Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7728-7753
https://doi.org/10.28919/jmcs/6575
ISSN: 1927-5307

ECCENTRIC DOMINATION NUMBER OF SOME CYCLE RELATED GRAPHS

S. K. VAIDYA[1,*], D. M. VYAS[2]

[1]Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India
[2]Department of Applied Science and Humanities, V.V.P. Engineering College, Kalawad Road, Rajkot - 360005, Gujarat, India

Abstract. In a graph G, a vertex u is said to be an eccentric vertex of a vertex v if the distance between u and v is equals to the eccentricity of vertex v. A dominating set D of a graph $G=(V,E)$ is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of the minimal eccentric dominating sets of graph G is said to be eccentric domination number, denoted by $\gamma_{ed}(G)$. The eccentric domination numbers of some cycle related graphs have been investigated.

Keywords: dominating set; eccentric dominating set; eccentric domination number.

2010 AMS Subject Classification: 05C38, 05C69, 05C76.

1. Introduction

The domination in graphs is one of the most rapidly growing fields within and out side of graph theory. Many researchers have been attracted to work on it due to its diversified applications in the various fields like linear algebra and optimization, design and analysis of communication networks as well as surveillance.

*Corresponding author
E-mail address: samirkvaidya@yahoo.co.in
Received July 29, 2021

There are many domination models available in the literature. Total domination [2], independent domination [3], eccentric domination [5], equitable domination [6], restrained domination [7], global domination [11], steiner domination [12] are among worth to mention.

Throughout this work, a graph $G = (V,E)$, we mean a finite, simple and connected graph with vertex set $V = \{v_1, v_2, v_3, \cdots, v_n\}$ and edge set E. For any graph theoretic terminology and notation we refer to West [13] while the terms related to domination are used in the sense of Haynes *et al.* [4].

2. Preliminaries

Definition 2.1. A set $D \subseteq V$ of vertices in a graph $G = (V,E)$ is said to be a **dominating set** if every vertex in $V - D$ is adjacent to at least one vertex in D.

Definition 2.2. A dominating set D is said to be a **minimal dominating set** if no proper subset $D' \subset D$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. The minimum cardinality of a set from $MDS(G)$ is called **domination number** of graph G and is denoted by $\gamma(G)$.

Definition 2.3. Let G be a connected graph and v be a vertex of G. The **eccentricity of** v is denoted by $e(v)$ is defined by $e(v) = max\{d(u,v) : u \in V\}$. The **radius of graph** G is defined as $rad(G) = min\{e(v) : v \in V\}$ while the **diameter of graph** G is defined as $diam(G) = max\{e(v) : v \in V\}$.

In a graph G, a vertex u is said to be an eccentric vertex of a vertex v if $d(u,v) = e(v) =$ eccentricity of vertex v. The eccentric set of a vertex v is denoted by $E(v)$ and is defined as $E(v) = \{u \in V(G) : d(u,v) = e(v)\}$.

Definition 2.4. A set $D \subseteq V(G)$ is an **eccentric dominating set of** G if D is a dominating set of G and for every vertex $v \in V - D$, there exists at least one eccentric vertex of v in D.

An eccentric dominating set D of graph G is a minimal eccentric dominating set if no proper subset $D' \subset D$ is an eccentric dominating set of graph G.

The cardinality of a minimal eccentric dominating set of a graph G is called eccentric domination number of G which is denoted as $\gamma_{ed}(G)$.

The concept of eccentric domination was introduced by Janakiraman *et al* [5].

3. Some Definitions and Existing Results

Definition 3.1. [8] The m**-shadow graph** $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbours of the corresponding g vertex v in G_j, $1 \leq i, j \leq m$.

Definition 3.2. [10] The **extended** m**-shadow graph** $D_m^*(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbours of the corresponding vertex v and with v in G_j, $1 \leq i, j \leq m$.

Definition 3.3. [8] The $m-$**splitting graph** $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G.

Definition 3.4. [9] The **extended** $m-$**splitting graph** $Spl_m^*(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G and also adjacent to v.

Definition 3.5. [1]

The **square of graph** G**, denoted by** G^2, is defined to be the graph with the same vertex set as G and in which two vertices u and v are joined by an edge if and only if in G we have $1 \leq d(u, v) \leq 2$.

Janakiraman *et al* [5] have proved the following results.

Theorem 3.6. [5] For any complete graph K_n, $\gamma_{ed}(K_n) = 1$.

Theorem 3.7. [5] For any cycle C_n,

$$\gamma_{ed}(C_n) = \begin{cases} \dfrac{n}{2} & ; n \text{ is even} \\ \dfrac{n}{3} & ; n = 3m \text{ and } n \text{ is odd} \\ \left\lceil \dfrac{n}{3} \right\rceil & ; n = 3m+1 \text{ and } n \text{ is odd} \\ \left\lceil \dfrac{n}{3} \right\rceil + 1 & ; n = 3m+2 \text{ and } n \text{ is odd} \end{cases}$$

In the following Theorem 4.1, we have improved the results proved in Theorem 3.7.

4. Main Results

Theorem 4.1. For any cycle C_n, $\gamma_{ed}(C_n) = \begin{cases} \dfrac{n}{2} & ;n \text{ is even} \\ 2\left\lceil \dfrac{n+3}{6} \right\rceil - 1 & ;n \text{ is odd} \end{cases}$

Proof. Let $V = V(C_n) = \{v_1, v_2, \cdots, v_n\}$ is the set of all vertices of cycle $C_n, \forall n \geq 3$ and let D is a minimal eccentric dominating set of C_n.

