambda_2}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\gamma^3)^{\lambda_1+\lambda_2}(1-\theta_\gamma^3)^{\lambda_1+\lambda_2}}}, (\vartheta^{\lambda_1+\lambda_2}).e^{i2\pi(\theta_\vartheta^{\lambda_1+\lambda_2})}\right)$$
$$= \left(\sqrt[3]{1-(1-\gamma^3)^{\lambda_1}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\gamma^3)^{\lambda_1}}}, \vartheta^{\lambda_1}.e^{i2\pi(\theta_\vartheta^{\lambda_1})}\right) \oplus$$
$$\left(\sqrt[3]{1-(1-\gamma^3)^{\lambda_2}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\gamma^3)^{\lambda_2}}}, \vartheta^{\lambda_2}.e^{i2\pi(\theta_\vartheta^{\lambda_2})}\right)$$
$$= \lambda_1\mathfrak{A}_1 \oplus \lambda_2\mathfrak{A}$$
$$(v)(\mathfrak{A}_1 \otimes \mathfrak{A}_2)^\lambda = \left((\gamma_1\gamma_2)^\lambda.e^{i2\pi(\theta_{\gamma_1}\theta_{\gamma_2})^\lambda}, \sqrt[3]{1-(1-\vartheta_1^3-\vartheta_2^3+\vartheta_1^3\vartheta_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\vartheta_1}^3-\theta_{\vartheta_2}^3+\theta_{\vartheta_1}^3\theta_{\vartheta_2}^3)^\lambda}}\right)$$
$$= \left((\gamma_1^\lambda\gamma_2^\lambda).e^{i2\pi(\theta_{\gamma_1}^\lambda\theta_{\gamma_2}^\lambda)}, \sqrt[3]{1-(1-\vartheta_1^3)^\lambda(1-\vartheta_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\vartheta_1}^3)^\lambda(\theta_{\vartheta_2}^3)^\lambda}}\right)$$
$$= \left(\gamma_1^\lambda.e^{i2\pi(\theta_{\gamma_1}^\lambda)}, \sqrt[3]{1-(1-\vartheta_1^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\vartheta_1}^3)^\lambda}}\right) \otimes$$
$$\left(\gamma_2^\lambda.e^{i2\pi(\theta_{\gamma_2}^\lambda)}, \sqrt[3]{1-(1-\vartheta_2^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\vartheta_2}^3)^\lambda}}\right)$$
$$= (\mathfrak{A}_1)^\lambda \otimes (\mathfrak{A}_2)^\lambda$$
$$(vi)\mathfrak{A}^{\lambda_1} \otimes \mathfrak{A}^{\lambda_2} = \left(\gamma^{\lambda_1}.e^{i2\pi(\theta_\gamma^{\lambda_1})}, \sqrt[3]{1-(1-\vartheta^3)^{\lambda_1}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^{\lambda_1}}}\right) \otimes$$
$$\left(\gamma^{\lambda_2}.e^{i2\pi(\theta_\gamma^{\lambda_2})}, \sqrt[3]{1-(1-\vartheta^3)^{\lambda_2}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^{\lambda_2}}}\right)$$
$$= \left(\gamma^{\lambda_1+\lambda_2}.e^{i2\pi(\theta_\gamma)^{\lambda_1+\lambda_2}}, \sqrt[3]{1-(1-\vartheta^3)^{\lambda_1+\lambda_2}}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^{\lambda_1+\lambda_2}}}\right)$$
$$= \mathfrak{A}^{\lambda_1+\lambda_2}$$

Theorem 3.3 For three CFFNs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and $\mathfrak{A}_3$, and $\lambda, \lambda_1, \lambda_2 > 0$, the following are valid

$(i)\mathfrak{A}_1 \cap \mathfrak{A}_2 = \mathfrak{A}_2 \cap \mathfrak{A}_1$ $\quad(ii)\mathfrak{A}_1 \cup \mathfrak{A}_2 = \mathfrak{A}_2 \cup \mathfrak{A}_1$

$(iii)\mathfrak{A}_1 \cap (\mathfrak{A}_2 \cap \mathfrak{A}_3) = (\mathfrak{A}_1 \cap \mathfrak{A}_2) \cap \mathfrak{A}_3$ $\quad(iv)\mathfrak{A}_1 \cup (\mathfrak{A}_2 \cup \mathfrak{A}_3) = (\mathfrak{A}_1 \cup \mathfrak{A}_2) \cup \mathfrak{A}_3$

$(v)\lambda(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \lambda\mathfrak{A}_1 \cup \lambda\mathfrak{A}_2$ $\quad(vi)(\mathfrak{A}_1 \cup \mathfrak{A}_2)^\lambda = \mathfrak{A}_1^\lambda \cup \mathfrak{A}_2^\lambda$

Proof: We present the proofs of (i),(iii) and (v) for the CFFNs $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}_3$ and $\lambda > 0$, according to Definition (3.2) and (3.3) we have obtain

$(i)(\mathfrak{A}_1 \cap \mathfrak{A}_2 = \left(min(\gamma_1, \gamma_2)e^{i2\pi min(\theta_{\gamma_1}, \theta_{\gamma_2})}, max(\vartheta_1, \vartheta_2).e^{i2\pi max(\theta_{\vartheta_1}, \theta_{\vartheta_2})}\right)$
$= \left(min(\gamma_2, \gamma_1).e^{i2\pi min(\theta_{\gamma_2}, \theta_{\gamma_1})}, max(\vartheta_2, \vartheta_1).e^{i2\pi max(\theta_{\vartheta_2}, \theta_{\vartheta_1})}\right)$
$= \mathfrak{A}_2 \cap \mathfrak{A}_1$

$(iii)\mathfrak{A}_1 \cup (\mathfrak{A}_2 \cup \mathfrak{A}_3) = \mathfrak{A}_1 \cap \left(min(\gamma_2, \gamma_3).e^{i2\pi min(\theta_{\gamma_2}, \theta_{\gamma_3})}, max(\vartheta_2, \vartheta_3).e^{i2\pi max(\theta_{\vartheta_2}, \theta_{\vartheta_3})}\right)$
$= \left((min(\gamma_1, min(\gamma_2, \gamma_3)).e^{i2\pi\left(min(\theta_{\gamma_1}, min(\theta_{\gamma_2}, \theta_{\gamma_3})\right)}, (max(\vartheta_1, max(\vartheta_2, \vartheta_3)).e^{i2\pi\left(max(\theta_{\vartheta_1}, max(\theta_{\vartheta_2}, \theta_{\vartheta_3}))\right)}\right)$
$= \left((min(min(\gamma_1, \gamma_2), \gamma_3).e^{i2\pi\left(min(min(\theta_{\gamma_1}, \theta_{\gamma_2}), \theta_{\gamma_3}\right)}, (max(max(\vartheta_1, \vartheta_2), \vartheta_3).e^{i2\pi\left(max(max(\theta_{\vartheta_1}, \theta_{\vartheta_2}), \theta_{\vartheta_3}\right)}\right)$
$= \left(min(\gamma_1, \gamma_2).e^{i2\pi min(\theta_{\gamma_1}, \theta_{\gamma_2})}, max(\vartheta_1, \vartheta_2).e^{i2\pi max(\theta_{\vartheta_1}, \theta_{\vartheta_2})}\right) \cap \mathfrak{A}_3$
$= (\mathfrak{A}_1 \cap \mathfrak{A}_2) \cap \mathfrak{A}_3$

$(v)\lambda(\mathfrak{A}_1 \cup \mathfrak{A}_2) = \lambda\left(max(\gamma_1, \gamma_2).e^{i2\pi max(\theta_{\gamma_1}, \theta_{\gamma_2})}, min(\vartheta_1, \vartheta_2).e^{i2\pi min(\theta_{\vartheta_1}, \theta_{\vartheta_2})}\right)$
$= \left(\sqrt[3]{1-(1-max(\gamma_1, \gamma_2)^{\lambda}}.e^{i2\pi\sqrt[3]{1-(1-max(\theta_{\gamma_1}, \theta_{\gamma_2})^{\lambda}}}, min(\vartheta_1^{\lambda}, \vartheta_2^{\lambda}).e^{i2\pi min(\theta_{\vartheta_1}^{\lambda}, \theta_{\vartheta_2}^{\lambda})}\right)$

