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ON MATRICES WHOSE MINKOWSKI INVERSE IS IDEMPOTENT IN MINKOWSKI SPACE

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Abstract. In this paper, we have extended the concept of class of square matrices which have Minkowski inverse is idempotent in Minkowski space. A number of original characteristics of the class are derived and new properties identified.

Keywords: generalized inverse; partial isometry; star-dagger; idempotent matrix; projector; Hartwig-Spindelbock decomposition; Minkowski adjoint; Minkowski space.

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1. INTRODUCTION

Throughout this paper, Let us denote the set of complex matrices as $\mathbb{C}^{m\times n}$ and $\mathbb{C}^n$ represent complex n-tuples. The symbols B^*, $B^\dagger$, $B^\sim$, $B^{ⓜ}$, $R(B)$ and $N(B)$ denote the conjugate transpose, Moore-Penrose inverse, Minkowski adjoint, Minkowski inverse, range space and null space of a matrix B respectively. The components of this complex vector in $\mathbb{C}^n$

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is represented as $u = (u_0, u_1, u_2, ..., u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, ..., -u_{n-1})$. Clearly the Minkowski metric matrix is given by

$$G= \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \tag{1}$$

$G = G^*$ and $G^2 = I_n$. In [13], Minkowski inner product on $\mathbb{C}^n$ is defined by $(u, v) = [u, Gv]$, where $[.,.]$ denotes the conventional Hilbert space inner product. $\mathscr{M}$ denotes the Minkowski space, which is a space with Minkowski inner product.

In 2000 Meenakshi [7] presented the concept of Minkowski inverse of a matrix represented as $A \in \mathbb{C}^{m\times n}$. Also presented a unique solution to the following four matrix equations:

$$AXA = A,\ XAX = X,\ (AX)^{\sim} = AX,\ (XA)^{\sim} = XA \tag{2}$$

where $A^{\sim}$ denotes the Minkowski adjoint of the matrix A in $\mathscr{M}$.

However, the Minkowski inverse of a matrix does not exists always as that of Moore-Penrose inverse of a matrix. It is proved that the Minkowski inverse of a matrix $A \in \mathbb{C}^{m\times n}$ exists if and only if $rk(AA^{\sim}) = rk(A^{\sim}A) = rk(A)$. A matrix $A \in \mathbb{C}^n$ is said to be m-symmetric if $A = A^{\sim}$.

The Moore-Penrose inverse belongs to one of the most important notions of matrix analysis, whose significance is well reflected by a great number of applied research areas where it is exploited.

2. PRELIMINARIES

The symbols $\mathbb{C}_n^{mS}$, $\mathbb{C}_n^{GN}$, $\mathbb{C}_n^{mEP}$, $\mathbb{C}_n^{mbi-GN}$, $\mathbb{C}_n^{mbi-D}$, $\mathbb{C}_n^{mbi-EP}$, $\mathbb{C}_n^{mC}$, $\mathbb{C}_n^{mSD}$, $\mathbb{C}_n^{PIm}$, $\mathbb{C}_n^{mIS}$, $\mathbb{C}_n^{mI}$
will stand for the sets consisting of m-symmetric, G-normal, m-EP, m-bi-G-normal, m-bi-dagger, m-bi-EP, m-core, m-star-dagger, m-partial isometry, m-idempotent and m-symmetric, m-idempotent in Minkowski space respectively, i.e.,

$\mathbb{C}_n^{mS} = \{B \in \mathbb{C}_{n,n} : B = B^{\sim}\}$,

$\mathbb{C}_n^{GN} = \{B \in \mathbb{C}_{n,n} : BB^{\sim} = B^{\sim}B\}$,

$\mathbb{C}_n^{mEP} = \{B \in \mathbb{C}_{n,n} : BB^{ⓜ} = B^{ⓜ}B\} = \{B \in \mathbb{C}_{n,n} : R(B) = R(B^{\sim})\}$,

$\mathbb{C}_n^{mbi-GN} = \{B \in \mathbb{C}_{n,n} : BB^{\sim}B^{\sim}B = B^{\sim}BBB^{\sim}\}$,