To dominate all vertices of C_n, we have to take one middle vertex from every three consecutive vertices of C_n in D.

Case-1: n is an even number.

Let $n = 2t$. Here, each vertex of cycle C_n has exactly one eccentric vertex and for each $k, 1 \leq k \leq t$ the eccentric vertex of the vertex v_k is v_{k+t} and vice versa.

So, we have to choose, either v_k or v_{k+t}, for $1 \leq k \leq t$, to construct the set D.

Case-(i): $n = 4$.

Then, $D = \{v_1, v_2\}$ is a minimal eccentric dominating set of C_4.

So, $\gamma_{ed}(C_4) = 2 = \dfrac{4}{2} = \dfrac{n}{2}$

Case-(ii): $n \geq 6$.

Here, we have to take one vertex from every two vertices, from v_1 to v_{t+1} and one vertex from every two vertices, from v_{t+2} to v_n.

Case-(a): t is odd.

Then, $t+1$ is an even number and $n-1$ is an odd number.

Then, $D = \{v_1, v_3, v_5, \cdots, v_t\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \cdots, v_{n-1}\}$

$= \{v_1, v_3, v_5, \cdots, v_t, v_{t+2}, v_{t+4}, \cdots, v_{n-1}\}$

$= \{v_1, v_3, v_5, \cdots, v_{n-1}\}$ is a minimal eccentric dominating set of C_n.

Thus, $\gamma_{ed}(C_n) = |D| = \dfrac{n}{2}$.

Case-(b): t is even.

Let, $t = 2u$, then $n = 2t = 2(2u) = 4u$.

Then, $D = \{v_1, v_3, v_5, \cdots, v_{t-1}\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \cdots, v_n\}$ is a minimal eccentric dominating set of C_n.

Thus, $\gamma_{ed}(C_n) = |D| = \left\lceil \frac{t-1}{2} \right\rceil + \left\lceil \frac{n-(t+1)}{2} \right\rceil = \left\lceil \frac{2u-1}{2} \right\rceil + \left\lceil \frac{4u-2u-1}{2} \right\rceil$

$$= \left\lceil \frac{2u-1}{2} \right\rceil + \left\lceil \frac{2u-1}{2} \right\rceil = 2\left\lceil \frac{2u-1}{2} \right\rceil = 2\left\lceil u - \frac{1}{2} \right\rceil$$

$= 2(u)$ (Since u is an integer)

$= t = \frac{n}{2}.$

Case-2: n **is an odd number.**

Let $n = 2t+1$.

Here, each vertex has exactly two eccentric vertices and for each $k, 1 \leq k \leq t$ the eccentric vertices of the vertex v_k are v_{k+t} and v_{k+t+1} and vice versa.

The eccentric vertices of the vertex v_{t+1} are v_n and v_1 and vice versa.

Case-(i): $n = 3$.

Then, $D = \{v_1\}$ is a minimal eccentric dominating set of C_3.

So, $\gamma_{ed}(C_3) = 1 = 2(1) - 1 = 2\left\lceil \frac{6}{6} \right\rceil - 1 = 2\left\lceil \frac{3+3}{6} \right\rceil - 1 = 2\left\lceil \frac{n+3}{6} \right\rceil - 1.$

Case-(ii): $n = 5$.

Then, $D = \{v_1, v_3, v_4\}$ is a minimal eccentric dominating set of C_5.

So, $\gamma_{ed}(C_5) = 3 = 2(2) - 1 = 2\left\lceil \frac{8}{6} \right\rceil - 1 = 2\left\lceil \frac{5+3}{6} \right\rceil - 1 = 2\left\lceil \frac{n+3}{6} \right\rceil - 1.$

Case-(iii): $n \geq 7$.

Here, we have to choose, one vertex from every three vertices, from v_1 to v_{t+2} and one vertex from every three vertices, from v_{t+3} to v_n, to construct the set D.

Then, $D = \{v_1, v_4, v_7, \cdots, v_s\} \cup \{v_{t+3}, v_{t+6}, v_{t+9}, \cdots, v_m\}$ is a minimal eccentric dominating set of C_n, where $t \leq s \leq t+2$ and $n-2 \leq m \leq n$.

Thus, $\gamma_{ed}(C_n) = |D| = \left\lceil \frac{t+2}{3} \right\rceil + \left\lceil \frac{n-(t+2)}{3} \right\rceil = \left\lceil \frac{\frac{n-1}{2}+2}{3} \right\rceil + \left\lceil \frac{n - \frac{n-1}{2} - 2}{3} \right\rceil$

$$= \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil = \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3-6}{6} \right\rceil$$

$$= \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3}{6} - 1 \right\rceil = \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3}{6} \right\rceil - 1$$

$$= 2\left\lceil \frac{n+3}{6} \right\rceil - 1.$$

Hence, $\gamma_{ed}(C_n) = \begin{cases} \dfrac{n}{2} & ;n \text{ is even} \\ 2\left\lceil \dfrac{n+3}{6} \right\rceil - 1 & ;n \text{ is odd} \end{cases}$ □

Illustration 4.2. Minimal eccentric dominating set of cycle $C_{11} = \{v_1, v_4, v_7, v_8, v_{11}\}$, is shown in Figure 1.

