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UNIQUENESS OF DIFFERENTIAL DIFFERENCE POLYNOMIALS OF *L*-FUNCTIONS AND MEROMORPHIC FUNCTIONS SHARING A POLYNOMIAL

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Abstract. In this article, we study the uniqueness of Differential difference polynomials of *L*-function and Differential difference polynomials of a meromorphic function concerning weighted sharing of a polynomial. Our result improves and generalizes results of Abhijit Banerjee, Saikat Bhattacharyya [1], N. Mandal, N. K. Datta [5].

Keywords: Nevanlinna theory; meromorphic function; *L*-function; differential difference polynomial; weighted sharing.

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1. INTRODUCTION

We presume that the reader is aware of Value Distribution of Nevanlinna theory [9, 10, 13]. It was Selberg who introduced the class called Selberg class. It is the set of all Dirichlet series. This class satisfies some axioms which leads to the definition of of L$-$function. In this paper, we make use of the definition of L$-$function and we redirect the reader to refer [1] to see more about the definition of L$-$function.

We define $\psi = \{g_1 : g_1$ is nonconstant meromorphic function$\}$, where nonconstant meromorphic functions are defined over C. In 19th century, Lahiri posed the inquiry regarding the

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relationship between f and g when two differential polynomials are expressed interms of f and g sharing non-zero complex values, refer([3]).

As an affirmative answer, Liu-Li-Yi obtained the condition of the uniqueness results for for $F = (f^n)^{(k)} - \alpha(z)$ and $L = (L^n)^{(k)} - \alpha(z)$ share $(0, \infty)$, refer ([4], Theorem A].

Recently, in 2018, to improve above theorem and to relax the nature of sharing the values, Sahoo and Halder used the concept of weighted sharing to prove uniqueness of F and L as defined earlier and corresponding conditions for l were obtained for different values of l, refer ([7], Theorem B].

In the same year, Hao-Chen obtained chain of theorems where differential polynomials are considered in more general way which highlights the uniqueness of g_1 and L for appropriate values of n and k sharing $(1, \infty)$, refer ([11], Theorem C, Theorem D) and also for sharing $(1, 0)$, refer ([11], Theorem E, Theorem F).

Inspired by these results, we prove the results as stated in section 3.

2. Preliminaries

If L is an L-function, then

(1) the relation between the characteristic function of L-function and the degree of L-function ‘d’ can be seen as follows

$$T(r, L) = \frac{d}{\pi} r log r + O(r), \tag{2.1}$$

refer([8]).

(2) The counting function for the poles of an L-function can be defined with the following relation

$$N(r, \infty, L) = S(r, L) = O(log r), \tag{2.2}$$

refer([5]).

Also, if $g_1 \in \psi$,

(1) the relation between g_1 and an L-function when they share $(\infty, 0)$ can be seen as

$$\overline{N}(r, \infty; g_1) = S(r, L) = O(log r), \tag{2.3}$$

refer ([6]).

(2) For $k \geq Z^+$

$$T(r, g_1^{(k)}) \leq T(r, g_1) + k\overline{N}(r, \infty; g_1) + S(r, g_1), \tag{2.4}$$

refer ([9]).

(3) Let a_0, $a_1 \ldots a_n$ be finite complex numbers such that $a_n \neq 0$, then

$$T(r, a_n g_1^n + a_{n-1} g_1^{n-1} + \ldots + a_1 g_1 + a_0) = nT(r, f) + S(r, f), \tag{2.5}$$

refer ([10])

(4) For $\alpha (\not\equiv 0, \infty)$ be a small function of g_1 then we have

$$\begin{aligned} T(r, g_1) \leq \overline{N}(r, \infty; g_1) + N(r, 0; g_1) + N(r, 0; g_1^{(k)} - \alpha) \\ -N\left(r, 0; \left(\frac{g_1^{(k)}}{\alpha}\right)'\right) + S(r, f), \end{aligned} \tag{2.6}$$

refer ([11]).