$\mathbb{C}_n^{mbi-D} = \{B \in \mathbb{C}_{n,n} : (B^{ⓜ})^2 = (B^2)^{ⓜ}\}$,

$\mathbb{C}_n^{mbi-EP} = \{B \in \mathbb{C}_{n,n} : BB^{ⓜ}B^{ⓜ}B = B^{ⓜ}BBB^{ⓜ}\}$,

$\mathbb{C}_n^{mC} = \{B \in \mathbb{C}_{n,n} : rk(B^2) = rk(B)\}$,

$\mathbb{C}_n^{mSD} = \{B \in \mathbb{C}_{n,n} : B^{\sim} B^{ⓜ} = B^{ⓜ} B^{\sim}\}$,

$\mathbb{C}_n^{PIm} = \{B \in \mathbb{C}_{n,n} : B^{\sim} = B^{ⓜ}\}$,

$\mathbb{C}_n^{mIS} = \{B \in \mathbb{C}_{n,n} : B^2 = B = B^{\sim}\}$,

$\mathbb{C}_n^{mI} = \{B \in \mathbb{C}_{n,n} : B^2 = B\}$,

Let $B \in \mathscr{M}$ has singular value decomposition given by $B = V\Sigma U^*$. Taking Minkowski adjoint on both sides we get $B^{\sim} = RDS^{\sim}$, where $R = G_1 U$, $V = S$ are unitary and $D = \Sigma G_2$ is a diagonal matrix. G_1, G_2 are Minkowski metric matrices of suitable order. Thus, corresponding to every matrix $B \in \mathscr{M}$ having a singular value decomposition, there corresponds a matrix $W = B^{\sim} = RDS^{\sim}$. Furthermore, if we assume that $UG_1 = G_1 U$ and $VG_2 = G_2 V$, then U and V are G-unitary, that is, $UU^{\sim} = U^{\sim}U = I$ and $VV^{\sim} = V^{\sim}V = I$.

Consider, $B = R \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} S^{\sim}$, where R and S are G-unitary and D is a diagonal subblock of rank r. Let $S^{\sim}R = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$. Then, it can be easily verified that $S^{\sim}R$ is G-unitary. Post-multiplying the above equality by $R^{\sim}$, we get $S^{\sim} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} R^{\sim}$. Using this representation of $S^{\sim}$, we have $B = R \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} R^{\sim}$. Since $S^{\sim}R$ is G-unitary. Thus $(S^{\sim}R)(S^{\sim}R)^{\sim} = I$ gives $EE^{\sim} - FG_1F^{\sim} = I$. We will use G_1 to denote the Minkowski metric matrix of order $n-r \times n-r$.

Lemma 2.1. [12] Let $B \in \mathscr{M}$ be of rank r. Then there exists unitary $U \in \mathbb{C}_{n,n}$ such that

$$B = U \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} U^{\sim}, \tag{3}$$

where $D = diag(\sigma_1 I_{r1}, ..., \sigma_t I_{r_t})$ is the diagonal matrix of singular values of B, $\sigma_1 > \sigma_2 > ... > \sigma_t > 0$, $r_1 + r_2 + ... + r_t = r$, and $E \in \mathbb{C}_{r,r}$, $F \in \mathbb{C}_{r,n-r}$ satisfy $EE^{\sim} - FG_1F^{\sim} = I_r$.
(4)

From (3) it follows that

$$B^{\sim} = U \begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix} U^{\sim}, \tag{5}$$

$$B^{ⓜ} = U \begin{pmatrix} E^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim}, \tag{6}$$

Lemma 2.2. [12] Let $H \in \mathscr{M}$ be of rank r and have representation (3). Then:

(i) $H \in \mathbb{C}_n^{mP}$ if and only if $D = I, J = I$, and $K = 0$

(ii) $H \in \mathbb{C}_n^{PIm}$ if and only if $D = I_r$,

(iii) $H \in \mathbb{C}_n^{GN}$ if and only if $K = 0, D^2J = JD^2$,

(iv) $H \in \mathbb{C}_n^{MIA}$ if and only if $D^2J^{\sim} = J^{\sim}D^2$,

(v) $H \in \mathbb{C}_n^{EPm}$ if and only if J is G-unitary and $K = 0$,

(vi) $H \in \mathbb{C}_n^{GmP}$ if and only if $J^3 = I$ and $K = 0$,

(vii) $H \in \mathbb{C}_n^{HGmP}$ if and only if $(JD)^3 = (DJ)^3 = I_r$ and $K = 0$,

(viii) H is nilpotent of index 2 if and only if $J = 0$.