Where, $\gamma_{ed}(C_{11}) = 5 = 2(3) - 1 = 2\left\lceil \frac{14}{6} \right\rceil - 1 = 2\left\lceil \frac{11+3}{6} \right\rceil - 1$

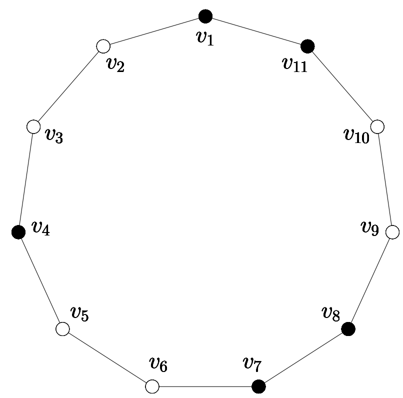

FIGURE 1. Minimal eccentric dominating set of cycle C_{11}

Theorem 4.3. $\gamma_{ed}[D^*_m(C_n)] = \begin{cases} 1 & ;n = 3 \\ n & ;n \text{ is even, } n > 3 \\ \left\lceil \dfrac{n}{2} \right\rceil & ;n \text{ is odd, } n > 3 \end{cases}$

Proof. Let $C_n^1, C_n^2, C_n^3, \cdots, C_n^m$ be the m copies of cycle C_n in the extended $m-$shadow graph of C_n.

Let $\{v_1^i, v_2^i, v_3^i, \cdots, v_n^i,\}$ is the vertex set of cycle $C_n^i, 1 \leq i \leq m$ and the set V is the set of all vertices of all copies of cycle C_n.

Let the set D is a minimal eccentric dominating set of the extended $m-$shadow graph of C_n, i.e. $D^*_m(C_n)$.

Now, in the graph $D^*_m(C_n)$, for $1 \leq i \leq m$ the vertices v_j^i are adjacent to the vertices v_{j-1}^k and v_{j+1}^k, for all $1 \leq k \leq m$ and $2 \leq j \leq n-1$ and v_j^i are also adjacent to $v_j^l, 1 \leq i \neq l \leq m$.

For $j=1$, the vertices v_1^i are adjacent to the vertices v_2^k and $v_n^k, 1 \leq k \leq m$ and v_1^i are also adjacent to the vertices $v_1^l, 1 \leq i \neq l \leq m$.

And similarly for $j=n$, the vertices v_n^i are adjacent to the vertices v_{n-1}^k and $v_1^k, 1 \leq k \leq m$ and v_n^i are also adjacent to the vertices $v_n^l, 1 \leq i \neq l \leq m$.

If n is an even number and $n=2t$ then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and vice versa, for $1 \leq i,j \leq m$ and $1 \leq k \leq t$ and If n is an odd number and $n=2t+1$ then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and v_{k+t+1}^j and vice versa, for $1 \leq i,j \leq m$ and $1 \leq k \leq t$.

Looking to the adjacency of vertices of the graph $D_m^*(C_n)$, it is possible to dominate all the vertices of the graph $D_m^*(C_n)$, just by considering all the vertices of any one copy of cycle C_n and we can also choose the eccentric vertices from any one copy of cycle C_n.

So, Without lose of generality, let we take the vertices from C_n^1 to construct the set D.

Case-1: $n=3$

Then, $D=\{v_1^1\}$ is a minimal eccentric dominating set of the graph $D_m^*(C_3)$, as the vertex v_1^1 dominates the all vertices of $V-D$ and distance between any two vertices in $D_m^*(C_3)$ is 1, so v_1^1 is also eccentric vertex of all vertices of $V-D$.

Thus, $\gamma_{ed}[D_m^*(C_3)]=1$

Case-2: n **is an even number,** $n>3$**.**

Let $n=2t$ then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and vice versa, for $1 \leq i,j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, we have to choose one vertex from v_k^i and one vertex from v_{k+t}^j, for $1 \leq i,j \leq m$, to construct the set D. Without lose of generality, we may choose the vertices from the vertex set of cycle C_n^1. Then, $D=\{v_1^1, v_2^1, v_3^1, \cdots, v_n^1\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[D_m^*(C_n)]=|D|=n$

Case-3: n **is an odd number,** $n>3$**.**

Let $n=2t+1$ then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and v_{k+t+1}^j and vice versa, for $1 \leq i,j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices v_{t+1}^i are v_n^j and v_1^j and vice versa, for $1 \leq i,j \leq m$.

So, for each $k, 1 \leq k \leq t$, we have to choose one vertex from v_k^i and one vertex from v_{k+t}^j or v_{k+t+1}^j, for $1 \leq i, j \leq m$, to construct the set D. Without lose of generality, we may choose the vertices from the vertex set of cycle C_n^1.

To construct the set D, we have to choose one vertex for every two vertices of cycle C_n^1. Then, $D = \{v_1^1, v_3^1, v_5^1, \cdots, v_n^1\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[D_m^*(C_n)] = |D| = \dfrac{n+1}{2} = \left(\dfrac{n}{2} + \dfrac{1}{2}\right) = \left\lceil \dfrac{n}{2} \right\rceil$, since $\left(\dfrac{n}{2} + \dfrac{1}{2}\right)$ is an integer.

Hence, $\gamma_{ed}[D_m^*(C_n)] = \begin{cases} 1 & ; n = 3 \\ n & ; n \text{ is even}, \ n > 3 \\ \left\lceil \dfrac{n}{2} \right\rceil & ; n \text{ is odd}, \ n > 3 \end{cases}$

$\square$

Illustration 4.4. Minimal eccentric dominating set of extended 3-Shadow graph of $C_5 = \{v_1, v_3, v_5\}$, is shown in Figure 2.