$\lambda\mathfrak{A}_1 \cup \lambda\mathfrak{A}_2 = \left(\sqrt[3]{1-(1-\gamma_1^3)^{\lambda}}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_1}^3)^{\lambda}}}, \vartheta_1^{\lambda}.e^{i2\pi(\theta_{\vartheta_1}^{\lambda})}\right) \cup$
$\left(\sqrt[3]{1-(1-\gamma_2^3)^{\lambda}}.e^{i2\pi\sqrt[3]{1-(1-\theta_{\gamma_2}^3)^{\lambda}}}, \vartheta_2^{\lambda}.e^{i2\pi(\theta_{\vartheta_2}^{\lambda})}\right)$
$= \left(max\left(\sqrt[3]{1-(1-\gamma_1^3)^{\lambda}}, \sqrt[3]{1-(1-\gamma_2^3)^{\lambda}}\right).e^{i2\pi min(\sqrt[3]{1-(1-\theta_{\gamma_1}^3)^{\lambda}}, \sqrt[3]{1-(1-\theta_{\gamma_2}^3)^{\lambda}})},\right.$
$\left.min(\vartheta_1^{\lambda}, \vartheta_2^{\lambda}).e^{i2\pi min(\theta_{\vartheta_1}^{\lambda}, \theta_{\vartheta_2}^{\lambda})}\right)$
$= \sqrt[3]{1-(1-max(\gamma_1^3, \gamma_2^3))^{\lambda}}.e^{i2\pi\sqrt[3]{1-(1-max(\theta_{\gamma_1}^3, \theta_{\gamma_2}^3))^{\lambda}}}, min(\vartheta_1^{\lambda}, \vartheta_2^{\lambda}).e^{i2\pi min(\theta_{\vartheta_1}^{\lambda}, \theta_{\vartheta_2}^{\lambda})}\Big)$
$= \lambda(\mathfrak{A}_1 \cup \mathfrak{A}_2)$

Theorem 3.4 Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and $\mathfrak{A}_3$ be three CFFNs and $\lambda > 0$, then we have,
$(i)(\mathfrak{A}_1 \cap \mathfrak{A}_2)^c = \mathfrak{A}_1^c \cup \mathfrak{A}_2^c$ $(ii)(\mathfrak{A}_1 \cup \mathfrak{A}_2)^c = \mathfrak{A}_1^c \cap \mathfrak{A}_2^c$ $(iii)(\mathfrak{A}_1 \oplus \mathfrak{A}_2)^c = \mathfrak{A}_1^c \otimes \mathfrak{A}_2^c$
$(iv)(\mathfrak{A}_1 \otimes \mathfrak{A}_2)^c = \mathfrak{A}_1^c \oplus \mathfrak{A}_2^c$ $(v)(\mathfrak{A}^c)^{\lambda} = (\lambda\mathfrak{A})^c$ $(vi)\lambda(\mathfrak{A}^c) = (\mathfrak{A}^{\lambda})^c$

Proof: we present the proofs of (i)(iii) and (v). For the three CFFNs $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}_3$, and $\lambda > 0$ according to Definition (3.2) and (3.3) we obtain

$(i)(\mathfrak{A}_1 \cap \mathfrak{A}_2)^c = \left(min(\gamma_1, \gamma_2).e^{i2\pi min(\theta_{\gamma_1}, \theta_{\gamma_2})}, max(\vartheta_1, \vartheta_2).e^{i2\pi max(\theta_{\vartheta_1}, \theta_{\vartheta_2})}\right)^c$
$= \left(max(\vartheta_1, \vartheta_2).e^{i2\pi max(\theta_{\vartheta_1}, \theta_{\vartheta_2})}, min(\gamma_1, \gamma_2).e^{i2\pi min(\theta_{\gamma_1}}\right)$
$= \left(\vartheta_1.e^{i2\pi\theta_{\vartheta_1}}, \gamma_1.e^{i2\pi\theta_{\gamma_1}}\right) \cup \left(\vartheta_2.e^{i2\pi\theta_{\vartheta_2}}, \gamma_2.e^{i2\pi\theta_{\gamma_2}}\right)$
$= \mathfrak{A}_1^c \cup \mathfrak{A}_2^c$

$(iii)(\mathfrak{A}_1 \oplus \mathfrak{A}_2)^c = \left(\sqrt[3]{\gamma_1^3+\gamma_2^3-\gamma_1^3\gamma_2^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_1}^3+\theta_{\gamma_2}^3-\theta_{\gamma_1}^3\theta_{\gamma_2}^3}}, (\vartheta_1\vartheta_2).e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})}\right)^c$

$= \left((\vartheta_1\vartheta_2).e^{i2\pi(\theta_{\vartheta_1}\theta_{\vartheta_2})}, \sqrt[3]{\gamma_1^3+\gamma_2^3-\gamma_1^3\gamma_2^3}.e^{i2\pi\sqrt[3]{\theta_{\gamma_1}^3+\theta_{\gamma_2}^3-\theta_{\gamma_1}^3\theta_{\gamma_2}^3}}\right)$

$= \left(\vartheta_1.e^{i2\pi\theta_{\vartheta_1}}, \gamma_1.e^{i2\pi\theta_{\gamma_1}}\right) \otimes \left(\vartheta_2.e^{i2\pi\theta_{\vartheta_2}}, \gamma_2.e^{i2\pi\theta_{\gamma_2}}\right)$

$= \mathfrak{A}_1^c \otimes \mathfrak{A}_2^c$

$(v)(\mathfrak{A}^c)^\lambda = \left(\vartheta.e^{i2\pi\theta_\vartheta}, \gamma.e^{i2\pi\theta_\gamma}\right)^\lambda$

$= \left(\gamma^\lambda.e^{i2\pi(\theta_\gamma^\lambda)}, \sqrt[3]{1-(1-\vartheta^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^\lambda}}\right)$

$= \left(\sqrt[3]{1-(1-\vartheta^3)^\lambda}.e^{i2\pi\sqrt[3]{1-(1-\theta_\vartheta^3)^\lambda}}, \gamma^\lambda.e^{i2\pi(\theta_\gamma^\lambda)}\right)$

$= (\mathfrak{A}^\lambda)^c$

Definition 3.4 For any CFFN $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$ the score function of $\mathfrak{A}$ is defined as $\check{\check{S}}(\mathfrak{A}) = \frac{1}{2}\left[(\gamma^3-\vartheta^3)+(\theta_\gamma^3-\theta_\vartheta^3)\right]$ where $\check{\check{S}}(\mathfrak{A}) \in [-1,1]$. For any two CFFNs $\mathfrak{A}_1$,$\mathfrak{A}_2$, if $\check{\check{S}}(\mathfrak{A}_1) < \check{\check{S}}(\mathfrak{A}_2)$, then $\mathfrak{A}_1 < \mathfrak{A}_2$. If $\check{\check{S}}(\mathfrak{A}_1) > \check{\check{S}}(\mathfrak{A}_2)$, then $\mathfrak{A}_1 > \mathfrak{A}_2$ If $\check{\check{S}}(\mathfrak{A}_1) = \check{\check{S}}(\mathfrak{A}_2)$, then $\mathfrak{A}_1 = \mathfrak{A}_2$. For example if $\mathfrak{A}_1 = (0.7.e^{i2\pi(0.5)}, 0.7.e^{i2\pi(0.5)})$ and $\mathfrak{A}_2 = (0.8.e^{i2\pi(0.7)}, 0.8.e^{i2\pi(0.7)})$ then $\check{\check{S}}(\mathfrak{A}_1) = \check{\check{S}}(\mathfrak{A}_2) = 0$. More generally, if any two CFFN satisfying $\mathfrak{A}_1 = \mathfrak{A}_2$ then their scores are 0. But there CFFNs are not identical. To overcome this difficulty, we define an accuracy function for CFFNs as follows.

