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COMMON FIXED POINT THEOREM FOR TWO SELFMAPS OF A G-METRIC SPACE

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Abstract. In this paper we prove a common fixed point theorem for two compatible self maps of a G-metric space.

Keywords: G-metric space; compatible mappings; fixed point; associated sequence of a point relative to two self maps; contractive modulus.

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1. Introduction

Sessa[9] introduced the notion of weakly commuting maps as a generalization of commuting maps. Later G.Jungck[4, 5] proposed compatibility as a further generalization of weakly commuting maps.

Among all generalizations[1,2,3,8] of metric spaces, G- metric spaces initiated by Zead Mustafa and Brailey Sims[6, 7] are noteworthy, as several results are established by many researchers on these.

The purpose of this paper is to prove a common fixed point theorem for two compatible self maps of a G -metric space.

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2. PRELIMINARIES

Definition 2.1. Let X be a non empty set and $G : X^3 \to [0,\infty)$ be a function satisfying

$(G1)$ $G(x,y,z) = 0$ if $x = y = z$

$(G2)$ $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$

$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$

$(G4)$ $G(x,y,z) = G(\sigma(x,y,z))$ for all $x,y,z \in X$ where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$ and

$(G5)$ $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then G is called a G-metric on X and the pair (X,G) is called a G- metric space.

Definition 2.2. [7]: Let (X,G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be G-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number N for which $G(x_n,x_n,x_0) < \varepsilon$ for all $n \geq N$.

Definition 2.3. [7]: Let (X,G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be G-Cauchy if for each $\varepsilon > 0$ there exists is a natural number N such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$.

Note that every G-convergent sequence in a G-metric space (X,G) is G-Cauchy.

Definition 2.4. [7]: A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G)

Definition 2.5. Let f and g be two self maps of a G-metric space (X,G) such that $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, then the functions f and g are said to be compatible.

Clearly commuting pairs of selfmaps are compatible but not conversely.

Definition 2.6. A function $\phi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$

Definition 2.7. Let f and g be self maps of a non-empty set X and let $x_0 \in X$, if we can find a sequence $\{x_n\}$ in X satisfying that $fx_n = gx_{n-1}$ for $n \geq 0$ then $\{x_n\}$ is called an associated sequence of x_0 relative to the self maps f and g.

3. Main Result

Theorem 3.1. *Suppose f is continuous selfmap of a G-metric space (X, G), then f has a fixed point in X if and only if there is a contractive modulus ϕ and a continuous selfmap g of X such that*

(i) f *and* g *are compatible*

(ii) $G(gx, gy, gy) \leq \phi(M(x,y))$ *for all* $x, y \in X$
where $M(x,y) = \max\{G(fx, fy, fy), G(gx, fy, fy), G(fx, gy, gy)\}$
and

(iii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that the sequence $\{fx_n\}$ converges to some point t of X. Further gt is the unique common fixed point of f and g

Proof. To prove the necessary part, suppose that f has a fixed point, say 'a', $a \in X$, then $fa = a$. Define $g : X \to X$ by $gx = a$ for all $x \in X$.
Now for any $x \in X$, we have $(gf)x = g(fx) = a$ and $(fg)x = fgx = fa = a$,giving that $fg = gf$, so that f and g are compatible. Now let ϕ be a contractive modulus, then $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ and for any $x, y \in X$
$G(gx, gy, gy) = G(a, a, a) = 0 \leq \phi(G(fx, fy, fy)$.
Further an associated sequence of $x_0 = a$ relative to the selfmaps f and g is given by $x_n = a$ for $n = 0, 1, 2, 3 \cdots$, and since the sequence $\{fx_n\}$ is a constant sequence converging to a, which is a point in X. Thus the condition (i) (ii) and (iii) of the theorem are satisfied.
Conversely, suppose that there is contractive modulus ϕ and a continuous selfmap g on X satisfying the conditions (i),(ii) and (iii) of the theorem.
From the condition (iii) of the theorem, there is an associated sequence $\{x_n\}$ of x_0 such that $fx_n = gx_{n-1}$ for $n = 1, 2, 3 \cdots$ it follows that $gx_n = fx_{n+1} \to t$ as $n \to \infty$.
From the condition (i) of the theorem and since $fx_n \to t, gx_n \to t$ as $n \to \infty$,
we have $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$
Using the continuity of G, f and g, we get $G(ft, gt, gt) = 0$ gives $ft = gt$.
To show that $fgt = gft$, take $z_n = t$ for $n = 1, 2, 3 \cdots$ so that $fz_n \to ft$ and $gz_n \to gt$ as $n \to \infty$.