Where, $\gamma_{ed}[D_3^*(C_5)] = 3 = \left\lceil \dfrac{5}{2} \right\rceil$

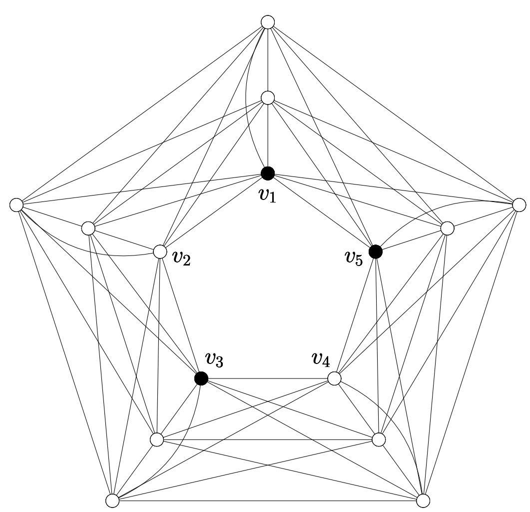

FIGURE 2. Minimal eccentric dominating set of $D_3^*(C_5)$

Theorem 4.5. $\gamma_{ed}[Spl_m^*(C_n)] = \begin{cases} n & ; n \text{ is even} \\ \left\lceil \dfrac{n}{2} \right\rceil & ; n \text{ is odd} \end{cases}$

Proof. Let $v_1, v_2, v_3, \cdots, v_n$ is the vertex set of cycle C_n.

Let $v_1^i, v_2^i, v_3^i, \cdots, v_n^i$ are the m copies of the vertex set of cycle $C_n, 1 \leq i \leq m$ in the extended $m-$splitting graph of C_n.

Let the set D is a minimal eccentric dominating set of the extended $m-$splitting graph of C_n, i.e. $Spl_m^*(C_n)$.

Now, in the graph $Spl_m^*(C_n)$, for $1 \leq i \leq m$ the vertices v_j and v_j^i are adjacent to the vertices v_{j-1} and v_{j+1} of cycle C_n, and v_j^i are also adjacent to v_j, for all $2 \leq j \leq n-1$.

For $j = 1$, the vertices v_1 and v_1^i are adjacent to the vertices v_2 and v_n and v_1^i are also adjacent to the vertex $v_1, 1 \leq i \leq m$.

Similarly for $j = n$, the vertices v_n and v_n^i are adjacent to the vertices v_{n-1} and v_1 and v_n^i are also adjacent to the vertex $v_n, 1 \leq i \leq m$.

Case-1: n **is an even number.**

Case-(i) $n = 4$

For the graph $Spl_m^*(C_4)$, the eccentric vertices of the vertices v_k and v_k^i are v_{k+2} and v_{k+2}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq 2$ and all the vertices v_k^i are the eccentric vertices to one another, for $1 \leq k \leq 4$ and $1 \leq i \leq m$.

So, for each $k, 1 \leq k \leq 2$, we have to choose one vertex from v_k or v_k^i and one vertex from v_{k+2} or v_{k+2}^j, for $1 \leq i, j \leq m$, to construct the set D.

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle C_4 to construct the set D. Then, $D = \{v_1, v_2, v_3, v_4\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[Spl_m^*(C_4)] = |D| = 4 = n$.

Case-(ii) $n > 4$

Let $n = 2t$ then the eccentric vertices of the vertices v_k and v_k^i are v_{k+t} and v_{k+t}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, we have to choose one vertex from v_k or v_k^i and one vertex from v_{k+t} or v_{k+t}^j, for $1 \leq i, j \leq m$, to construct the set D.

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle C_n to construct the set D. Then, $D = \{v_1, v_2, v_3, \cdots, v_n\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[Spl_m^*(C_n)] = |D| = n$

Case-2: n **is an odd number.**

Case-(i) $n = 3$

For the graph $Spl_m^*(C_3)$, all the vertices other than v_k are the eccentric vertices of the vertex v_k, for $1 \leq k \leq 3$ and all the vertices v_k^i are the eccentric vertices to one another, for $1 \leq k \leq 3$ and $1 \leq i \leq m$.

Moreover, for any k, $1 \leq k \leq 3$, the vertex v_k dominates the all vertices of the graph $Spl_m^*(C_3)$.

So, to construct the set D, we have to choose one vertex from v_k and one vertex from v_l^i, for $1 \leq k,l \leq 3$ and $1 \leq i \leq m$.

Then, without lose of generality, we may take, $D = \{v_1, v_1^1\}$, which is a minimal eccentric dominating set of $Spl_m^*(C_3)$.

Thus, $\gamma_{ed}[Spl_m^*(C_3)] = |D| = 2 = \left\lceil \frac{3}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

Case-(ii) $n = 5$

For the graph $Spl_m^*(C_5)$, the eccentric vertices of the vertices v_k and v_k^i are v_{k+2}, v_{k+2}^j, v_{k+3} and v_{k+3}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq 2$ and the eccentric vertices of the vertices v_3 and v_3^i are v_5, v_5^j, v_1 and v_1^j and vice versa, for $1 \leq i, j \leq m$. Moreover, all the vertices v_k^i are the eccentric vertices to one another, for $1 \leq k \leq 5$ and $1 \leq i \leq m$.

So, for each $k, 1 \leq k \leq 2$, we can choose one vertex from v_k or v_k^i and one vertex from v_{k+2} or v_{k+2}^j or v_{k+3} or v_{k+3}^j, for $1 \leq i, j \leq m$ and we can choose one vertex from v_3 or v_3^i and one vertex from v_5 or v_5^j or v_1 or v_1^j, for $1 \leq i, j \leq m$, to construct the set D.

