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PATHWAY FRACTIONAL INTEGRAL OPERATOR ASSOCIATED WITH THE j-GENERALIZED p-k MITTAG-LEFFLER FUNCTION

MEENA KUMARI GURJAR[1,*], PREETI CHHATTRY[2], SHOBHA SHUKLA[2]
[1]Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, Rajasthan, India-342011
[2]C.V. Raman University, Kota, Bilaspur, Chhattisgarh, India-495113

Abstract: In this paper, we present composition of the pathway fractional integral $P_{0+}^{(\eta,\sigma)}$ with the j-generalized p-k Mittag-Leffler (M-L) function. We also find out some special cases of the main results with those earlier ones.

Keywords: Pathway fractional integral operators, Mittag-Leffler function.
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1. INTRODUCTION

Mittag-Leffler (M-L) functions play a vital role in determining the solutions of fractional differential and integral equations which are associated with an extensive variety of problems in diverse areas of mathematics and mathematical physics. Some functions which defined via power series in the whole complex plane $\mathbb{C}$ are popularly known as Mittag-Leffler (M-L) functions. The M-L functions are generalization of the exponential functions.

The Swedish mathematician Mittag-Leffler [11] introduced the so-called Mittag-Leffler function

*Corresponding author
E-mail address: meenanetj@gmail.com
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$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \ , \quad (\alpha, z \in \mathbb{C}; \Re(\alpha) > 0) \tag{1.1}$$

A generalization of $E_{\alpha}(z)$ was studied by Wiman [18] and known as generalized Mittag-Leffler function or Wiman's function in the following form

$$E_{\propto,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \ , \quad (z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0) \tag{1.2}$$

In 1971, Prabhakar [14] introduced the Mittag-Leffler function $E_{\propto,\beta}^{\gamma}(z)$ in the form

$$E_{\propto,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \ , \tag{1.3}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0$ and $(\gamma)_n$ the Pochhammer symbol given by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2) \dots (\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$$

In 2007, Shukla & Prajapati [17] established the function as

$$E_{\propto,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{1.4}$$

where $z, \propto, \beta, \gamma \in \mathbb{C}$; $\Re(\propto) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$ and $q \epsilon (0,1) \cup N$.

Another generalization of the Mittag-Leffler function called k-Mittag-Leffler function has been introduced by Dorrego & Cerutti [2] defined as

$$E_{k,\propto,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\propto n+\beta)} \frac{z^n}{n!} \tag{1.5}$$

where $k > 0$; $\alpha, \beta, \gamma, z \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\beta) > 0$ and $(\gamma)_{n,k}$ the Pochhammer k-symbol given by Diaz & Pariguan [1] as

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k) \dots (\gamma+(n-1)k) = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} \tag{1.6}$$

and Γ_k the k-Gamma function given by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} \ dt \ , \ (\Re(z) > 0).$$

Nisar et al. [13] introduce the generalized k-Mittag-Leffler function as

$$E_{k,\propto,\beta,\delta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\propto n+\beta)} \frac{z^n}{(\delta)_n} \tag{1.7}$$

Where $k \in \Re$; $\alpha, \beta, \gamma, \delta, z \in \mathbb{C}; \Re(\propto) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\delta \neq 0, -1, \dots$ and nq is a positive integer.

A new generalization of the k-Mittag –Leffler function has been defined by Gehlot [3] as given below:

$$GE_{k,\propto,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{1.8}$$

where $k \in \Re$; $\alpha, \beta, \gamma, z \in \mathbb{C}$; $\Re(\propto) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$; $q\epsilon(0,1) \cup N$ and $(\gamma)_{nq,k}$ is defined as (1.6) and the generalized Pochammer symbol $(\gamma)_{nq}$ defined as

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n, \text{ if } q\epsilon N$$