Definition 3.5 Let $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$, then the accuracy function of $ac(\mathfrak{A})$ denoted by $ac(\mathfrak{A})$ is defined as $ac(\mathfrak{A}) = \frac{1}{2}\left[(\gamma^3+\vartheta^3)+(\theta_\gamma^3+\theta_\vartheta^3)\right]$. Clearly, $ac(\mathfrak{A}) \in [0,1]$. if $ac(\mathfrak{A}_1) < ac(\mathfrak{A}_2)$, then $\mathfrak{A}_1 < \mathfrak{A}_2$. If $ac(\mathfrak{A}_1) > ac(\mathfrak{A}_2)$, then $\mathfrak{A}_1 > \mathfrak{A}_2$. If $ac(\mathfrak{A}_1) = ac(\mathfrak{A}_2)$, then $\mathfrak{A}_1 \sim \mathfrak{A}_2$.

4. AGGREGATION OF COMPLEX FERMATEAN FUZZY SET

In this section, we define a new concept complex Fermatean Fuzzy weighted average, complex Fermatean Fuzzy weighted geometric operator, complex Fermatean Fuzzy weighted power average and complex Fermatean Fuzzy weighted power geometric operators and discuss some of their properties.

Definition 4.1 Let $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1,2,...,n)$ be 'n' CFFNs and $\chi = (\chi_1,\chi_2,...,\chi_n)^T$ be the weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n}\chi_\iota = 1$. Then a complex Fermatean fuzzy weighted average (CFFWA) operator is a function $CFFWA : \mathfrak{A}^n \rightarrow \mathfrak{A}$, where

$$CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)=\left(\sum_{\iota=1}^{n}\chi_\iota\gamma_\iota.e^{i2\pi\sum_{\iota=1}^{n}\chi_\iota\theta_{\gamma_\iota}},\sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota.e^{i2\pi\sum_{\iota=1}^{n}\chi_\iota\theta_{\vartheta_\iota}}\right) \quad (4.1)$$

Definition 4.2 Let $\mathfrak{A}_\iota=(\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}},\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota=1,2,...,n)$ be a number of CFFNs and $\chi=(\chi_1,\chi_2,...,\chi_n)^T$ be the weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n}\chi_\iota=1$. Then a complex Fermatean fuzzy weighted geometric (CFFWG) operator is a function $CFFWG:\mathfrak{A}^n\to\mathfrak{A}$, where

$$CFFWG(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)=\left(\prod_{\iota=1}^{n}\gamma_\iota^{\chi_\iota}.e^{i2\pi\prod_{\iota=1}^{n}\theta_{\gamma_\iota}^{\chi_\iota}},\prod_{\iota=1}^{n}\vartheta_\iota^{\chi_\iota}.e^{i2\pi\prod_{\iota=1}^{n}\theta_{\vartheta_\iota}^{\chi_\iota}}\right) \quad (4.2)$$

Definition 4.3 Let $\mathfrak{A}_\iota=(\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}},\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota=1,2,...,n)$ be a number of CFFNs and $\chi=(\chi_1,\chi_2,...,\chi_n)^T$ be the weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n}\chi_\iota=1$. Then a complex Fermatean fuzzy weighted power average (CFFWPA) operator is a function $CFFWPA:\mathfrak{A}^n\to\mathfrak{A}$, where

$$CFFWPA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)$$
$$=\left(\left(\sum_{\iota=1}^{n}\chi_\iota\gamma_\iota^3\right)^{\frac{1}{3}}.e^{i2\pi\left(\sum_{\iota=1}^{n}\chi_\iota\theta_{\gamma_\iota}^3\right)^{\frac{1}{3}}},\left(\sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota^3\right)^{\frac{1}{3}}.e^{i2\pi\left(\sum_{\iota=1}^{n}\chi_\iota\theta_{\vartheta_\iota}^3\right)^{\frac{1}{3}}}\right) \quad (4.3)$$

Definition 4.4 Let $\mathfrak{A}_\iota=(\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}},\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota=1,2,...,n)$ be a number of CFFNs and $\chi=(\chi_1,\chi_2,...,\chi_n)^T$ be the weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n}\chi_\iota=1$. Then a complex Fermatean fuzzy weighted power geometric (CFFWPG) operator is a function $CFFWPG:\mathfrak{A}^n\to\mathfrak{A}$, where

$$CFFWPG(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)$$
$$=\left(\left(1-\prod_{\iota=1}^{n}(1-\gamma_\iota^3)^{\chi_\iota}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\prod_{\iota=1}^{n}(1-\theta_{\gamma_\iota}^3)^{\chi_\iota}\right)^{\frac{1}{3}}},\left(1-\prod_{\iota=1}^{n}(1-\vartheta_\iota^3)^{\chi_\iota}\right)^{\frac{1}{3}}.e^{i2\pi\left(1-\prod_{\iota=1}^{n}(1-\theta_{\vartheta_\iota}^3)^{\chi_\iota}\right)^{\frac{1}{3}}}\right) \quad (4.4)$$

Example 4.1 Let $\mathfrak{A}_1=(0.8.e^{i2\pi(0.9)},0.6.e^{i2\pi(0.3)})$, $\mathfrak{A}_2=(0.7.e^{i2\pi(0.5)},0.5.e^{i2\pi(0.8)})$ and $\mathfrak{A}_3=(0.6.e^{i2\pi(0.8)},0.4.e^{i2\pi(0.6)})$ be three Fermatean fuzzy values, and suppose that $\chi=(0.1.0.3,0.4)^T$ is weight vector of $\mathfrak{A}_\iota$ $(\iota=1,2,3)$ then

(i) $CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3)=\Big((0.8\times0.1+0.7\times0.3+0.6\times0.4).e^{i2\pi(0.9\times0.1+0.3\times0.3+0.8\times0.4)},$
$(0.6\times0.1+0.5\times0.3+0.4\times0.4).e^{i2\pi(0.3\times0.1+0.8\times0.3+0.6\times0.4)}\Big)=$
$\Big(0.45e^{i2\pi(0.5)},0.37e^{i2\pi(0.51)}\Big)$

(ii) $CFFWG(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3)=\Big((0.8^{0.1}\times0.7^{0.3}\times0.6^{0.4}).e^{i2\pi(0.9^{0.1}\times0.5^{0.3}\times0.8^{0.4})},$
$(0.6^{0.1}\times0.5^{0.3}\times0.4^{0.4}).e^{i2\pi(0.3^{0.1}\times0.8^{0.3}\times0.6^{0.4})}\Big)=\Big(0.69e^{i2\pi(0.61)},0.53e^{i2\pi(0.66)}\Big)$

(iii) $CFFWPA(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3)=\Big(\big(0.8^3\times0.1+0.7^3\times0.3+0.6^3\times0.4\big)^{\frac{1}{3}}.$

$e^{i2\pi\left(0.9^3\times0.1+0.5^3\times0.3+0.8^3\times0.4\right)^{\frac{1}{3}}}, \left(0.6^3\times0.1+0.5^3\times0.3+0.4^3\times0.4\right)^{\frac{1}{3}}.$

$e^{i2\pi\left(0.3^3\times0.1+0.8^3\times0.3+0.6^3\times0.4\right)^{\frac{1}{3}}}\Big) = \left(0.61e^{i2\pi(0.66)}, 0.43e^{i2\pi(0.83)}\right)$

(iv) $CFFWPG(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3) =$

$\Big(\left(1-(1-0.8^3)^{0.1}\times1-(1-0.7^3)^{0.3}\times1-(1-0.6^3)^{0.4}\right)^{\frac{1}{3}}$

$.e^{i2\pi\left(1-(1-0.9^3)^{0.1}\times1-(1-0.5^3)^{0.3}\times1-(1-0.8^3)^{0.4}\right)^{\frac{1}{3}}}, \Big(1-(1-0.6^3)^{0.1}\times1-(1-0.5^3)^{0.3}\times1-(1-$

$0.4^3)^{0.4}\Big)^{\frac{1}{3}}.e^{i2\pi\left(1-(1-0.3^3)^{0.1}\times1-(1-0.8^3)^{0.3}\times1-(1-0.6^3)^{0.4}\right)^{\frac{1}{3}}}\Big) =$

$\left(0.08e^{i2\pi(0.22)}, 0.12e^{i2\pi(0.03)}\right)$

Theorem 4.1 Let $\mathfrak{A}_\imath = (\gamma_\imath.e^{i2\pi\theta_{\gamma_\imath}}, \vartheta_\imath.e^{i2\pi\theta_{\vartheta_\imath}})$ $(\imath = 1,2,...,n)$ be a number of CFFNs, $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$ is also CFFN and $\chi = (\chi_1,\chi_2,...,\chi_n)^T$ be weight vector of $\mathfrak{A}_\imath$ with $\sum_{\imath=1}^{n}\chi_\imath = 1$, then

(i) $CFFWA(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},...,\mathfrak{A}_n\oplus\mathfrak{A}) \geq CFFWA(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A})$

(ii) $CFFWG(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},...,\mathfrak{A}_n\oplus\mathfrak{A}) \geq CFFWA(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A})$

(iii) $CFFWPA(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},...,\mathfrak{A}_n\oplus\mathfrak{A}) \geq CFFWA(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A})$

(iv) $CFFWPG(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},...,\mathfrak{A}_n\oplus\mathfrak{A}) \geq CFFWPA(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A})$

Proof: we prove (i) and (iv). The other assertions are proved analogously.