Lemma 2.3. Let $B \in \mathbb{C}_{n,n}$ be of rank r and have representation (3). Then:

(i) B is m-symmetric if and only if $F = 0, E^{\sim}D = DE$,

(ii) B is bi-normal if and only if $E^{\sim}D^2F = 0$ and, additionally, $E^{\sim}D^2E$ and D commute,

(iii) B is bi-dagger if and only if E is a partial isometry and, additionally, $E^{\sim}E$ and D commute,

(iv) B is bi-EP if and only if E is a partial isometry,

(v) B is star-dagger if and only if $ED = DE$,

(vi) B is a partial isometry if and only if $D = I_r$,

(vii) B is idempotent and m-symmetric if and only if $D = I_r, E = I_r$,

(viii) B is idempotent if and only if $DE = I_r$.

Proof: (i) Since B is m-symmetric$\Leftrightarrow B = B^{\sim}$.

$$\Leftrightarrow U \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow DE = E^{\sim}D; DF = 0 \Rightarrow F = 0.$

(ii) Since B is bi-normal $\Leftrightarrow BB^{\sim}B^{\sim}B = B^{\sim}BBB^{\sim}$.

$$\Leftrightarrow U \begin{pmatrix} DEE^{\sim}DE^{\sim}D^2E - DFG_1F^{\sim}DE^{\sim}D^2E & DEE^{\sim}DE^{\sim}D^2F - DFG_1F^{\sim}DE^{\sim}D \\ 0 & 0 \end{pmatrix} U^{\sim}$$

$$= U \begin{pmatrix} E^{\sim}DDEDEE^{\sim}D - E^{\sim}DDEDFG_1F^{\sim}D & 0 \\ -G_1F^{\sim}DDEDEE^{\sim}D + G_1F^{\sim}DDEDFG_1F^{\sim}D & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow DEE^{\sim}DE^{\sim}DDE$

$-DFG_1F^{\sim}DE^{\sim}DDE = E^{\sim}DDEDEE^{\sim}D - E^{\sim}DDEDFG_1F^{\sim}D;$ (i)

$\Leftrightarrow DEE^{\sim}DE^{\sim}DDF - DFG_1F^{\sim}DE^{\sim}DDF = 0;$ (ii)

$\Leftrightarrow -G_1F^{\sim}DDEDEE^{\sim}D + G_1F^{\sim}DDEDFG_1F^{\sim}D = 0;$ (iii)

From equation (i), we have

$\Leftrightarrow D(EE^{\sim} - FG_1F^{\sim})DE^{\sim}DDE = E^{\sim}DDED(EE^{\sim} - FG_1F^{\sim})D$ (Using equation(4))

$\Leftrightarrow DI_rDE^{\sim}DDE = E^{\sim}DDEDI_rD$

$\Leftrightarrow D^2E^{\sim}D^2E = E^{\sim}D^2ED^2$

Therefore $E^{\sim}D^2E$ and D commute.

From equation (ii), we have

$\Leftrightarrow D(EE^{\sim} - FG_1F^{\sim})DE^{\sim}DDF = 0$ (Using equation(4))

$\Leftrightarrow DI_rDE^{\sim}D^2F = 0$

$\Leftrightarrow D^2E^{\sim}D^2F = 0$

$\Leftrightarrow E^{\sim}D^2F = 0.$

(iii) Since B is bi-dagger$\Leftrightarrow B^{\sim}B^{ⓜ}B^{ⓜ}B^{\sim} = B^{ⓜ}B^{\sim}B^{\sim}B^{ⓜ}$.

$$\Leftrightarrow U \begin{pmatrix} E^{\sim}DE^{\sim}D^{-1}E^{\sim}D^{-1}E^{\sim}D & 0 \\ -G_1F^{\sim}DE^{\sim}D^{-1}E^{\sim}D^{-1}E^{\sim}D & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D^{-1}E^{\sim}DE^{\sim}DE^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1}E^{\sim}DE^{\sim}DE^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}DE^{\sim}D^{-1}E^{\sim}D^{-1}E^{\sim}D = E^{\sim}D^{-1}E^{\sim}DE^{\sim}DE^{\sim}D^{-1};$ (i)

$\Leftrightarrow -G_1F^{\sim}DE^{\sim}D^{-1}E^{\sim}D^{-1}E^{\sim}D = -G_1F^{\sim}D^{-1}E^{\sim}DE^{\sim}DE^{\sim}D^{-1};$ (ii)

From equation (i), we have

$\Leftrightarrow D = D^{-1}$

$\Leftrightarrow D^2 = I_r$

$\Leftrightarrow D = I_r$

Hence E is a partial isometry.