Gehlot [5] introduced the p-k Mittag –Leffler function which is defined as

$$_pE_{k,\propto,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{nq,k}}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{z^n}{n!} \tag{1.9}$$

where $k, p \in \Re^+ - \{0\}$; $z \in \mathbb{C}$; $\alpha, \beta, \gamma \in \mathbb{C}/kZ^-$ with $\Re(\propto) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$; $q\epsilon(0,1) \cup N$ and $p^{(\gamma)_{nq,k}}$ is Pochammer (p-k)symbol and $p^{\Gamma_k(x)}$ gamma function is defined by Gehlot [4] as

$$p^{(\gamma)_{n,k}} = \left[\frac{\gamma p}{k}\right]\left[\frac{\gamma p}{k} + p\right] \ldots \left[\frac{\gamma p}{k} + (n-1)p\right] = \prod_{i=0}^{n-1} \left[\frac{\gamma p}{k} + ip\right] \tag{1.10}$$

and $p^{\Gamma_k(x)}$ gamma function is defined as

$$p^{\Gamma_k(x)} = \frac{1}{k} \lim_{n\to\infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}-1}}{p^{(x)_{n,k}}} \tag{1.11}$$

$$p^{\Gamma_k(x)} = \left(\frac{p}{k}\right)^{x/k} \Gamma_k(x) = \frac{(p)^{x/k}}{k} \Gamma\left(\frac{x}{k}\right) \tag{1.12}$$

Luque [7] introduced the L-Mittag-Leffler function defined for $z \in \mathbb{C}$; $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\propto) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0, j\epsilon N_0$ by the series

$$L_{\propto,\beta}^{\gamma,j}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}}{\Gamma(\alpha n+\beta)} \frac{z^n}{(n+j)!} \tag{1.13}$$

Gehlot [6] investigated the j-generalized p-k Mittag-Leffler function as

$$_p^jE_{k,\propto,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{z^n}{(n+j)!} \tag{1.14}$$

where $k, p \in \Re^+ - \{0\}$;$z \in \mathbb{C}$; $\alpha, \beta, \gamma \in \mathbb{C}/kZ^-$with $\Re(\propto) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$;$q\epsilon(0,1) \cup N$; $j\epsilon N_0$ and $p^{(\gamma)_{nq,k}}$ is two parameter Pochammer symbol given by (1.10) and $p^{\Gamma_k(x)}$is the two parameter gamma function given by (1.11) and (1.12).

The fractional calculus is a field of applied mathematics that deals with the fractional derivatives and fractional integrals of arbitrary orders. During the last few decades, many researchers have applied fractional calculus to all fields of science such as engineering and mathematics. The researchers have developed significant contributions in the field of fractional calculus such as fractional derivatives of constant and variable orders, global existence solution of differential equations; an alternative method for solving generalized differential equations of fractional order, a new type of fractional derivative formula containing the normalized sine function without singular kernel.

The pathway model is introduced by Mathai [8] and further studied by Mathai & Haubold [9] & [10]. Recently, pathway function integral operator introduced by Nair [12].

Let $f(x) \in L(a,b), \eta \in \mathbb{C}, \Re(\eta) > 0,\ a > 0$ and the pathway parameter $\sigma < 1$, then the pathway fractional integral operator is given as follows:

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\sigma)}\right]} \left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}} f(t)\, \mathrm{d}t \tag{1.15}$$

For real scalar σ, the pathway model for scalar random variables is represented by the following probability density function (p.d.f):

$$f\ (x) = c\ |x|^{\nu-1} \left[\ 1 - a\ (1-\sigma)|x|^{\xi}\right]^{\frac{\lambda}{(1-\sigma)}} \tag{1.16}$$

assigned that $-\infty < x < \infty, \xi > 0$, $\lambda \geq 0, \left[1 - a(1-\sigma)|x|^{\xi}\right] > 0$ and $\nu > 0$, where c and σ denotes the normalizing constant and pathway parameter respectively. For real σ, the normalizing constant c is as follows:

$$c = \begin{cases} \dfrac{1}{2} \dfrac{\xi[a(1-\sigma)]^{\frac{\nu}{\xi}}\, \Gamma\left(\frac{\nu}{\xi} + \frac{\lambda}{1-\sigma} + 1\right)}{\Gamma\left(\frac{\nu}{\xi}\right) \Gamma\left(\frac{\lambda}{1-\sigma} + 1\right)}\ , & (\sigma < 1). \\ \dfrac{1}{2} \dfrac{\xi[a(\sigma-1)]^{\frac{\nu}{\xi}} \Gamma\left(\frac{\lambda}{\sigma-1}\right)}{\Gamma\left(\frac{\nu}{\xi}\right)\ \Gamma\left(\frac{\lambda}{\sigma-1} - \frac{\nu}{\xi}\right)}\ , & \left(\frac{1}{\sigma-1} - \frac{\nu}{\xi} > 0, \qquad \sigma > 1\right) \\ \dfrac{1}{2} \dfrac{\xi(a\lambda)^{\frac{\nu}{\xi}}}{\Gamma\left(\frac{\nu}{\xi}\right)} & (\sigma \to 1). \end{cases} \tag{1.17}$$