(i) For any $\mathfrak{A}_\imath = (\gamma_\imath.e^{i2\pi\theta_{\gamma_\imath}}, \vartheta_\imath.e^{i2\pi\theta_{\vartheta_\imath}})$ $(\imath = 1,2,...,n)$ and $\mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$, we can get $\sqrt[3]{\gamma_\imath^3+\gamma^3-\gamma_\imath^3\gamma^3} \geq \sqrt[3]{2\gamma_\imath^3\gamma^3-\gamma_\imath^3\gamma^3} = \gamma_\imath^3\gamma^3$ and the phase term $\sqrt[3]{\theta_{\gamma_\imath}^3+\theta_\gamma^3-\theta_{\gamma_\imath}^3\theta_\gamma^3} \geq \sqrt[3]{2\theta_{\gamma_\imath}^3\theta_\gamma^3-\theta_{\gamma_\imath}^3\theta_\gamma^3} = \theta_{\gamma_\imath}^3\theta_\gamma^3$ and $\sqrt[3]{\vartheta_\imath^3+\vartheta^3-\vartheta_\imath^3\vartheta^3} \geq \sqrt[3]{2\vartheta_\imath^3\vartheta^3-\vartheta_\imath^3\vartheta^3} = \vartheta_\imath^3\vartheta^3$ and the phase term $\sqrt[3]{\theta_{\vartheta_\imath}^3+\theta_\vartheta^3-\theta_{\vartheta_\imath}^3\theta_\vartheta^3} \geq \sqrt[3]{2\theta_{\vartheta_\imath}^3\theta_\vartheta^3-\theta_{\vartheta_\imath}^3\theta_\vartheta^3} = \theta_{\vartheta_\imath}^3\theta_\vartheta^3$,i.e., $\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\gamma_\imath^3+\gamma^3-\gamma_\imath^3\gamma^3} \geq \sum_{\imath=1}^{n}\chi_\imath\gamma_\imath^3\gamma^3$ and the phase term $\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\theta_{\gamma_\imath}^3+\theta_\gamma^3-\theta_{\gamma_\imath}^3\theta_\gamma^3} \geq \sum_{\imath=1}^{n}\chi_\imath\theta_{\gamma_\imath}^3\theta_\gamma^3$ and $\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\vartheta_\imath^3+\vartheta^3-\vartheta_\imath^3\vartheta^3} \geq \sum_{\imath=1}^{n}\chi_\imath\vartheta_\imath^3\vartheta^3$ and the phase term $\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\theta_{\vartheta_\imath}^3+\theta_\vartheta^3-\theta_{\vartheta_\imath}^3\theta_\vartheta^3} \geq \sum_{\imath=1}^{n}\chi_\imath\theta_{\vartheta_\imath}^3\theta_\vartheta^3$. Since $FFWA(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},...,\mathfrak{A}_n\oplus\mathfrak{A}) =$

$$\left(\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\gamma_\imath^3+\gamma^3-\gamma_\imath^3\gamma^3}.e^{i2\pi\sum_{\imath=1}^{n}\chi_\imath\sqrt[3]{\theta_{\gamma_\imath}^3+\theta_\gamma^3-\theta_{\gamma_\imath}^3\theta_\gamma^3}}, \sum_{\imath=1}^{n}\chi_\imath\vartheta_\imath^3\vartheta^3.e^{i2\pi\sum_{\imath=1}^{n}\chi_\imath\theta_{\vartheta_\imath}^3\theta_\vartheta^3}\right) \text{ and}$$

$$FFWA(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A}) = \left(\sum_{\imath=1}^{n}\chi_\imath\gamma_\imath^3\gamma^3.e^{i2\pi\sum_{\imath=1}^{n}\chi_\imath\theta_{\gamma_\imath}^3\theta_\gamma^3},\right.$$

$\sum_{\iota=1}^{n}\chi_\iota\sqrt[3]{\vartheta_\iota^3+\vartheta^3-\vartheta_\iota^3\vartheta^3}.e^{i2\pi\sum_{\iota=1}^{n}\chi_\iota\sqrt[3]{\theta_{\vartheta_\iota}^3+\theta_\vartheta^3-\theta_{\vartheta_\iota}^3\theta_\vartheta^3}}\Big)$. According to Definition (3.4), the proof follows.

(iv) For any $\mathfrak{A}_\iota=(\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}},\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota=1,2,...,n)$ and $\mathfrak{A}=(\gamma.e^{i2\pi\theta_\gamma},\vartheta.e^{i2\pi\theta_\vartheta})$, we can get $\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3\geq 2\gamma_\iota^3\gamma^3-\gamma_\iota^3\gamma^3=\gamma_\iota^3\gamma^3$ and the phase term $\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3\geq 2\theta_{\gamma_\iota}^3\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3=\theta_{\gamma_\iota}^3\theta_\gamma^3\Rightarrow 1-(\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3)\geq 1-(\gamma_\iota^3\gamma^3)$ and the phase term $1-(\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3)\geq 1-(\theta_{\gamma_\iota}^3\theta_\gamma^3)\Rightarrow\big(1-(\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}\geq\big(1-(\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}$ and the phase term

$\Big(1-(\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\geq\Big(1-(\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\Rightarrow\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}\geq$

$\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}$ and the phase term $\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\geq\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}$

$\Rightarrow 1-\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}\geq 1-\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}$ and the phase term

$1-\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\geq 1-\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}$.

Similarly, $1-\prod_{\iota=1}^{n}\big(1-(\vartheta_\iota^3+\vartheta^3-\vartheta_\iota^3\vartheta^3)\big)^{\chi_\iota}\geq 1-\prod_{\iota=1}^{n}\big(1-(\vartheta_\iota^3\vartheta^3)\big)^{\chi_\iota}$ and the phase term

$1-\prod_{\iota=1}^{n}\big(1-(\theta_{\vartheta_\iota}^3+\theta_\vartheta^3-\theta_{\vartheta_\iota}^3\theta_\vartheta^3)\big)^{\chi_\iota}\geq 1-\prod_{\iota=1}^{n}\big(1-(\theta_{\vartheta_\iota}^3\theta_\vartheta^3)\big)^{\chi_\iota}$.