From equation (i) simplifying we have

$\Leftrightarrow D^2EE^{\sim} = EE^{\sim}D^2EE^{\sim}$

$\Leftrightarrow DEE^{\sim} = EE^{\sim}D$

$\Leftrightarrow EE^{\sim}$ and D commute.

(iv) Since B is bi-EP $\Leftrightarrow BB^{ⓜ}B^{ⓜ}B = B^{ⓜ}BBB^{ⓜ}$.

$$\Leftrightarrow U\begin{pmatrix} DEE^{\sim}D^{-1}E^{\sim}E - DFG_1F^{\sim}D^{-1}E^{\sim}E & DEE^{\sim}D^{-1}E^{\sim}F - DFG_1F^{\sim}D^{-1}E^{\sim}F \\ 0 & 0 \end{pmatrix}U^{\sim}$$
$$= U\begin{pmatrix} E^{\sim}D^{-1}DEDEE^{\sim}D^{-1} - E^{\sim}D^{-1}DEDFG_1F^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1}DEDEE^{\sim}D^{-1} + G_1F^{\sim}D^{-1}DEDFG_1F^{\sim}D^{-1} & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow DEE^{\sim}D^{-1}E^{\sim}D^{-1}DE - DFG_1F^{\sim}D^{-1}E^{\sim}D^{-1}DE$

$= E^{\sim}D^{-1}DEDEE^{\sim}D^{-1} - E^{\sim}D^{-1}DEDFG_1F^{\sim}D^{-1};$ (i)

$\Leftrightarrow DEE^{\sim}D^{-1}E^{\sim}D^{-1}DF - DFG_1F^{\sim}D^{-1}E^{\sim}D^{-1}DF = 0;$ (ii)

$\Leftrightarrow -G_1F^{\sim}D^{-1}DEDEE^{\sim}D^{-1} + G_1F^{\sim}D^{-1}DEDFG_1F^{\sim}D^{-1} = 0;$ (iii)

From equation (i), we have

$\Leftrightarrow D(EE^{\sim} - FG_1F^{\sim})D^{-1}E^{\sim}D^{-1}DE = E^{\sim}D^{-1}DED(EE^{\sim} - FG_1F^{\sim})D^{-1}$(Using equation(4))

$\Leftrightarrow DI_rD^{-1}E^{\sim}D^{-1}DE = E^{\sim}D^{-1}DEDI_rD^{-1}$

$\Leftrightarrow D = I_r$

Hence E is a partial isometry.

(v) Since B is star-dagger$\Leftrightarrow B^{\sim}B^{ⓜ} = B^{ⓜ}B^{\sim}$.

$$\Leftrightarrow U\begin{pmatrix} E^{\sim}DE^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}DE^{\sim}D^{-1} & 0 \end{pmatrix}U^{\sim} = U\begin{pmatrix} E^{\sim}D^{-1}E^{\sim}D & 0 \\ -G_1F^{\sim}D^{-1}E^{\sim}D & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$$\Leftrightarrow E^{\sim}DE^{\sim}D^{-1} = E^{\sim}D^{-1}E^{\sim}D \qquad \text{(i)}$$

$$\Leftrightarrow -G_1F^{\sim}DE^{\sim}D^{-1} = -G_1F^{\sim}D^{-1}E^{\sim}D \qquad \text{(ii)}$$