It is noted that if $\sigma < 1$, we have $\left[1 - a(1-\sigma)|x|^{\xi}\right] > 0$ and (1.16) can be considered as member of the extended generalized type-1 beta family. Also, the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f. are particular case of the pathway density function in (1.16), for $\sigma < 1$.

For instance, $\sigma > 1$ settion $(1-\sigma) = -(\sigma - 1)$ in (1.15) gives

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{-a(\sigma-1)}\right]} \left[1 + \frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}} f(t)\, \mathrm{d}t \tag{1.18}$$

and

$$f(x) = c|x|^{\nu-1} \left[1 + a\,(\sigma-1)|x|^{\xi}\right]^{-\frac{\lambda}{(\sigma-1)}} \tag{1.19}$$

provided that $-\infty < x < \infty, \xi > 0$, $\lambda \geq 0$, and $\sigma > 1$ which represents the extended generalized type-2 beta model for real x. Further the type-2 beta density, the F density, the student-t density and many other density functions are particular cases of the density function defined in (1.19). Moreover, if $\sigma \to 1$, then the operator (1.15) reduces to the Laplace integral transform. Similarly, if $\sigma = 0, a = 1$ and λ is replaced by $\lambda - 1$, then pathway operator (1.15) reduces to the well-known Riemann-Liouville fractional integral operator.

The main objective of this study is to obtain pathway fractional integral operators associated with j-generailzed p-k Mittag-Leffler functions.

2. PATHWAY FRACTIONAL INTEGRATION OF j-GENERALIZED p-k MITTAG-LEFFLER FUNCTION

In this section, we derive the pathway integration formulas involving the j-generalizd p-k Mittag-Leffler function from (1.14).

Theorem 2.1. Let $\rho, \beta, \eta, z \in \mathbb{C}; \alpha, \beta, \gamma \in \mathbb{C}/kZ^{-}; k, p \in \Re^{+} - \{0\}, \{\Re(\propto), \Re(\beta), \Re(\gamma)\} > 0$, $q\epsilon(0,1) \cup N; j\epsilon N_0; \Re\left(1 + \frac{\eta}{1-\sigma}\right) > 0, \sigma < 1.$ Then the following formula holds true:

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\, {}_{p}^{j}E_{k,\propto,\beta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\right](x) = x^{\eta+\frac{\beta}{k}} \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\, p^{\left(1+\frac{\eta}{1-\sigma}\right)}}{[a(1-\sigma)]^{\frac{\beta}{k}}}\, {}_{p}^{j}E_{k,\propto,\beta+k\left(1+\frac{\eta}{1-\sigma}\right)}^{\gamma,q}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\propto}{k}}\right] \tag{2.1}$$

Proof: By applying (1.15), we have

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\,{}_{p}^{j}E_{k,\propto,\beta,\delta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\right]= x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\sigma)}\right]} t^{\frac{\beta}{k}-1}\left[1-\frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}}\,{}_{p}^{j}E_{k,\propto,\beta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\,\mathrm{d}t$$

For convenience, we assume that $\mathfrak{J}_1$ in the place of the left hand integral of the above term and also using (1.14), it gives

$$\mathfrak{J}_1 = x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\sigma)}\right]} t^{\frac{\beta}{k}-1}\left[1-\frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}}\left\{\sum_{n=0}^{\infty}\frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}}\frac{\left(\omega t^{\frac{\propto}{k}}\right)^{n}}{(n+j)!}\right\}\mathrm{d}t$$

After interchanging the order of integration and summation which is permissible under the conditions stated in the theorem, we get

$$\mathfrak{J}_1 = x^{\eta}\sum_{n=0}^{\infty}\frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}}\frac{(\omega)^{n}}{(n+j)!}\int_{0}^{\left[\frac{x}{a(1-\sigma)}\right]}\left[1-\frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}} t^{\frac{\beta}{k}+\frac{\propto n}{k}-1}\,\mathrm{d}t$$