Now $CFFWPG(\mathfrak{A}_1\oplus\mathfrak{A},\mathfrak{A}_2\oplus\mathfrak{A},......,\mathfrak{A}_n\oplus\mathfrak{A})=$

$$\Bigg(\Big(1-\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3+\gamma^3-\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}.e^{i2\pi\Big(1-\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3+\theta_\gamma^3-\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\Big)^{\frac{1}{3}}},$$

$$\Big(1-\prod_{\iota=1}^{n}\big(1-(\vartheta_\iota^3\vartheta^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}.e^{i2\pi\Big(1-\prod_{\iota=1}^{n}\big(1-(\theta_{\vartheta_\iota}^3\theta_\vartheta^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}}\Bigg)$$ and

$CFFWPG(\mathfrak{A}_1\otimes\mathfrak{A},\mathfrak{A}_2\otimes\mathfrak{A},...,\mathfrak{A}_n\otimes\mathfrak{A})=$

$$\Bigg(\Big(1-\prod_{\iota=1}^{n}\big(1-(\gamma_\iota^3\gamma^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}.e^{i2\pi\Big(1-\prod_{\iota=1}^{n}\Big(1-(\theta_{\gamma_\iota}^3\theta_\gamma^3)\Big)^{\chi_\iota}\Big)^{\frac{1}{3}}},$$

$$\Big(1-\prod_{\iota=1}^{n}\big(1-(\vartheta_\iota^3+\vartheta^3-\vartheta_\iota^3\vartheta^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}.e^{i2\pi\Big(1-\prod_{\iota=1}^{n}\big(1-(\theta_{\vartheta_\iota}^3+\theta_\vartheta^3-\theta_{\vartheta_\iota}^3\theta_\vartheta^3)\big)^{\chi_\iota}\Big)^{\frac{1}{3}}}\Bigg).$$

By Definition (3.4), the proof is follows.

Theorem 4.2 Let $\mathfrak{A}_\iota=(\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}},\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota=1,2,...,n)$ be a number of CFFNs, $\mathfrak{A}=(\gamma.e^{i2\pi\theta_\gamma},\vartheta.e^{i2\pi\theta_\vartheta})$ is also CFFN and $\chi=(\chi_1,\chi_2,...,\chi_n)^T$ be weight vector of $\mathfrak{A}_\iota$ with

$\sum_{\iota=1}^{n} \chi_\iota = 1$, then

(i) $CFFWA(\lambda\mathfrak{A}_1, \lambda\mathfrak{A}_2, ..., \lambda\mathfrak{A}_n) \geq CFFWA(\mathfrak{A}_1^\lambda, \mathfrak{A}_2^\lambda, ..., \mathfrak{A}_n^\lambda)$

(ii) $CFFWG(\lambda\mathfrak{A}_1, \lambda\mathfrak{A}_2, ..., \lambda\mathfrak{A}_n) \geq CFFWG(\mathfrak{A}_1^\lambda, \mathfrak{A}_2^\lambda, ..., \mathfrak{A}_n^\lambda)$

(iii) $CFFWPA(\lambda\mathfrak{A}_1, \lambda\mathfrak{A}_2, ..., \lambda\mathfrak{A}_n) \geq CFFWPA(\mathfrak{A}_1^\lambda, \mathfrak{A}_2^\lambda, ..., \mathfrak{A}_n^\lambda)$

(iv) $CFFWPG(\lambda\mathfrak{A}_1, \lambda\mathfrak{A}_2, ..., \lambda\mathfrak{A}_n) \geq CFFWPG(\mathfrak{A}_1^\lambda, \mathfrak{A}_2^\lambda, ..., \mathfrak{A}_n^\lambda)$

(v) $\lambda CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \geq (CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^\lambda$

(vi) $\lambda CFFWG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \geq (CFFWG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^\lambda$

(vii) $\lambda CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \geq (CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^\lambda$

(viii) $\lambda CFFWPG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \geq (CFFWPG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^\lambda$

Proof: we give the proof of (i). The other assertions are proved analogously.

(i) For any $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1, 2, ..., n)$,

we have $CFFWA(\lambda\mathfrak{A}_1, \lambda\mathfrak{A}_2, ..., \lambda\mathfrak{A}_n)$

$$= \Big(\sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\gamma_\iota^3)^\lambda}.e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\theta_{\gamma_\iota}^3)^\lambda}}, \sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota^\lambda.e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota}^\lambda} \Big),$$

$$CFFWA(\mathfrak{A}_1^\lambda, \mathfrak{A}_2^\lambda, ..., \mathfrak{A}_n^\lambda) = \Big(\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota^\lambda.e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota}^\lambda}, \sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\vartheta_\iota^3)^\lambda}.$$

$e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\theta_{\vartheta_\iota}^3)^\lambda}} \Big)$. In the upcoming, we mean $f(\gamma_\iota)$ $= 1-(1-\gamma_\iota^3)^\lambda - (\gamma_\iota^3)^\lambda, f(\theta_{\gamma_\iota}) = 1-(1-\theta_{\gamma_\iota}^3)^\lambda - (\theta_{\gamma_\iota}^3)^\lambda$ and show that $f(\gamma_\iota), f(\theta_{\gamma_\iota}) \geq 0$. Utilizing Newton generalized binomial theorem, we get $(1-\gamma_\iota^3)^\lambda - (\gamma_\iota^3)^\lambda \leq (1-\gamma_\iota^3+\gamma_\iota^3)^\lambda =$ $1, (1-\theta_{\gamma_\iota}^3)^\lambda - (\theta_{\gamma_\iota}^3)^\lambda \leq (1-\theta_{\gamma_\iota}^3+\theta_{\gamma_\iota}^3)^\lambda = 1$ Thus $f(\gamma_\iota), f(\theta_{\gamma_\iota}) \geq 0$, i.e., $(1-(1-\gamma_\iota^3)^\lambda - (\gamma_\iota^3)^\lambda \geq$ $0, 1-(1-\theta_{\gamma_\iota}^3)^\lambda - (\theta_{\gamma_\iota}^3)^\lambda \geq 0)$. $\Rightarrow 1-(1-\gamma_\iota^3)^\lambda \geq (\gamma_\iota^3)^\lambda, 1-(1-\theta_{\gamma_\iota}^3)^\lambda \geq (\theta_{\gamma_\iota}^3)^\lambda$, which implies $\sqrt[3]{1-(1-\gamma_\iota^3)^\lambda} \geq \gamma_\iota^\lambda, \sqrt[3]{1-(1-\theta_{\gamma_\iota}^3)^\lambda} \geq \theta_{\gamma_\iota}^\lambda$. Thus we have $\sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\gamma_\iota^3)^\lambda} \geq \sum_{\iota=1}^{n} \chi_\iota \gamma_\iota^\lambda$, similarly $\sum_{\iota=1}^{n} \chi_\iota \sqrt[3]{1-(1-\vartheta_\iota^3)^\lambda} \geq \sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota^\lambda$. According to Definition (3.4), the proof follows.

Theorem 4.3 Let $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1, 2, ..., n)$ be a number of CFFNs, and $\chi = (\chi_1, \chi_2, ..., \chi_n)^T$ be weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n} \chi_\iota = 1$, then

(i) $CFFWA(\mathfrak{A}_1^c, \mathfrak{A}_2^c, ..., \mathfrak{A}_n^c) \geq (CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^c$

(ii) $CFFWG(\mathfrak{A}_1^c, \mathfrak{A}_2^c, ..., \mathfrak{A}_n^c) \geq (CFFWG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))^c$

(iii) $CFFWPA(\mathfrak{A}_1^c,\mathfrak{A}_2^c,...,\mathfrak{A}_n^c) \geq (CFFWPA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n))^c$

(iv) $CFFWPG(\mathfrak{A}_1^c,\mathfrak{A}_2^c,...,\mathfrak{A}_n^c) \geq (CFFWPG(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n))^c$

Proof: We give the proof of (i). The other assertions are proved analogously.

(i) For any $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1,2,...,n)$, we get

$CFFWA(\mathfrak{A}_1^c,\mathfrak{A}_2^c,...,\mathfrak{A}_n^c) = \left(\sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}}, \sum_{\iota=1}^{n}\chi_\iota\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}\right)$ and

$$(CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n))^c = \left(\sum_{\iota=1}^{n}\chi_\iota\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}}\right)^c$$

$$= \left(\sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}}, \sum_{\iota=1}^{n}\chi_\iota\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}\right) = CFFWA(\mathfrak{A}_1^c,\mathfrak{A}_2^c,...,\mathfrak{A}_n^c)$$

Theorem 4.4 (Boundedness) Let $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1,2,...,n)$ be a number of CFFNs and $\chi = (\chi_1,\chi_2,...,\chi_n)^T$ be weight vector of $\mathfrak{A}_\iota$ with $\sum_{\iota=1}^{n}\chi_\iota = 1$, then

(i) $\mathfrak{A}_{min} \leq CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n) \leq \mathfrak{A}_{max}$

(ii) $\mathfrak{A}_{min} \leq CFFWG(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n) \leq \mathfrak{A}_{max}$

(iii) $\mathfrak{A}_{min} \leq CFFWPA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n) \leq \mathfrak{A}_{max}$

(iv) $\mathfrak{A}_{min} \leq CFFWPG(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n) \leq \mathfrak{A}_{max}$

Proof: We present the proof(i) and (iii).