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle C_5 to construct the set D. Moreover, to construct the set D, we have to choose one vertex for every two vertices of cycle C_5. Then, $D = \{v_1, v_3, v_5\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[Spl_m^*(C_5)] = |D| = 3 = \left\lceil \frac{5}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

Case-(iii) $n > 5$.

Let $n = 2t+1$ then for the graph $Spl_m^*(C_n)$, the eccentric vertices of the vertices v_k and v_k^i are v_{k+t}, v_{k+t}^j, v_{k+t+1} and v_{k+t+1}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices v_{t+1} and v_{t+1}^i are v_n, v_n^j, v_1 and v_1^j and vice versa, for $1 \leq i, j \leq m$.

So, for each $k, 1 \leq k \leq t$, we can choose one vertex from v_k or v_k^i and one vertex from v_{k+t} or v_{k+t}^j or v_{k+t+1} or v_{k+t+1}^j, for $1 \leq i, j \leq m$ and we can choose one vertex from v_{t+1} or v_{t+1}^i and one vertex from v_n or v_n^j or v_1 or v_1^j, for $1 \leq i, j \leq m$, to construct the set D.

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle C_n to construct the set D. Moreover, to construct the set D, we have to choose one vertex for every two vertices of cycle C_n. Then, $D = \{v_1, v_3, v_5, \cdots, v_n\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[Spl_m^*(C_n)] = |D| = \dfrac{n+1}{2} = \left(\dfrac{n}{2} + \dfrac{1}{2}\right) = \left\lceil \dfrac{n}{2} \right\rceil$, since $\left(\dfrac{n}{2} + \dfrac{1}{2}\right)$ is an integer

.

Hence, $= \gamma_{ed}[Spl_m^*(C_n)] = \begin{cases} n & ; n \textit{ is even} \\ \left\lceil \dfrac{n}{2} \right\rceil & ; n \textit{ is odd} \end{cases}$

□

Illustration 4.6. Minimal eccentric dominating set of extended 2-Splitting graph of $C_6 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, is shown in Figure 3.

Where, $\gamma_{ed}[Spl_2^*(C_6)] = 6$

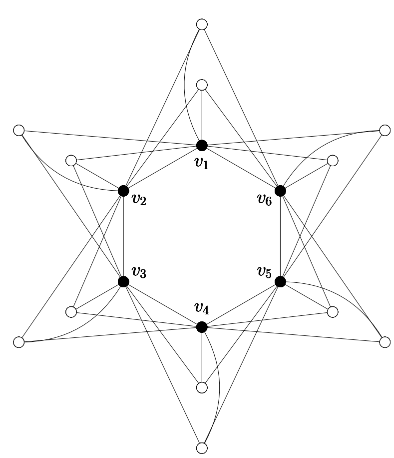

FIGURE 3. Minimal eccentric dominating set of Extended 2-Splitting graph of C_6

Theorem 4.7. For any square of a cycle C_n,

$$\gamma_{ed}(C_n^2) = \begin{cases} \dfrac{n}{2} & ; n \equiv 2(mod 4) \\ 2\left\lceil \dfrac{n+5}{10} \right\rceil - 1 & ; n \equiv 1(mod 4) \\ \left\lceil \dfrac{n+1}{6} \right\rceil + \left\lceil \dfrac{n-3}{6} \right\rceil & ; n \equiv 3(mod 4) \\ \left\lceil \dfrac{n+2}{8} \right\rceil + \left\lceil \dfrac{n-4}{8} \right\rceil & ; n \equiv 0(mod 4) \end{cases}$$

Proof. Let $V = V(C_n^2) = \{v_1, v_2, \cdots, v_n\}$ is the set of all vertices of graph $C_n^2, \forall n \geq 3$ and let D is a minimal eccentric dominating set of C_n^2.

To dominate all vertices of C_n^2, we have to take one vertex from every five consecutive vertices of C_n^2 in D.

Case-1: $n \equiv 2(mod 4)$

Let $n = 4u+2$ and $t = 2u+1$ then $n = 2t$, where n is an even number and t is an odd number. Here, each vertex has exactly one eccentric vertex and for each $k, 1 \leq k \leq t$ the eccentric vertex of the vertex v_k is v_{k+t} and vice versa.

So, we have to choose, either v_k or v_{k+t}, for $1 \leq k \leq t$, to construct the set D. Moreover, we have to take one vertex from every two vertices, from v_1 to v_{t+1} and one vertex from every two vertices, from v_{t+2} to v_n.

Then, $D = \{v_1, v_3, v_5, \cdots, v_t\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \cdots, v_{n-1}\} = \{v_1, v_3, v_5, \cdots, v_{n-1}\}$ is a minimal eccentric dominating set of C_n^2.

Thus, $\gamma_{ed}(C_n^2) = |D| = \dfrac{n}{2}$.

Case-2: $n \equiv 1(mod 4)$

Case-(i): $n = 5$

Then, the graph C_5^2 is same as the complete graph K_5.

So, $\gamma_{ed}(C_5^2) = \gamma_{ed}(K_5) = 1$ (Using Theorem 3.6)

$$= 2(1) - 1 = 2\left\lceil \frac{10}{10} \right\rceil - 1 = 2\left\lceil \frac{5+5}{10} \right\rceil - 1 = 2\left\lceil \frac{n+5}{10} \right\rceil - 1.$$

Case-(ii): $n > 5$

Let $n = 4u+1$ and $t = 2u$ then $n = 2t+1$, where n is an odd number and t is an even number. Here, each vertex has exactly four eccentric vertices and for each $k, 1 \leq k \leq t-1$ the eccentric vertices of the vertex v_k are $v_{k+t-1}, v_{k+t}, v_{k+t+1}$ and v_{k+t+2} and vice versa.