Now, we simplify the above equation with help of well-known beta function formula and using p-k Gamma function equation (1.11), we have

$$\mathfrak{J}_1 = x^{\eta}\sum_{n=0}^{\infty}\frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}}\frac{(\omega)^{n}}{(n+j)!}\left[\frac{x}{a(1-\sigma)}\right]^{\frac{\beta}{k}+\frac{\propto n}{k}}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1+\frac{\eta}{1-\sigma}\right)}$$

$$=\frac{x^{\eta+\frac{\beta}{k}}}{[a(1-\sigma)]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{p^{(\gamma)_{(n+j)q,k}}}{(n+j)!\;p^{\Gamma_k(\alpha n+\beta)}}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right]^{n}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1+\frac{\eta}{1-\sigma}\right)}$$

$$=\frac{x^{\eta+\frac{\beta}{k}}\,\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{k\,p^{(\gamma)}{}_{(n+j)_{q,k}}}{(n+j)!\;p^{\frac{\alpha n+\beta}{k}}}\times\frac{\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\propto}{k}}\right]^{n}}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1+\frac{\eta}{1-\sigma}\right)}$$

Again, on applying (1.14), we get the desired results of Theorem 2.1.

Corollary 2.2. If we put $p=k, j=0$ in Theorem 2.1, then we get the following known result given by Ram and Choudhary [16] contains generalized k-Mittag-Leffler function defined in (1.8):

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\,GE_{k,\propto,\beta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\right](x)= x^{\eta+\frac{\beta}{k}}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)k^{\left(1+\frac{\eta}{\sigma-1}\right)}}{[a(1-\sigma)]^{\frac{\beta}{k}}}\,GE_{k,\propto,\beta+k\left(1+\frac{\eta}{1-\sigma}\right)}^{\gamma,q}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\propto}{k}}\right]$$

Corollary 2.3. If we substitute $p = k, q = 1, j = 0$ in Theorem 2.1, then we get the following known result given by Nisar et al. [13] contains k- Mittag-Leffler function defined in (1.5):

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\,E_{k,\propto,\beta}^{\gamma}\left(\omega t^{\frac{\propto}{k}}\right)\right](x)$$

$$= x^{\eta+\frac{\beta}{k}}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)k^{\left(1+\frac{\eta}{\sigma-1}\right)}}{[a(1-\sigma)]^{\frac{\beta}{k}}}\,E_{k,\propto,\beta+k\left(1+\frac{\eta}{1-\sigma}\right)}^{\gamma}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\propto}{k}}\right]$$

Corollary 2.4. If we take $p = k = 1, j = 0$ in Theorem 2.1, then we get the following known result derived by Rahman et al. [15]:

$$P_{0+}^{(\eta,\sigma)}\left[t^{\beta-1}\,E_{\propto,\beta}^{\gamma,q}(\omega t^{\propto})\right](x) = x^{\eta+\beta}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\beta}}\,E_{\propto,\beta+\left(1+\frac{\eta}{1-\sigma}\right)}^{\gamma,q}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\propto}\right]$$

Corollary 2.5. If we take $p = k = q = 1, j = 0$ in Theorem 2.1, then we get the following known result given by Nair [12]:

$$P_{0+}^{(\eta,\sigma)}\left[t^{\beta-1}\,E_{\propto,\beta}^{\gamma}(\omega t^{\propto})\right](x) = x^{\eta+\beta}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\beta}}\,E_{\propto,\beta+\left(1+\frac{\eta}{1-\sigma}\right)}^{\gamma}\left[\omega\left(\frac{x}{a(1-\sigma)}\right)^{\propto}\right]$$

In the similar manner, we can find several other new and known results by substituting different values of parameters.