For any $\mathfrak{A}_\iota = (\gamma_\iota.e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota.e^{i2\pi\theta_{\vartheta_\iota}})$ $(\iota = 1,2,...,n)$, we can get $\gamma^- \leq \gamma_\iota \leq \gamma^+$, $\theta_\gamma^- \leq \theta_{\gamma_\iota} \leq \theta_\gamma^+$ and $\vartheta^- \leq \vartheta_\iota \leq \vartheta^+$, $\theta_\vartheta^- \leq \theta_{\vartheta_\iota} \leq \theta_\vartheta^+$ $(\iota = 1,2,...,n)$. Suppose that $\mathfrak{A}_{min} = (\gamma^-.e^{i2\pi\theta_\gamma^-}, \vartheta^+.e^{i2\pi\theta_\vartheta^+})$ and $\mathfrak{A}_{max} = (\gamma^+.e^{i2\pi\theta_\gamma^+}, \vartheta^-.e^{i2\pi\theta_\vartheta^-})$.

(i) $\sum_{\iota=1}^{n}\chi_\iota\gamma^- \leq \sum_{\iota=1}^{n}\chi_\iota\gamma_\iota \leq \sum_{\iota=1}^{n}\chi_\iota\gamma^+$, $\sum_{\iota=1}^{n}\chi_\iota\theta_\gamma^- \leq \sum_{\iota=1}^{n}\chi_\iota\theta_{\gamma_\iota} \leq \sum_{\iota=1}^{n}\chi_\iota\theta_\gamma^+$ and $\sum_{\iota=1}^{n}\chi_\iota\vartheta^- \leq \sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota \leq \sum_{\iota=1}^{n}\chi_\iota\vartheta^+$, $\sum_{\iota=1}^{n}\chi_\iota\theta_\vartheta^- \leq \sum_{\iota=1}^{n}\chi_\iota\theta_{\vartheta_\iota} \leq \sum_{\iota=1}^{n}\chi_\iota\theta_\vartheta^+$.

Now $score(\mathfrak{A}_{min}) = \frac{1}{2}\left[\left((\gamma^-)^3 - (\vartheta^+)^3\right) + \left((\theta_\gamma^-)^3 - (\theta_\vartheta^+)^3\right)\right]$

$$= \frac{1}{2}\left[\left((\sum_{\iota=1}^{n}\chi_\iota\gamma^-)^3 - (\sum_{\iota=1}^{n}\chi_\iota\vartheta^+)^3\right) + \left((\sum_{\iota=1}^{n}\chi_\iota\theta_\gamma^-)^3 - (\sum_{\iota=1}^{n}\chi_\iota\theta_\vartheta^+)^3\right)\right],$$

$score(\mathfrak{A}_{max}) = \frac{1}{2}\left[\left((\gamma^+)^3 - (\vartheta^-)^3\right) + \left((\theta_\gamma^+)^3 - (\theta_\vartheta^-)^3\right)\right]$

$$= \frac{1}{2}\left[\left((\sum_{\iota=1}^{n}\chi_\iota\gamma^+)^3 - (\sum_{\iota=1}^{n}\chi_\iota\vartheta^-)^3\right) + \left((\sum_{\iota=1}^{n}\chi_\iota\theta_\gamma^+)^3 - (\sum_{\iota=1}^{n}\chi_\iota\theta_\vartheta^-)^3\right)\right],$$

$$score(CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)) = \frac{1}{2}\left[\left((\sum_{\iota=1}^{n}\chi_\iota\gamma_\iota)^3 - (\sum_{\iota=1}^{n}\chi_\iota\vartheta_\iota)^3\right) + \left((\sum_{\iota=1}^{n}\chi_\iota\theta_{\gamma_\iota})^3 - (\sum_{\iota=1}^{n}\chi_\iota\theta_{\vartheta_\iota})^3\right)\right].$$

Consequently, $score(\mathfrak{A}_{min}) \leq score(CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)) \leq score(\mathfrak{A}_{max})$

(iii) $\gamma^{-}.e^{i2\pi\theta_{\gamma}^{-}} = \Big(\sum_{\imath=1}^{n}\chi_\imath(\gamma^{-})^3\Big)^{\frac{1}{3}}.e^{i2\pi\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{-})^3\Big)^{\frac{1}{3}}}$, $\gamma^{+}.e^{i2\pi\theta_{\gamma}^{+}} = \Big(\sum_{\imath=1}^{n}\chi_\imath(\gamma^{+})^3\Big)^{\frac{1}{3}}.$ $e^{i2\pi\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{+})^3\Big)^{\frac{1}{3}}}$ and $\vartheta^{-}.e^{i2\pi\theta_{\vartheta}^{-}} = \Big(\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{-})^3\Big)^{\frac{1}{3}}.e^{i2\pi\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{-})^3\Big)^{\frac{1}{3}}}$ $\vartheta^{+}.e^{i2\pi\theta_{\vartheta}^{+}} =$ $\Big(\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{+})^3\Big)^{\frac{1}{3}}.e^{i2\pi\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{+})^3\Big)^{\frac{1}{3}}}$. Hence $\Big(\sum_{\imath=1}^{n}\chi_\imath(\gamma^{-})^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath\gamma_\imath^3\Big)^{\frac{1}{3}} \leq$ $\Big(\sum_{\imath=1}^{n}\chi_\imath(\gamma^{+})^3\Big)^{\frac{1}{3}}$ $\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{-})^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath\theta_{\gamma_\imath}^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{+})^3\Big)^{\frac{1}{3}}$ and $\Big(\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{-})^3\Big)^{\frac{1}{3}}$ $\leq \Big(\sum_{\imath=1}^{n}\chi_\imath\vartheta_\imath^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{+})^3\Big)^{\frac{1}{3}}$, $\Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{-})^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath\theta_{\vartheta_\imath}^3\Big)^{\frac{1}{3}} \leq \Big(\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{+})^3\Big)^{\frac{1}{3}}$.

Then $score(\mathfrak{A}_{min}) = \frac{1}{2}\Big[\Big((\gamma^{-})^3 - (\vartheta^{+})^3\Big) + \Big((\theta_{\gamma}^{-})^3 - (\theta_{\vartheta}^{+})^3\Big)\Big]$

$= \frac{1}{2}\Big[\Big(\Big((\sum_{\imath=1}^{n}\chi_\imath(\gamma^{-})^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{+})^3)^{\frac{1}{3}}\Big)^3\Big) + \Big(\Big((\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{-})^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{-})^3)^{\frac{1}{3}}\Big)^3\Big)\Big]$

$score(\mathfrak{A}_{max}) = \frac{1}{2}\Big[\Big((\gamma^{+})^3 - (\vartheta^{-})^3\Big) + \Big((\theta_{\gamma}^{+})^3 - (\theta_{\vartheta}^{-})^3\Big)\Big]$

$= \frac{1}{2}\Big[\Big(\Big((\sum_{\imath=1}^{n}\chi_\imath(\gamma^{+})^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath(\vartheta^{-})^3)^{\frac{1}{3}}\Big)^3\Big) + \Big(\Big((\sum_{\imath=1}^{n}\chi_\imath(\theta_{\gamma}^{+})^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath(\theta_{\vartheta}^{+})^3)^{\frac{1}{3}}\Big)^3\Big)\Big]$

$score(CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))$

$= \frac{1}{2}\Bigg[\Big((\sum_{\imath=1}^{n}\chi_\imath\gamma_\imath)^3 - (\sum_{\imath=1}^{n}\chi_\imath\vartheta_\imath)^3\Big) + \Big((\sum_{\imath=1}^{n}\chi_\imath\theta_{\gamma_\imath})^3 - (\sum_{\imath=1}^{n}\chi_\imath\theta_{\vartheta_\imath})^3\Big)\Bigg]$

$= \frac{1}{2}\Big[\Big(\Big((\sum_{\imath=1}^{n}\chi_\imath\gamma_\imath^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath\vartheta_\imath^3)^{\frac{1}{3}}\Big)^3\Big) + \Big(\Big((\sum_{\imath=1}^{n}\chi_\imath\theta_{\gamma_\imath}^3)^{\frac{1}{3}}\Big)^3 - \Big((\sum_{\imath=1}^{n}\chi_\imath\theta_{\vartheta_\imath}^3)^{\frac{1}{3}}\Big)^3\Big)\Big]$.