post multiply by D, from equation (i), we have

$$\Leftrightarrow E^{\sim}DE^{\sim}D^{-1}D = E^{\sim}D^{-1}E^{\sim}DD$$

$$\Leftrightarrow E^{\sim}DE^{\sim} = E^{\sim}D^{-1}E^{\sim}D^2$$

$$\Leftrightarrow DE^{\sim} = EE^{\sim}D^{-1}E^{\sim}D^2$$

$$\Leftrightarrow DE^{\sim} = I_rD^{-1}E^{\sim}D^2$$

$$\Leftrightarrow DE^{\sim} = D^{-1}E^{\sim}D^2$$

pre multiply by D, we have

$$\Leftrightarrow DDE^{\sim} = DD^{-1}E^{\sim}D^2$$

$$\Leftrightarrow D^2E^{\sim} = E^{\sim}D^2$$

Taking square roots on both sides

$$\Leftrightarrow DE^{\sim} = E^{\sim}D$$

Taking Minkowski adjoint on both sides, we have

$$\Leftrightarrow DE = ED.$$

(vi) Since B is a partial isometry$\Leftrightarrow B^{\sim} = B^{ⓜ}$.

$$\Leftrightarrow U\begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix}U^{\sim} = U\begin{pmatrix} E^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1} & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$$\Leftrightarrow E^{\sim}D = E^{\sim}D^{-1}; \qquad \text{(i)}$$

$$\Leftrightarrow -G_1F^{\sim}D = -G_1F^{\sim}D^{-1}; \qquad \text{(ii)}$$

From equation (i), we have

$$\Leftrightarrow D = EE^{\sim}D^{-1}$$

$$\Leftrightarrow DD = I_r$$

$\Leftrightarrow D^2 = I_r$

$\Leftrightarrow D = I_r$.

(vii) Since B is idempotent$\Leftrightarrow B^2 = B$.

$$\Leftrightarrow U \begin{pmatrix} DEDE & DEDF \\ 0 & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} U^{\sim}$$

$\Leftrightarrow DEDE = DE$; (i)

$\Leftrightarrow DEDF = DF$; (ii)

and m-symmetric$\Leftrightarrow B = B^{\sim}$.

$$\Leftrightarrow U \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}D = DE$; (iii)

$\Leftrightarrow DF = 0$; (iv)

$\Leftrightarrow -G_1F^{\sim}D = 0$; (v)

From equation (i), we have

$\Leftrightarrow (DE)^2 = DE$

$\Leftrightarrow DE = I$

$\Leftrightarrow D = I$ and $E = I$.

(viii) Since B is idempotent$\Leftrightarrow B^2 = B$.

$$\Leftrightarrow U \begin{pmatrix} DEDE & DEDF \\ 0 & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow DEDE = DE$; (i)

$\Leftrightarrow DEDF = DF$; (ii)

From equation (i), we have

$\Leftrightarrow (DE)^2 = DE$

$\Leftrightarrow DE = I_r$.

Hence the proof.

Lemma 2.4. Let $B \in \mathbb{C}_{n,n}$ be of the form in (3). Then:

(i) $B^{ⓜ}$ is idempotent if and only if $D = E$.

(ii) $B^{ⓜ}$ is bi-normal if and only if $E^{\sim}D^{-2}F = 0$ and, additionally, $E^{\sim}D^{-2}E$ and D^{-2} commute.

Proof: (i) Since $B^{ⓜ}$ is idempotent$\Leftrightarrow (B^{ⓜ})^2 = B^{ⓜ}$.

$$\Leftrightarrow U \begin{pmatrix} E^{\sim}D^{-1}E^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1}E^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$$\Leftrightarrow E^{\sim}D^{-1}E^{\sim}D^{-1} = E^{\sim}D^{-1} \Rightarrow E^{\sim}D^{-1}E^{\sim} = E^{\sim} \tag{i}$$

$$\Leftrightarrow -G_1F^{\sim}D^{-1}E^{\sim}D^{-1} = -G_1F^{\sim}D^{-1} \Rightarrow -G_1F^{\sim}D^{-1}E^{\sim} = -G_1F^{\sim} \tag{ii}$$

pre multiply by E and F from equations (i) and (ii), we have

$$\Leftrightarrow EE^{\sim}D^{-1}E^{\sim} = EE^{\sim} \tag{iii}$$

$$\Leftrightarrow -FG_1F^{\sim}D^{-1}E^{\sim} = -FG_1F^{\sim} \tag{iv}$$

Adding equations (iii) and (iv), we have

$$\Leftrightarrow (EE^{\sim} - FG_1F^{\sim})D^{-1}E^{\sim} = (EE^{\sim} - FG_1F^{\sim}) \qquad \text{(Using equation(4))}$$

$\Leftrightarrow I_rD^{-1}E^{\sim} = I_r$

$\Leftrightarrow D^{-1}E^{\sim} = I_r$

pre multiply by D, we have

$\Leftrightarrow DD^{-1}E^{\sim} = D$

$\Leftrightarrow E^{\sim} = D$

Taking Minkowski adjoint on both sides, we have

$\Leftrightarrow E = D$.