For h_1 being a transcendental meromorphic function of finite order then we have

$$T(r, h_1(z + c)) = T(r, h_1) + S(r, h_1), \tag{2.7}$$

refer ([2])

Suppose f and g be two transcendental meromorphic function and

(1) $H \not\equiv 0$. We have

$$\begin{aligned} \frac{1}{2}[T(r, f) + T(r, g)] \leq \left(\frac{k}{2} + 2\right)[\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)] \\ +N_{k+2}(r, 0; f) + N_{k+2}(r, 0; g) - \left(l - \frac{3}{2}\right)\overline{N}_*(r, 1; F, G) \\ +S(r, f) + S(r, g), \end{aligned} \tag{2.8}$$

where $F = \frac{f^{(k)}}{Q}$ and $G = \frac{g^{(k)}}{Q}$, refer ([1])

(2) Either $f^{(k)} g^{(k)} \equiv Q^2$ or $f \equiv g$, whenever f and g satisfies one of the following conditions,

(i) $l \geq 2$ and

$$\Delta_1 = \left(\frac{k}{2}+2\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g) > k+5. \tag{2.9}$$

(ii) $l = 1$ and

$$\begin{aligned}\Delta_2 = \left(\frac{3k}{4}+\frac{9}{4}\right)&\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\ &+\frac{1}{4}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\} > \frac{3k}{2}+6.\end{aligned} \tag{2.10}$$

(ii) $l = 0$ and

$$\begin{aligned}\Delta_3 = \left(2k+\frac{7}{2}\right)&\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\ &+\frac{3}{2}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\} > 4k+11,\end{aligned} \tag{2.11}$$

refer ([1]).

3. Main Results

Theorem 1. *Let $g_1 \in \psi$, L be an L-function, m,d,k and v_j $(j = 1,2,\ldots,d)$ be nonnegative integers and c_j $(j = 1,2,\ldots,d$ be distinct finite complex numbers. Suppose that $[g_1^n(f-1)^m \prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)} - \eta^{(z)}$ and $[L^n(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)} - \eta^{(z)}$ share $(0,l)$. If $l \geq 2$ and*

$$(m+n) > \frac{5k}{2}+6+2m_2(k+2)+2m_1+\sigma,$$

or $l = 1$ and

$$(m+n) > \frac{13k}{4}+\frac{27}{4}+\left(\frac{5k}{2}+\frac{9}{2}\right)m_2+\frac{5}{2}m_1+\frac{3}{2}\sigma,$$

or $l = 0$ and

$$(m+n) > 7k+\frac{21}{2}+(5k+7)m_2+5m_1+4\sigma.$$

Then one of the following two cases holds

$$(i)\ [g_1^n(f-1)^m \prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)}[L^n(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)} = \eta^2(z).$$

$$(ii)\ [g_1^n(f-1)^m \prod_{j=1}^{d} g_1(z+c_j)^{v_j}] = [L^n(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{v_j}],$$

or $g_1 = tL$ for a constant t satisfying $g_1^{m+n} = 1$.

Proof. We set the functions F_1 and G_1 as follows

$$F_1 = \frac{F^{(k)}}{\eta(z)}, \quad G_1 = \frac{G^{(k)}}{\eta(z)},$$

where,

$$F = \left[g_1^n (g_1 - 1)^m \prod_{j=1}^{d} g_1(z + c_j)^{v_j} \right],$$

$$G = \left[L^n (L - 1)^m \prod_{j=1}^{d} L(z + c_j)^{v_j} \right].$$

Obviously as $F^{(k)} - \eta(z)$, $G^{(k)} - \eta(z)$ share $(0, l)$, therefore F_1, G_1 share $(1, l)$ and an L-function has at most one pole $z = 1$ in the complex plane, we deduce by (2.5), (2.6) and Valiron-Mokhonko's lemma (see [12]) that,

$$\begin{aligned}
(n + m + \sigma)T(r, L) + S(r, g_1) &= T(r, G), \\
&\leq \overline{N}(r, \infty; G) + N(r, 0; G) + \overline{N}(r, 1; G_1) - N(r, 0; G_1') + S(r, g_1), \\
&\leq \overline{N}(r, \infty; G) + N_{k+1}(r, 0; G) + \overline{N}(r, 1; G_1) - N_0(r, 0; G_1') + S(r, g_1), \\
&\leq \overline{N}(r, \infty; L) + (k + 1)\overline{N}(r, 0; G) + \overline{N}(r, 1; G_1) + S(r, g_1), \\
&\leq (k + 1)(n + m + \sigma)T(r, L) + \overline{N}(r, 1; F_1) + S(r, g_1).
\end{aligned}$$

$$-k(n + m + \sigma)T(r, L) \leq T(r, F^{(k)}) + S(r, g_1) \tag{3.1}$$

By (2.1), we see that L is a transcendental meromorphic function, combining this with (3.1), ([13],Theorem 1.5) and the assumption of the lower bound of $(m + n)$, we deduce that $F^{(k)}$ and so g_1 is a transcendental meromorphic function.