Now, we establish the following theorem by assuming the case that $\lambda > 1$ and using equation (1.18)

Theorem 2.6. Let $\rho,\beta,\eta,z \in \mathbb{C}; \alpha,\beta,\gamma \in \mathbb{C}/kZ^{-}; k,p \in \Re^{+}-\{0\}$, $\{\Re(\propto),\Re(\beta),\Re(\gamma)\} > 0$, $q\epsilon(0,1)\cup N; j\epsilon N_0; \Re\left(1+\frac{\eta}{1-\sigma}\right) > 0, \sigma > 1$. Then the following formula holds true:

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\,{}_{p}^{j}E_{k,\propto,\beta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\right](x)$$

$$= x^{\eta+\frac{\beta}{k}}\frac{\Gamma\left(1-\frac{\eta}{\sigma-1}\right)p^{\left(1-\frac{\eta}{\sigma-1}\right)}}{[-a(\sigma-1)]^{\frac{\beta}{k}}}\,{}_{p}^{j}E_{k,\propto,\beta+k\left(1-\frac{\eta}{\sigma-1}\right)}^{\gamma,q}\left[\omega\left(\frac{x}{-a(\sigma-1)}\right)^{\frac{\propto}{k}}\right]$$

Proof: By applying (1.15), we have

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}\,{}_{p}^{j}E_{k,\propto,\beta,\delta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\right] = x^{\eta}\int_{0}^{\left[\frac{x}{-a(\sigma-1)}\right]} t^{\frac{\beta}{k}-1}\left[1+\frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}}\,{}_{p}^{j}E_{k,\propto,\beta}^{\gamma,q}\left(\omega t^{\frac{\propto}{k}}\right)\,\mathrm{d}t$$

For convenience, we assume that $\mathfrak{J}_2$ in the place of the left hand integral of the above term and also using (1.14), it gives

$$\mathfrak{J}_2 = x^{\eta} \int_0^{\left[\frac{x}{-a(\sigma-1)}\right]} t^{\frac{\beta}{k}-1} \left[1+\frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}} \left\{\sum_{n=0}^{\infty} \frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{\left(\omega t^{\frac{\propto}{k}}\right)^n}{(n+j)!}\right\} \mathrm{d}t$$

After interchanging the order of integration and summation which is permissible under the conditions stated in the theorem, we get

$$\mathfrak{J}_2 = x^{\eta} \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{(\omega)^n}{(n+j)!} \int_0^{\left[\frac{x}{-a(\sigma-1)}\right]} \left[1+\frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}} t^{\frac{\beta}{k}+\frac{\propto n}{k}-1} \mathrm{d}t$$

Now, we simplify the above equation with help of well-known beta function formula and using p-k Gamma function equation (1.11), we have

$$\mathfrak{J}_2 = x^{\eta} \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{(n+j)q,k}}}{p^{\Gamma_k(\alpha n+\beta)}} \frac{(\omega)^n}{(n+j)!} \left[\frac{x}{-a(\sigma-1)}\right]^{\frac{\beta}{k}+\frac{\propto n}{k}} \frac{\Gamma\left(1-\frac{\eta}{\sigma-1}\right)\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1-\frac{\eta}{\sigma-1}\right)}$$

$$= \frac{x^{\eta+\frac{\beta}{k}}}{[-a(\sigma-1)]^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{(n+j)q,k}}}{(n+j)!\ p^{\Gamma_k(\alpha n+\beta)}} \left[\omega\left(\frac{x}{-a(\sigma-1)}\right)^{\frac{\rho}{k}}\right]^n \frac{\Gamma\left(1-\frac{\eta}{\sigma-1}\right)\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1-\frac{\eta}{\sigma-1}\right)}$$

$$= \frac{x^{\eta+\frac{\beta}{k}}\ \Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[-a(\sigma-1)]^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{k\, p^{(\gamma)}{}_{(n+j)_{q,k}}}{(n+j)!\ p^{\frac{\alpha n+\beta}{k}}} \times \frac{\left[\omega\left(\frac{x}{-a(\sigma-1)}\right)^{\frac{\propto}{k}}\right]^n}{\Gamma\left(\frac{\beta}{k}+\frac{\propto n}{k}+1-\frac{\eta}{\sigma-1}\right)}$$

Again, on applying (1.14), we get the desired results of Theorem 2.6.

A number of several other results of Theorem 2.6 can also be obtained in a similar manner by putting some specific values of parameters.

3. CONCLUSION

In this paper, we have discussed two pathway integration formulae associated with j-generalized p-k Mittag-Leffler function in its kernel. Some known special cases are also described as corollaries. Numerous results concerning images of generalized k-Mittag-Leffler function under pathway operator can be obtained by suitably specializing the parameters of j-generalized k-Mittag-Leffler function but for lake of space they are not presented.

CONFLICT OF INTEREST

The author(s) declare that there is no conflict of interests.

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