(ii) Since $B^{ⓜ}$ is bi-normal $\Leftrightarrow B^{ⓜ}(B^{ⓜ})^{\sim}(B^{ⓜ})^{\sim}B^{ⓜ} = (B^{ⓜ})^{\sim}B^{ⓜ}B^{ⓜ}(B^{ⓜ})^{\sim}$.

$$\Leftrightarrow U \begin{pmatrix} E^{\sim}D^{-2}ED^{-1}EE^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-2}ED^{-1}EE^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} D^{-2}E^{\sim}D^{-2}E & D^{-2}E^{\sim}D^{-2}F \\ 0 & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}D^{-2}ED^{-1}EE^{\sim}D^{-1} = D^{-2}E^{\sim}D^{-2}E;$ (i)

$\Leftrightarrow -G_1F^{\sim}D^{-2}ED^{-1}EE^{\sim}D^{-1} = 0;$ (ii)

$\Leftrightarrow D^{-2}E^{\sim}D^{-2}F = 0;$ (iii)

From equation (i), we have

$\Leftrightarrow E^{\sim}D^{-2}ED^{-1}I_rD^{-1} = D^{-2}E^{\sim}D^{-2}E$

$\Leftrightarrow E^{\sim}D^{-2}ED^{-2} = D^{-2}E^{\sim}D^{-2}E$

$\Leftrightarrow E^{\sim}D^{-2}E$ and D^{-2} commute.

From equation (iii), we have

$\Leftrightarrow D^{-2}E^{\sim}D^{-2}F = 0$

$\Leftrightarrow E^{\sim}D^{-2}F = 0.$

Hence the proof.

3. MAIN RESULTS

Theorem 3.1. Let $B \in \mathbb{C}_{n,n}$. Then $B^{ⓜ}$ is idempotent if and only if any of the following statements is satisfied

(i) $B^{\sim}B^{ⓜ} = B^{\sim}$,

(ii) $B^{ⓜ}B^{\sim} = B^{\sim}$,

(iii) $(BB^{\sim})^{ⓜ}$ is an inner inverse of B,

(iv) $(BB^{\sim})^{ⓜ}$ is an outer inverse of B.

Proof: (i) Since $B^{ⓜ}$ is idempotent $\Leftrightarrow B^{\sim}B^{ⓜ} = B^{\sim}$.

$$\Leftrightarrow U\begin{pmatrix} E^{\sim}DE^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}DE^{\sim}D^{-1} & 0 \end{pmatrix}U^{\sim} = U\begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}DE^{\sim}D^{-1} = E^{\sim}D$ (i)

$\Leftrightarrow -G_1F^{\sim}DE^{\sim}D^{-1} = -G_1F^{\sim}D$ (ii)

pre multiplying by E and F and adding by equations (i) and (ii), we have

$\Leftrightarrow EE^{\sim}DE^{\sim}D^{-1} - FG_1F^{\sim}DE^{\sim}D^{-1} = (EE^{\sim} - FG_1F^{\sim})D$

$\Leftrightarrow (EE^{\sim} - FG_1F^{\sim})DE^{\sim}D^{-1} = (EE^{\sim} - FG_1F^{\sim})D$ (Using equation(4))

$\Leftrightarrow I_rDE^{\sim}D^{-1} = I_rD$

$\Leftrightarrow DE^{\sim}D^{-1} = D$

post multiply by D, we have

$\Leftrightarrow DE^{\sim}D^{-1}D = DD$

$\Leftrightarrow DE^{\sim} = D^2$

pre multiplying by D^{-1}, we have

$\Leftrightarrow D^{-1}DE^{\sim} = D^{-1}DD$

$\Leftrightarrow E^{\sim} = D$

Taking Minkowski adjoint on both sides, we have

$\Leftrightarrow E = D.$

(ii) Since $B^{ⓜ}$ is idempotent $\Leftrightarrow B^{ⓜ}B^{\sim} = B^{\sim}$.