The eccentric vertices of the vertex v_t are v_{n-2}, v_{n-1}, v_n and v_1 and vice versa.

The eccentric vertices of the vertex v_{t+1} are v_{n-1}, v_n, v_1 and v_2 and vice versa.

The eccentric vertices of the vertex v_{t+2} are v_n, v_1, v_2 and v_3 and vice versa.

So, we have to choose, one vertex from every five vertices, from v_1 to v_{t+3} and one vertex from every five vertices, from v_{t+4} to v_n, to construct the set D.

Then, $D = \{v_1, v_6, v_{11}, \cdots, v_s\} \cup \{v_{t+4}, v_{t+9}, v_{t+14}, \cdots, v_m\}$ is a minimal eccentric dominating set of C_n^2, where $s \leq t+3$ and $m \leq n$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \left\lceil \frac{t+3}{5} \right\rceil + \left\lceil \frac{n-(t+3)}{5} \right\rceil = \left\lceil \frac{\frac{n-1}{2}+3}{5} \right\rceil + \left\lceil \frac{n-\frac{n-1}{2}-3}{5} \right\rceil$

$$= \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n-5}{10} \right\rceil = \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{(n+5)-10}{10} \right\rceil$$

$$= \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n+5}{10} - 1 \right\rceil = \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n+5}{10} \right\rceil - 1$$

$$= 2\left\lceil \frac{n+5}{10} \right\rceil - 1.$$

Case-3: $n \equiv 3(mod 4)$

Case-(i): $n = 3$

Then, the graph C_3^2 is same as the graph C_3.

So, $\gamma_{ed}(C_3^2) = \gamma_{ed}(C_3) = 1$ (Using Theorem 4.1)

$$= 1 + 0 = \left\lceil \frac{4}{6} \right\rceil + \left\lceil \frac{0}{6} \right\rceil = \left\lceil \frac{3+1}{6} \right\rceil + \left\lceil \frac{3-3}{6} \right\rceil = \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil.$$

Case-(ii): $n > 3$

Let $n = 4u+3$ and $t = 2u+1$ then $n = 2t+1$, where n and t are both odd numbers.

Here, each vertex has exactly two eccentric vertices and for each $k, 1 \leq k \leq t$ the eccentric vertices of the vertex v_k are v_{k+t} and v_{k+t+1} and vice versa.

The eccentric vertices of the vertex v_{t+1} are v_n and v_1 and vice versa.

So, we have to choose, one vertex from every three vertices, from v_1 to v_{t+1} and one vertex from every three vertices, from v_{t+3} to v_n, to construct the set D.

Then, $D = \{v_1, v_4, v_7, \cdots, v_s\} \cup \{v_{t+3}, v_{t+6}, v_{t+9}, \cdots, v_m\}$ is a minimal eccentric dominating set of C_n^2, where $s \leq t+1$ and $m \leq n$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \left\lceil \frac{t+1}{3} \right\rceil + \left\lceil \frac{n-(t+2)}{3} \right\rceil = \left\lceil \frac{\frac{n-1}{2}+1}{3} \right\rceil + \left\lceil \frac{n-\frac{n-1}{2}-2}{3} \right\rceil$

$$= \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil.$$

Case-4: $n \equiv 0(mod4)$

Case-(i): $n = 4$

Then, the graph C_4^2 is same as the complete graph K_4.

So, $\gamma_{ed}(C_4^2) = \gamma_{ed}(K_4) = 1$ (Using Theorem 3.6)

$$= 1+0 = \left\lceil \frac{6}{8} \right\rceil + \left\lceil \frac{0}{8} \right\rceil = \left\lceil \frac{4+2}{8} \right\rceil + \left\lceil \frac{4-4}{8} \right\rceil = \left\lceil \frac{n+2}{8} \right\rceil + \left\lceil \frac{n-4}{8} \right\rceil.$$

Case-(ii): $n > 4$

Let $n = 4u$ and $t = 2u$ then $n = 2t$, where n and t are both even numbers.

Here, each vertex has exactly three eccentric vertices and for each $k, 1 \leq k \leq t-1$ the eccentric vertices of the vertex v_k are v_{k+t-1}, v_{k+t} and v_{k+t+1} and vice versa.

The eccentric vertices of the vertex v_t are v_{n-1}, v_n and v_1 and vice versa.

The eccentric vertices of the vertex v_{t+1} are v_n, v_1 and v_2 and vice versa.

So, we have to choose, one vertex from every four vertices, from v_1 to v_{t+1} and one vertex from every four vertices, from v_{t+3} to v_n, to construct the set D.

Then, $D = \{v_1, v_5, v_9, \cdots, v_s\} \cup \{v_{t+3}, v_{t+7}, v_{t+11}, \cdots, v_m\}$ is a minimal eccentric dominating set of C_n^2, where $s \leq t+1$ and $m \leq n$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \left\lceil \frac{t+1}{4} \right\rceil + \left\lceil \frac{n-(t+2)}{4} \right\rceil = \left\lceil \frac{\frac{n}{2}+1}{4} \right\rceil + \left\lceil \frac{n-\frac{n}{2}-2}{4} \right\rceil =$
$\left\lceil \frac{n+2}{8} \right\rceil + \left\lceil \frac{n-4}{8} \right\rceil.$

$$\text{Hence, } \gamma_{ed}(C_n^2) = \begin{cases} \dfrac{n}{2} & ; n \equiv 2(mod 4) \\ 2\left\lceil \dfrac{n+5}{10} \right\rceil - 1 & ; n \equiv 1(mod 4) \\ \left\lceil \dfrac{n+1}{6} \right\rceil + \left\lceil \dfrac{n-3}{6} \right\rceil & ; n \equiv 3(mod 4) \\ \left\lceil \dfrac{n+2}{8} \right\rceil + \left\lceil \dfrac{n-4}{8} \right\rceil & ; n \equiv 0(mod 4) \end{cases}$$