Using (2.5), we have

$$\begin{aligned}
\Theta(\infty, F) &= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; F)}{T(r, F)}, \\
&= 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \infty; F)}{(m + n + \sigma)T(r, F) + O(1)}, \\
&\geq 1 - \frac{1}{m + n + \sigma}
\end{aligned} \tag{3.2}$$

$$\delta_{k+2}(0,F) = 1 - \limsup_{r\to\infty} \frac{N_{k+2}(r,0;F)}{T(r,F)},$$

$$\geq 1 - \limsup_{r\to\infty} \frac{N_{k+2}(r,0;g_1^n) + N_{k+2}(r,0;(g_1-1)^m) + N_{k+2}(r,0;\phi)}{(m+n+\sigma)T(r,F) + O(1)}, \tag{3.3}$$

$$\geq 1 - \frac{(k+2) + m_2(k+2) + m_1 + \sigma}{m+n+\sigma}.$$

Similarly,

$$\delta_{k+2}(0,G) \geq 1 - \frac{(k+2) + m_2(k+2) + m_1 + \sigma}{m+n+\sigma}, \tag{3.4}$$

$$\delta_{k+1}(0,F) \geq 1 - \frac{(k+1) + m_2(k+1) + m_1 + \sigma}{m+n+\sigma}, \tag{3.5}$$

$$\delta_{k+1}(0,G) \geq 1 - \frac{(k+1) + m_2(k+1) + m_1 + \sigma}{m+n+\sigma}. \tag{3.6}$$

Since an L-function has at most one pole at $z = 1$ in the complex plane, we have

$$N(r,L) \leq logr + O(1).$$

So using (2.1) we deduce that

$$\Theta(\infty,G) = 1. \tag{3.7}$$

Case 1: Let $l \geq 2$

By using (2.9), (3.2)-(3.4) and (3.7), we obtain

$$(m+n) > \frac{5k}{2} + 6 + 2m_2(k+2) + 2m_1 + \sigma.$$

We have $\Delta_1 > k+5$. Thus by (2.9), we get either $F^{(k)}G^{(k)} = \eta^2(z)$ or $F \equiv G$. Let $F \equiv G$, i.e,

$$g_1^n(g_1-1)^m \prod_{j=1}^{d} g_1(z+c_j)^{\nu_j} = L^n(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{\nu_j}. \tag{3.8}$$

Now, we set

$$H = \frac{g_1}{L}. \tag{3.9}$$

If H is a nonconstant meromorphic function, then we get (3.8). Suppose H is a constant i,e. $H = t = \frac{g_1}{L}$. Then from (3.9), we get

$$(tL)^n(tL-1)^m \prod_{j=1}^{d}(tL)(z+c_j)^{\nu_j} = L^n(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{\nu_j}$$

$$t^{n+\sigma}\left[t^m L^m + \binom{m}{1}(-1)t^{m-1}L^{m-1} + \binom{m}{2}(-1)^2 t^{m-2}L^{m-2} + \ldots + (-1)^m\right]$$
$$= \left[L^m + \binom{m}{1}(-1)L^{m-1} + \binom{m}{2}(-1)^2 L^{m-2} + \ldots + (-1)^m\right],$$
$$t^{n+m+\sigma} = t^{n+m-1+\sigma} = t^{n+m-2+\sigma} = \ldots = t^{n+\sigma} = 1.$$

So we know $t = 1$, then $g_1 = tL$ for a constant t satisfying $t^{n+m+\sigma} = 1$.

Case 2: Let $l = 1$.