Consequently, $score(\mathfrak{A}_{min}) \leq score(CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n)) \leq score(\mathfrak{A}_{max})$. The other assertions (ii) and (iv) are proved analogously.

Theorem 4.5 (Idempotency) Let $\mathfrak{A}_\imath = (\gamma_\imath.e^{i2\pi\theta_{\gamma_\imath}}, \vartheta_\imath.e^{i2\pi\theta_{\vartheta_\imath}})$ $(\imath = 1, 2, ..., n)$ be a number of CFFNs and $\mathfrak{A}_\imath = \mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$, $\chi = (\chi_1, \chi_2, ..., \chi_n)^T$ be weight vector of $\mathfrak{A}_\imath$ with $\sum_{\imath=1}^{n}\chi_\imath = 1$, then

(i) $CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = \mathfrak{A}$

(ii) $CFFWG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = \mathfrak{A}$

(ii) $CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = \mathfrak{A}$

(iv) $CFFWPG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = \mathfrak{A}$

Proof: We present the proof (i) and (iii).

(i) Since $\mathfrak{A}_\imath = \mathfrak{A} = (\gamma.e^{i2\pi\theta_\gamma}, \vartheta.e^{i2\pi\theta_\vartheta})$ $(\imath = 1, 2, ..., n)$,

then $CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = CFFWA(\mathfrak{A}, \mathfrak{A}, ..., \mathfrak{A})$

$$= \Big(\sum_{\iota=1}^{n} \chi_\iota \gamma . e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \theta_\gamma}, \sum_{\iota=1}^{n} \chi_\iota \vartheta . e^{i2\pi \sum_{\iota=1}^{n} \chi_\iota \theta_\vartheta} \Big) = \mathfrak{A}$$

(iii) $CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) = CFFWA(\mathfrak{A}, \mathfrak{A}, ..., \mathfrak{A})$

$$= \Bigg(\Big(\sum_{\iota=1}^{n} \chi_\iota \gamma^3 \Big)^{\frac{1}{3}} . e^{i2\pi \Big(\sum_{\iota=1}^{n} \chi_\iota \theta_\gamma^3 \Big)^{\frac{1}{3}}}, \Big(\sum_{\iota=1}^{n} \chi_\iota \vartheta^3 \Big)^{\frac{1}{3}} . e^{i2\pi \Big(\sum_{\iota=1}^{n} \chi_\iota \theta_\vartheta^3 \Big)^{\frac{1}{3}}} \Bigg)$$

$$= (\gamma . e^{i2\pi\theta_\gamma}, \vartheta . e^{i2\pi\theta_\vartheta}) = \mathfrak{A}$$

The other assertions (ii) and (iv) are proved analogously.

Theorem 4.6 (Monotonicity) Let $\mathfrak{A}_\iota = (\gamma_\iota . e^{i2\pi\theta_{\gamma_\iota}}, \vartheta_\iota . e^{i2\pi\theta_{\vartheta_\iota}})$ and $\mathtt{B}_\iota = (\delta_\iota . e^{i2\pi\theta_{\delta_\iota}}, \nu_\iota . e^{i2\pi\theta_{\nu_\iota}})$ $(\iota = 1, 2, ..., n)$ be two number of CFFNs, if $\gamma_\iota \leq \delta_\iota, \theta_{\gamma_\iota} \leq \theta_{\delta_\iota}$ and $\vartheta_\iota \geq \nu_\iota, \theta_{\vartheta_\iota} \geq \theta_{\nu_\iota}$ for all ι, then

(i) $CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)$

(ii) $CFFWG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)$

(iii) $CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)$

(iv) $CFFWPG(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)$

Proof: we present the proof of (i) and (iii). Since $\gamma_\iota \leq \delta_\iota, \theta_{\gamma_\iota} \leq \theta_{\delta_\iota}$ and $\vartheta_\iota \geq \nu_\iota, \theta_{\vartheta_\iota} \geq \theta_{\nu_\iota}$ $(\iota = 1, 2, ..., n)$ then

(i) $\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota \leq \sum_{\iota=1}^{n} \chi_\iota \delta_\iota, \sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota} \leq \sum_{\iota=1}^{n} \chi_\iota \theta_{\delta_\iota}$ and $\sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota \geq \sum_{\iota=1}^{n} \chi_\iota \nu_\iota, \sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota} \geq \sum_{\iota=1}^{n} \chi_\iota \theta_{\nu_\iota}$.

$$score(CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n)) = \tfrac{1}{2} \Bigg[\Big((\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota)^3 - (\sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota)^3 \Big) + \Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota})^3 - (\sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota})^3 \Big) \Bigg],$$

$$score(CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)) = \tfrac{1}{2} \Bigg[\Big((\sum_{\iota=1}^{n} \chi_\iota \delta_\iota)^3 - (\sum_{\iota=1}^{n} \chi_\iota \nu_\iota)^3 \Big) + \Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\delta_\iota})^3 - (\sum_{\iota=1}^{n} \chi_\iota \theta_{\nu_\iota})^3 \Big) \Bigg].$$

Consequently, $score(CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n)) \leq score(CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n))$

Therefore, $CFFWA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n)$

(iii) $\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota^3 \leq \sum_{\iota=1}^{n} \chi_\iota \delta_\iota^3, \sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota}^3 \leq \sum_{\iota=1}^{n} \chi_\iota \theta_{\delta_\iota}^3$ and $\sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota^3 \geq \sum_{\iota=1}^{n} \chi_\iota \nu_\iota^3, \sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota}^3 \geq \sum_{\iota=1}^{n} \chi_\iota \theta_{\nu_\iota}^3$.

$score(CFFWPA(\mathfrak{A}_1, \mathfrak{A}_2, ..., \mathfrak{A}_n))$

$$= \tfrac{1}{2} \Bigg[\Bigg(\Big((\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota^3)^{\frac{1}{3}} \Big)^3 - \Big((\sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota^3)^{\frac{1}{3}} \Big)^3 \Bigg) + \Bigg(\Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota}^3)^{\frac{1}{3}} \Big)^3 - \Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota}^3)^{\frac{1}{3}} \Big)^3 \Bigg) \Bigg].$$

$$= \Big(\sum_{\iota=1}^{n} \chi_\iota \gamma_\iota^3 - \sum_{\iota=1}^{n} \chi_\iota \vartheta_\iota^3 \Big) + \Big(\sum_{\iota=1}^{n} \chi_\iota \theta_{\gamma_\iota}^3 - \sum_{\iota=1}^{n} \chi_\iota \theta_{\vartheta_\iota}^3 \Big)$$

$score(CFFWPA(\mathtt{B}_1, \mathtt{B}_2, ..., \mathtt{B}_n))$

$$= \tfrac{1}{2} \Bigg[\Bigg(\Big((\sum_{\iota=1}^{n} \chi_\iota \delta_\iota^3)^{\frac{1}{3}} \Big)^3 - \Big((\sum_{\iota=1}^{n} \chi_\iota \nu_\iota^3)^{\frac{1}{3}} \Big)^3 \Bigg) + \Bigg(\Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\delta_\iota}^3)^{\frac{1}{3}} \Big)^3 - \Big((\sum_{\iota=1}^{n} \chi_\iota \theta_{\nu_\iota}^3)^{\frac{1}{3}} \Big)^3 \Bigg) \Bigg].$$

$$= \Big(\sum_{\iota=1}^{n}\chi_\iota\delta_\iota^3 - \sum_{\iota=1}^{n}\chi_\iota\nu_\iota^3\Big) + \Big(\sum_{\iota=1}^{n}\chi_\iota\theta_{\delta_\iota}^3 - \sum_{\iota=1}^{n}\chi_\iota\theta_{\nu_\iota}^3\Big).$$

Consequently,$score(CFFWPA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n)) \leq score(CFFWPA(\mathtt{B}_1,\mathtt{B}_2,...,\mathtt{B}_n))$.