□

Illustration 4.8. Minimal eccentric dominating set of $C_9^2 = \{v_1, v_6, v_8\}$, is shown in Figure 4.

$$\text{Where, } \gamma_{ed}(C_9^2) = 3 = 2(2) - 1 = 2\left\lceil \frac{14}{10} \right\rceil - 1 = 2\left\lceil \frac{9+5}{10} \right\rceil - 1$$

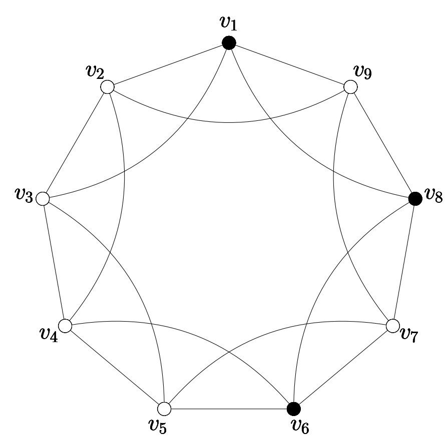

FIGURE 4. Minimal eccentric dominating set of cycle C_9^2

$$\textbf{Theorem 4.9. } \gamma_{ed}[D_m(C_n)] = \begin{cases} n & ; n = 3 \text{ or } n \text{ is an even number with } n \geq 6 \\ 2 & ; n = 4 \\ 3 & ; n = 5 \\ \left\lceil \dfrac{2n}{3} \right\rceil & ; n \text{ is an odd number with } n \geq 7 \end{cases}$$

Proof. Let $C_n^1, C_n^2, C_n^3, \cdots, C_n^m$ are the m copies of cycle C_n in the $m-$shadow graph of C_n. Let $\{v_1^i, v_2^i, v_3^i, \cdots, v_n^i,\}$ is the vertex set of cycle $C_n^i, 1 \leq i \leq m$ and the set V is the set of all vertices of all copies of cycle C_n.

Let the set D is a minimal eccentric dominating set of the $m-$shadow graph of C_n, i.e. $D_m(C_n)$. Now, in the graph $D_m(C_n)$, the vertices v_k^i are adjacent to the vertices v_{k-1}^j and v_{k+1}^j, for all $1 \leq i, j \leq m$ and $2 \leq k \leq n-1$.

For $k = 1$, the vertices v_1^i are adjacent to the vertices v_2^j and $v_n^j, 1 \leq i, j \leq m$.

And similarly for $k = n$, the vertices v_n^i are adjacent to the vertices v_{n-1}^j and $v_1^j, 1 \leq i, j \leq m$.

Case-1: $n = 3$

Then, for the graph $D_m(C_3)$, the vertices $v_k^1, v_k^2, v_k^3, \cdots, v_k^m$ are the eccentric vertices to each other for $1 \leq k \leq 3$.

So, for each $k, 1 \leq k \leq 3$, we have to choose one vertex from the vertices $v_k^1, v_k^2, v_k^3, \cdots, v_k^m$ to construct the set D. Moreover, any two vertices of any one copy of cycle C_3 will dominate to all remaining vertices of $D_m(C_3)$.

Without lose of generality, let we take the vertices from the vertex set of C_3^1 to construct the set D.

Then, $D = \{v_1^1, v_2^1, v_3^1\}$ is a minimal eccentric dominating set of the graph $D_m(C_3)$.

Thus, $\gamma_{ed}[D_m(C_3)] = 3 = n$

Case-2: $n = 4$

Then, for the graph $D_m(C_4)$, the vertices $v_k^1, v_k^2, v_k^3, \cdots, v_k^m, v_{k+2}^1, v_{k+2}^2, v_{k+2}^3, \cdots, v_{k+2}^m$ are the eccentric vertices to each other for $1 \leq k \leq 2$.

So, for each $k, 1 \leq k \leq 2$, we have to choose one vertex from the vertices $v_k^1, v_k^2, v_k^3, \cdots, v_k^m, v_{k+2}^1, v_{k+2}^2, v_{k+2}^3, \cdots, v_{k+2}^m$ to construct the set D. Moreover, any two consecutive vertices of any one copy of cycle C_4 will dominate to all remaining vertices of $D_m(C_4)$.

Without lose of generality, let we take the vertices from the vertex set of C_4^1 to construct the set D.

Then, $D = \{v_1^1, v_2^1\}$ is a minimal eccentric dominating set of the graph $D_m(C_4)$.

Thus, $\gamma_{ed}[D_m(C_4)] = 2$

Case-3: $n = 5$

Then, for the graph $D_m(C_5)$, the vertices $v_k^1, v_k^2, v_k^3, \cdots, v_k^m$ are the eccentric vertices to each other for $1 \leq k \leq 5$. Moreover, the eccentric vertices of the vertices v_k^i are v_{k+2}^j and v_{k+3}^j and vice versa, for $1 \leq k \leq 2$ and $1 \leq i, j \leq m$ and the eccentric vertices of the vertices v_3^i are v_5^j and v_1^j

and vice versa, for $1 \leq i, j \leq m$. Moreover, any three consecutive vertices of any one copy of cycle C_5 will dominate to all remaining vertices of $D_m(C_5)$.