By using (2.10), (3.2)-(3.7), we have

$$\Delta_2 = \left(\frac{3k}{4} + \frac{9}{4}\right)\{\Theta(\infty,F) + \Theta(\infty,G)\} + \delta_{k+2}(0,F),$$
$$+\delta_{k+2}(0,G) + \frac{1}{4}\{\delta_{k+1}(0,F) + \delta_{k+1}(0,G)\},$$

$$\Delta_2 \geq \frac{3k}{4} + 7 - \frac{\left(\frac{3k}{4}+\frac{9}{4}\right) + \frac{5k}{2} + \frac{9}{2} + \left(\frac{5k}{2}+\frac{9}{2}\right)m_2 + \frac{5}{2}m_1 + \frac{5}{2}\sigma}{m+n+\sigma}. \tag{3.10}$$

By (3.10) and the assumption

$$(m+n) > \frac{13k}{4} + \frac{27}{4} + \left(\frac{5k}{2} + \frac{9}{2}\right)m_2 + \frac{5}{2}m_1 + \frac{3}{2}\sigma.$$

We have $\Delta_2 > \frac{3k}{2} + 6$. Thus by (2.10) we get either $F^{(k)}G^{(k)} = \eta^2(z)$ or $F \equiv G$. Proceeding in the same manner as done in the case 1, we get conclusion.

Case 3: Let $l = 0$. By using (2.11), (3.2)-(3.7), we have

$$\Delta_3 \geq 4k + 12 - \frac{\left(2k+\frac{7}{2}\right) + 5k + 7 + (5k+7)m_2 + 5m_1 + 5\sigma}{m+n+\sigma}, \tag{3.11}$$

by (3.11) and the assumption

$$(m+n) > 7k + \frac{21}{2} + (5k+7)m_2 + 5m_1 + 4\sigma.$$

We have $\Delta_3 > 4k + 11$. Thus by (2.11), we get either $F^{(k)}G^{(k)} = \eta^2(z)$ or $F \equiv G$. Proceeding in the same manner as done in case 1, we get conclusion. □

For $n=0$, we obtain following result.

Corollary 1. *Let $g_1 \in \psi$, L be an L-function, m,d,k and v_j $(j=1,2,\dots,d)$ be nonnegative integers and c_j $(j=1,2,\dots,d$ be distinct finite complex numbers. Suppose that $[(g_1-1)^m \prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)}-\eta^{(z)}$ and $[(L-1)^m \prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)}-\eta^{(z)}$ share $(0,l)$. If*
$l \geq 2$ and

$$m>\frac{k}{2}+2+2m_2(k+2)+2m_1+\sigma,$$

or $l=1$ and

$$m>\frac{3k}{4}+\frac{9}{4}+\left(\frac{5k}{2}+\frac{9}{2}\right)m_2+\frac{5}{2}m_1+\frac{3}{2}\sigma,$$

or $l=0$ and

$$m>2k+\frac{7}{2}+(5k+7)m_2+5m_1+4\sigma.$$

Then one of the following two cases holds

$$(i)\ [(g_1-1)^m\prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)}[(L-1)^m\prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)}=\eta^2(z).$$

$$(ii)\ [(g_1-1)^m\prod_{j=1}^{d} g_1(z+c_j)^{v_j}]=\left[(L-1)^m\prod_{j=1}^{d} L(z+c_j)^{v_j}\right].$$

or $g_1=tL$ for a constant t satisfying $t^{m+\sigma}=1$.

For $m=0$, we obtain the following result.

Corollary 2. *Let $g_1 \in \psi$, L be an L-function, m,d,k and v_j $(j=1,2,\dots,d)$ be nonnegative integers and c_j $(j=1,2,\dots,d$ be distinct finite complex numbers. Suppose that $[g_1^n \prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)}-\eta^{(z)}$ and $[(L^n \prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)}-\eta^{(z)}$ share $(0,l)$. If*
$l \geq 2$ and

$$n>\frac{5k}{2}+6+\sigma,$$

or $l=1$ and

$$n>\frac{13k}{4}+\frac{27}{4}+\frac{3}{2}\sigma,$$

or $l=0$ and

$$n>7k+\frac{21}{2}+4\sigma.$$

Then one of the following two cases holds

$$(i)\ [g_1^n \prod_{j=1}^{d} g_1(z+c_j)^{v_j}]^{(k)} [L^n \prod_{j=1}^{d} L(z+c_j)^{v_j}]^{(k)} = \eta^2(z).$$

$$(ii)\ [g_1^n \prod_{j=1}^{d} g_1(z+c_j)^{v_j}] = \left[L^n \prod_{j=1}^{d} L(z+c_j)^{v_j} \right].$$

or $g_1 = tL$ *for a constant* t *satisfying* $t^{n+\sigma} = 1$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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