Therefore, $CFFWA(\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_n) \leq CFFWA(\mathtt{B}_1,\mathtt{B}_2,...,\mathtt{B}_n)$

The other assertions (ii)and (iv) are proved analogously.

5. MCDM Problem Using Various Operators in CFFNs Environment

In this section, we present an application using proposed operators to solve MCDM problems in CFFNs environment.

Let $\mathfrak{A} = \{\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_m\}$ be alternatives and $\mathtt{E} = \{\mathtt{E}_1,\mathtt{E}_2,...,\mathtt{E}_n\}$ be the parameters.

Let $\chi = \{\chi_1,\chi_2,...,\chi_n\}$be the weight of the parameters subject to the conditions $0 \leq \chi_\iota \leq 1$ $(\iota = 1,2,...,n)$ and $\sum_{\iota=1}^{n}\chi_\iota = 1$. We denote the values of the alternatives $\mathfrak{A}_i$ corresponding to each parameter $\mathtt{E}_j$ by $c_{ij} = (F_{ij},G_{ij})$, represented in CFFN. Then $\aleph = (c_{ij})_{m\times n}$ is a complex Fermatean fuzzy decision matrix(CFFDM). Now apply the different type of operators to combine the CFFNs in each row to a single CFFN. Finally, determine the ranking of the alternatives using score function. The maximum value denotes the best alternative among the given alternatives.

Algorithm for solving MCDM problem in CFFNs environment:

Step 1: Construct CFFDM using alternatives $\mathfrak{A} = \{\mathfrak{A}_1,\mathfrak{A}_2,...,\mathfrak{A}_m\}$ and parameters $\mathtt{E} = \{\mathtt{E}_1,\mathtt{E}_2,...,\mathtt{E}_n\}$.

Step 2: Aggregate the values by using the operators.

Step 3: Calculate the score values using Definition 3.4.

Step 4: Rank the alternatives in descending order and select the best alternative.

Case study: We present an application to select the best online educational application (OEA) which can facilitate students, teachers, and parents during the lockdown of schools. We study the selection process using different operators in the CFFN environment.

Case-I The system of traditional education methods has come to halt during this COVID-19 period. An alternate to classroom teaching is OEA. Nowadays, with the rise of internet usage and other new technologies, we can bridge the gap between classroom teaching and OEA. This study aims to select the best OEA to facilitate teachers, students, and parents during the lockdown of schools. Let an expert evaluate the four OEAs $(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3,\mathfrak{A}_4)$ based on parameters E_1= user friendly system , E_2=internet usage, E_3=question sessions, E_4=report for teachers. The weight for the parameters are (0.3,0.1,0.4,0.2) respectively.

The Method: We use the following method to determine the best OEA.

Step 1: The expert provides the information for the alternatives corresponding to each parameter in CFFDM form,

$$\aleph_{4\times4} = \begin{pmatrix} (0.71.e^{i2\pi(0.4)},0.3.e^{i2\pi(0.8)}) & (0.4.e^{i2\pi(0.6)},0.6.e^{i2\pi(0.7)}) & (0.5.e^{i2\pi(0.6)},0.6.e^{i2\pi(0.5)}) & (0.8.e^{i2\pi(0.4)},0.2.e^{i2\pi(0.9)}) \\ (0.5.e^{i2\pi(0.6)},0.7.e^{i2\pi(0.6)}) & (0.8.e^{i2\pi(0.4)},0.5.e^{i2\pi(0.6)}) & (0.4.e^{i2\pi(0.6)},0.5.e^{i2\pi(0.7)}) & (0.7.e^{i2\pi(0.5)},0.4.e^{i2\pi(0.7)}) \\ (0.9.e^{i2\pi(0.2)},0.6.e^{i2\pi(0.4)}) & (0.8.e^{i2\pi(0.6)},0.1.e^{i2\pi(0.8)}) & (0.6.e^{i2\pi(0.7)},0.4.e^{i2\pi(0.8)}) & (0.6.e^{i2\pi(0.8)},0.9.e^{i2\pi(0.3)}) \\ (0.6.e^{i2\pi(0.8)},0.7.e^{i2\pi(0.4)}) & (0.7.e^{i2\pi(0.8)},0.9.e^{i2\pi(0.2)}) & (0.7.e^{i2\pi(0.3)},0.3.e^{i2\pi(0.7)}) & (0.5.e^{i2\pi(0.6)},0.3.e^{i2\pi(0.7)}) \end{pmatrix}$$

Step 2: By using the discussed operators given in Definitions 4.1, 4.2, 4.3 and 4.4, aggregate the values in CFFDM and tabulate the results as shown in Table 1.

TABLE 1. Shows aggregation values for the operators.

	$\mathfrak{A}_1$	$\mathfrak{A}_2$
CFFWA	$(0.61.e^{i2\pi(0.5)},0.43.e^{i2\pi(0.69)})$	$(0.53.e^{i2\pi(0.56)},0.54.e^{i2\pi(0.66)})$
CFFWG	$(0.59.e^{i2\pi(0.48)},0.39.e^{i2\pi(0.66)})$	$(0.51.e^{i2\pi(0.55)},0.52.e^{i2\pi(0.65)})$
CFFWPA	$(0.64.e^{i2\pi(0.51)},0.49.e^{i2\pi(0.72)})$	$(0.56.e^{i2\pi(0.56)},0.56.e^{i2\pi(0.66)})$
CFFWPG	$(0.61.e^{i2\pi(0.5)},0.43.e^{i2\pi(0.69)})$	$(0.53.e^{i2\pi(0.56)},0.54.e^{i2\pi(0.66)})$

Step 3: Determine the score value by using Definition 3.4 and tabulate the results as shown in Table 2.

	$\mathfrak{A}_3$	$\mathfrak{A}_4$
CFFWA	$(0.71.e^{i2\pi(0.56)}, 0.53.e^{i2\pi(0.58)})$	$(0.63.e^{i2\pi(0.56)}, 0.48.e^{i2\pi(0.56)})$
CFFWG	$(0.69.e^{i2\pi(0.48)}, 0.46.e^{i2\pi(0.53)})$	$(0.62.e^{i2\pi(0.51)}, 0.42.e^{i2\pi(0.51)})$
CFFWPA	$(0.73.e^{i2\pi(0.64)}, 0.61.e^{i2\pi(0.65)})$	$(0.63.e^{i2\pi(0.63)}, 0.57.e^{i2\pi(0.6)})$
CFFWPG	$(0.71.e^{i2\pi(0.56)}, 0.53.e^{i2\pi(0.58)})$	$(0.63.e^{i2\pi(0.56)}, 0.48.e^{i2\pi(0.56)})$

TABLE 2. Shows the rank for alternatives.

	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{A}_4$	Rank
CFFWA	-0.026	-0.060	0.094	0.069	$\mathfrak{A}_3 > \mathfrak{A}_4 > \mathfrak{A}_1 > \mathfrak{A}_2$
CFFWG	-0.0156	-0.0000003	0.0964	0.078	$\mathfrak{A}_3 > \mathfrak{A}_4 > \mathfrak{A}_1 > \mathfrak{A}_2$
CFFWPA	-0.048	-0.055	0.074	0.049	$\mathfrak{A}_3 > \mathfrak{A}_4 > \mathfrak{A}_1 > \mathfrak{A}_2$
CFFWPG	-0.0000031	-0.0000003	0.000031	0.00000043	$\mathfrak{A}_3 > \mathfrak{A}_4 > \mathfrak{A}_1 > \mathfrak{A}_2$

Step 4: This means that OEA $\mathfrak{A}_3$ is the best educational tool when compared with other alternatives.

6. Conclusion

In this article, we have generalized the concept of FFS to CFFS. We have defined the concepts of CFFWA, CFFWG, CFFWPA and CFFWPG operators. We have presented an application to select the best OEA using these operators. As a direction for future research, we plan to apply the proposed concepts to other fuzzy environments.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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