Since $ft = gt$, f and g are compatible , we get $\lim_{n\to\infty} G(fgz_n, gfz_n, gfz_n) = 0$.
Using the continuity of G, f and g , we obtain $G(fgt, gft, gft) = o$ and hence $fgt = gft$
Consequently

$$ffft = fgt = gft = ggt \tag{1}$$

If possible suppose that $gt \neq ggt$, then $G(gt, ggt, ggt) > 0$ and hence

$$\phi(G(gt, ggt, ggt)) < G(gt, ggt, ggt) \tag{2}$$

But from (ii)of the theorem and (1) we get

$$G(gt, ggt, ggt) \leq \phi(M(t, gt))$$

where

$$\begin{aligned} M(t, gt) &= \max\{G(ft, fgt, fgt), G(gt, fgt, fgt), G(ft, ggt, ggt)\} \\ &= \max\{G(gt, ggt, ggt), G(gt, ggt, ggt), G(gt, ggt, ggt)\} \\ &= G(gt, ggt, ggt) \end{aligned}$$

That is $G(gt, ggt, ggt) \leq \phi((G(gt, ggt, ggt))$
which contradicts (2), hence $gt = ggt$.
Showing thatgt is a common fixed point of f and g.
Uniqueness: Suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$.
if possible suppose that $u \neq v$,then $G(u, v, v) \neq 0$ so that

$$\phi(G(u, v, v)) < G(u, v, v) \tag{3}$$

from the condition(ii) of the theorem, we have

$$\begin{aligned} G(u, v, v) &= G(gu, gv, gv) \\ &\leq \phi(M(u, v)) \\ &= \phi(\max\{G(fu, fv, fv), G(gu, fv, fv), G(fu, gv, gv)\}) \\ &= \phi(\max\{G(u, v, v), G(u, v, v), G(u, v, v)\}) \\ &= \phi(G(u, v, v)) \end{aligned}$$

implies $G(u,v,v) \leq \phi(G(u,v,v))$ which contradicts (3), hence $u = v$, proving the theorem completely. □

Corollary 3.2. *Suppose f and g are selfmaps of a G-metric space (X,G), if there is a contractive modulus ϕ and a positive integer k such that*

(i) $fg = gf$

(ii) $G(g^k x, g^k y, g^k y) \leq \phi(M(x,y))$ *for all* $x,y \in X$

where $M(x,y) = \max\{G(fx,fy,fy), G(gx,fy,fy), G(fx,gy,gy)\}$

and

(iii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g^k such that the sequence $\{fx_n\}$ converges to some point t of X. Then f and g have unique common fixed point in X

Proof. From the condition (i) of the corollary, we get $fg^k = g^k f$. Thus f and g^k are commuting and hence satisfying the hypothesis of Theorem 3.1, and therefore f and g^k have a unique common fixed point say b, then $g^k b = b = fb$.

Now $g^k gb = g^{k+1} b = gg^k b = gb$ and $fgb = gfb = gb$

this shows that gb is a common fixed point of f and g^k. The uniqueness of b implies that $gb = b$ since $fb = b$, b is a common fixed point of f and g.

To prove that f and g have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u,v \in X$, so that $g^k u = u$ and $g^k v = v$.

This shows that u,v are common fixed points of f and g^k. The uniqueness of common fixed point of f and g^k implies $u = v$ □

Corollary 3.3. *Suppose f is continuous selfmap of a G-metric space (X,G), then f has a fixed point in X if and only if there is a contractive modulus ϕ and a selfmap g of X such that*

(i) $fg = gf$

(ii) $G(gx,gy,gy) \leq \phi(M(x,y))$ *for all* $x,y \in X$

where $M(x,y) = \max\{G(fx,fy,fy), G(gx,fy,fy), G(fx,gy,gy)\}$

and

(iii) there is a point $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* x_0 *relative to the selfmaps* f *and* g *such that the sequence* $\{fx_n\}$ *converges to some point* t *of* X. *Further* gt *is the unique common fixed point of* f *and* g

Proof. From the fact that the commutativity implies the compatibility of a pair of selfmaps, proof of the corollary follows from the Theorem 3.1 □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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