$$\Leftrightarrow U \begin{pmatrix} E^{\sim}D^{-1}E^{\sim}D & 0 \\ -G_1F^{\sim}D^{-1}E^{\sim}D & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D & 0 \\ -G_1F^{\sim}D & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}D^{-1}E^{\sim}D = E^{\sim}D$ (i)

$\Leftrightarrow -G_1F^{\sim}D^{-1}E^{\sim}D = -G_1F^{\sim}D$ (ii)

pre multiplying by E and F and adding equations (i) and (ii), we have

$\Leftrightarrow EE^{\sim}D^{-1}E^{\sim}D - FG_1F^{\sim}D^{-1}E^{\sim}D = EE^{\sim}D - FG_1F^{\sim}D$

$\Leftrightarrow (EE^{\sim} - FG_1F^{\sim})D^{-1}E^{\sim}D = (EE^{\sim} - FG_1F^{\sim})D$ (Using equation(4))

$\Leftrightarrow I_rD^{-1}E^{\sim}D = I_rD$

$\Leftrightarrow D^{-1}E^{\sim}D = D$

post multiply by D^{-1}, we have

$\Leftrightarrow D^{-1}E^{\sim}DD^{-1} = DD^{-1}$

$\Leftrightarrow D^{-1}E^{\sim} = I_r$

$\Leftrightarrow E^{\sim} = D$

Taking Minkowski adjoint on both sides, we have

$\Leftrightarrow E = D.$

(iii) Since $B^{ⓜ}$ is idempotent $\Leftrightarrow (BB^{\sim})^{ⓜ}$ is an inner inverse of B.

$\Leftrightarrow B(BB^{\sim})^{ⓜ}B = B$

$$\Leftrightarrow U\begin{pmatrix} DED^{-2}DE & DED^{-2}DF \\ 0 & 0 \end{pmatrix}U^{\sim} = U\begin{pmatrix} DE & DF \\ 0 & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow DED^{-2}DE = DE;$ (i)

$\Leftrightarrow DED^{-2}DF = DF;$ (ii)

pre multiply by D^{-1} from equation (i), we have

$\Leftrightarrow D^{-1}DED^{-2}DE = D^{-1}DE$

$\Leftrightarrow ED^{-1}E = E$ (iii)

pre multiply by D^{-1} from equation (ii), we have

$\Leftrightarrow D^{-1}DED^{-2}DF = D^{-1}DF$

$\Leftrightarrow ED^{-1}F = F$ (iv)

post multiply by $E^{\sim}$ and $F^{\sim}$ and adding from equations (iii) and (iv), we have

$\Leftrightarrow ED^{-1}EE^{\sim} - ED^{-1}FG_1F^{\sim} = EE^{\sim} - FG_1F^{\sim}$

$\Leftrightarrow ED^{-1}(EE^{\sim} - FG_1F^{\sim}) = EE^{\sim} - FG_1F^{\sim}$ (Using equation(4))

$\Leftrightarrow ED^{-1}I_r = I_r$

$\Leftrightarrow ED^{-1} = I_r$

$\Leftrightarrow E = D.$

(iv) Since $B^{ⓜ}$ is idempotent$\Leftrightarrow (BB^{\sim})^{ⓜ}$ is an outer inverse of B.

$\Leftrightarrow (BB^{\sim})^{ⓜ}B(BB^{\sim})^{ⓜ} = (BB^{\sim})^{ⓜ}$

$$\Leftrightarrow U\begin{pmatrix} D^{-2}DED^{-2} & 0 \\ 0 & 0 \end{pmatrix}U^{\sim} = U\begin{pmatrix} D^{-2} & 0 \\ 0 & 0 \end{pmatrix}U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow D^{-2}DED^{-2} = D^{-2}$

$\Leftrightarrow DED^{-2} = I_r$

$\Leftrightarrow DED^{-1}D^{-1} = I_r$

postmultiply by D, we have

$\Leftrightarrow DED^{-1}D^{-1}D = D$

$\Leftrightarrow DED^{-1} = D$

post multiply by D, we have

$\Leftrightarrow DED^{-1}D = DD$

$\Leftrightarrow E = D^2$

pre multiply by D^{-1}, we have

$\Leftrightarrow D^{-1}DE = D^{-1}D^2$

$\Leftrightarrow E = D$.

Hence complete the proof.

Lemma 3.2. Let $B \in \mathbb{C}_{n,n}$ be of the form (3). Then $(B^2)^{ⓜ} = (B^{ⓜ})^2$ is satisfied if and only if $(DED)^{ⓜ} = D^{-1}E^{\sim}D^{-1}$.

Proof: Since $(B^2)^{ⓜ} = (B^{ⓜ})^2$.

$$\Leftrightarrow U \begin{pmatrix} E^{\sim}(DED)^{ⓜ} & 0 \\ -G_1F^{\sim}(DED)^{ⓜ} & 0 \end{pmatrix} U^{\sim} = U \begin{pmatrix} E^{\sim}D^{-1}E^{\sim}D^{-1} & 0 \\ -G_1F^{\sim}D^{-1}E^{\sim}D^{-1} & 0 \end{pmatrix} U^{\sim}$$

Equating the corresponding entries

$\Leftrightarrow E^{\sim}(DED)^{ⓜ} = E^{\sim}D^{-1}E^{\sim}D^{-1}$ (i)

$\Leftrightarrow -G_1F^{\sim}(DED)^{ⓜ} = -G_1F^{\sim}D^{-1}E^{\sim}D^{-1}$ (ii)

pre multiplying by E and F by equations (i) and (ii) and adding, we have

$\Leftrightarrow (EE^{\sim} - FG_1F^{\sim})(DED)^{ⓜ} = (EE^{\sim} - FG_1F^{\sim})D^{-1}E^{\sim}D^{-1}$ (Using equation(4))

$\Leftrightarrow (DED)^{ⓜ} = D^{-1}E^{\sim}D^{-1}$.

Hence complete the proof.

4. CONCLUSION

In this paper, we have concluded the algebraic structure of matrices whose Minkowski inverse is idempotent in Minkowski space.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Ben-Israel and TNE. Greville, Generalized Inverses: Theory and Applications. Springer, New York, 2nd Edition (2003).

[2] J. K. Baksalary, O. M. Baksalary and X. Liu, Further results on generalized and hypergeneralized projectors, Linear Algebra Appl. 429(5-6) (2008), 1038–1050.

[3] O. M. Baksalary and G. Trenkler, Revisitation of the product of two orthogonal projectors, Linear Algebra Appl. 430(10) (2009), 2813–2833.

[4] O. M. Baksalary and G. Trenkler, On matrices whose Moore-Penrose inverse is idempotent, Linear Multilinear Algebra (2020), https://doi.org/10.1080/03081087.2020.1781038.

[5] S. L. Campbell and C. D. Meyer, Generalized inverses of linear transformations, SIAM, Philadelphia, (2009).

[6] R. E. Hartwig and K. Spindelbock, Matrices for which A^* and $A^\dagger$ commute, Linear Multilinear Algebra 14(3) (1983), 241–256.

[7] A. R. Meenakshi, Generalized Inverses of Matrices in Minkowski Space, Proc. Nat. Sem. Algebra Appl. (2000), 1–14.

[8] A. R. Meenakshi, Range Symmetric Matrices in Minkowski Space, Bull. Malays. Math. Sci. Soc. 23 (2000), 45–52.

[9] C. D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, Philadelphia, (2000).

[10] A. R. Meenakshi and D. Krishnaswamy, On sums of range symmetric matrices in Minkowski space, Bull. Malays. Math. Sci. Soc. Second Ser. 25 (2002), 137–148.

[11] A. R. Meenakshi and D. Krishnaswamy, Product of Range Symmetric Block Matrices in Minkowski Space, Bull. Malays. Math. Sci. Soc. 29 (2006), 59–68.

[12] M. S. Lone, A study on projections and Range symmetric matrices in Minkowski space. Ph.D. Dissertation, Annamalai University, Annamalai Nagar, India (2018).

[13] M. Renardy, Singular value decomposition in Minkowski space, Linear Algebra Appl. 236 (1996), 53–58.