Without lose of generality, let we take the vertices from C_5^1 to construct the set D.

Then, $D = \{v_1^1, v_2^1, v_3^1\}$ is a minimal eccentric dominating set of the graph $D_m(C_5)$.

Thus, $\gamma_{ed}[D_m(C_5)] = 3$

Case-4: n **is an even number, with** $n \geq 6$**.**

Let $n = 2t$, then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, we have to choose one vertex from v_k^i and one vertex from v_{k+t}^j, for $1 \leq i, j \leq m$, to construct the set D.

Without lose of generality, we may choose the vertices from the vertex set of cycle C_n^1.

Then, $D = \{v_1^1, v_2^1, v_3^1, \cdots, v_n^1\}$ is a minimal eccentric dominating set, as they are dominating to all vertices and eccentric vertices of all vertices of the set $V - D$ of the graph $D_m(C_n)$.

Thus, $\gamma_{ed}[D_m(C_n)] = |D| = n$

Case-5: n **is an odd number, with** $n \geq 7$**.**

Let $n = 2t + 1$ then the eccentric vertices of the vertices v_k^i are v_{k+t}^j and v_{k+t+1}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices v_{t+1}^i are v_n^j and v_1^j and vice versa, for $1 \leq i, j \leq m$.

So, for each $k, 1 \leq k \leq t$, we can choose one vertex from v_k^i and one vertex from v_{k+t}^j or v_{k+t+1}^j, for $1 \leq i, j \leq m$, to construct the set D.

Without lose of generality, we may choose the vertices from the vertex set of cycle C_n^1.

To construct the set D, we have to choose two consecutive vertices for every three vertices of cycle C_n^1.

Case-(i): $t = 3s$**.**

Then, $n = 2t + 1 = 2(3s) + 1 = 6s + 1$

So, $D = \{v_1^1, v_2^1, v_4^1, v_5^1, v_7^1, v_8^1, \cdots, v_{6s-2}^1, v_{6s-1}^1, v_{6s+1}^1\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1^1, v_4^1, v_7^1, \cdots, v_{6s-2}^1\} = \{v_1^1, v_4^1, v_7^1, \cdots, v_{3l-2}^1\}$ where, $l = 2s$ and so, $|D'| = l = 2s$.

Moreover, $|D| = 2|D'| + 1 = 2(2s) + 1 = 4s + 1 = 4\left(\dfrac{n-1}{6}\right) + 1 = \dfrac{2n-2}{3} + 1 = \dfrac{2n-2+3}{3}$

$$= \frac{2n+1}{3} = \frac{2n}{3} + \frac{1}{3} = \left\lceil \frac{2n}{3} \right\rceil, \text{ since } \left(\frac{2n}{3} + \frac{1}{3} \right) \text{ is an integer.}$$

Thus, $\gamma_{ed}[D_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

Case-(ii): $t = 3s+1$**.**

Then, $n = 2t+1 = 2(3s+1)+1 = 6s+3$

So, $D = \{v_1^1, v_2^1, v_4^1, v_5^1, v_7^1, v_8^1, \cdots, v_{6s+1}^1, v_{6s+2}^1\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1^1, v_4^1, v_7^1, \cdots, v_{6s+1}^1\} = \{v_1^1, v_4^1, v_7^1, \cdots, v_{3l+1}^1\}$ where, $l = 2s$ and so, $|D'| = l+1 = 2s+1$.

Moreover, $|D| = 2|D'| = 2(2s+1) = 4s+2 = 4\left(\frac{n-3}{6}\right) + 2 = \frac{2n-6}{3} + 2 = \frac{2n-6+6}{3}$

$$= \frac{2n}{3} = \left\lceil \frac{2n}{3} \right\rceil, \text{ since } \left(\frac{2n}{3} \right) \text{ is an integer.}$$

Thus, $\gamma_{ed}[D_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

Case-(iii): $t = 3s+2$**.**

Then, $n = 2t+1 = 2(3s+2)+1 = 6s+5$

So, $D = \{v_1^1, v_2^1, v_4^1, v_5^1, v_7^1, v_8^1, \cdots, v_{6s+4}^1, v_{6s+5}^1\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1^1, v_4^1, v_7^1, \cdots, v_{6s+4}^1\} = \{v_1^1, v_4^1, v_7^1, \cdots, v_{3l+4}^1\}$ where, $l = 2s$ and so, $|D'| = l+2 = 2s+2$.

Moreover, $|D| = 2|D'| = 2(2s+2) = 4s+4 = 4\left(\frac{n-5}{6}\right) + 4 = \frac{2n-10}{3} + 4 = \frac{2n-10+12}{3}$

$$= \frac{2n+2}{3} = \frac{2n}{3} + \frac{2}{3} = \left\lceil \frac{2n}{3} \right\rceil, \text{ since } \left(\frac{2n}{3} + \frac{2}{3} \right) \text{ is an integer.}$$

Thus, $\gamma_{ed}[D_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

Thus, for all odd $n \geq 7$, $\gamma_{ed}[D_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

$$\text{Hence, } \gamma_{ed}[D_m(C_n)] = \begin{cases} n & ; n = 3 \text{ } or \text{ } n \text{ is an even number with } n \geq 6 \\ 2 & ; n = 4 \\ 3 & ; n = 5 \\ \left\lceil \dfrac{2n}{3} \right\rceil & ; n \text{ is an odd number with } n \geq 7 \end{cases}$$

□

Illustration 4.10. Minimal eccentric dominating set of 3-Shadow graph of $C_9 = \{v_1, v_2, v_4, v_5, v_7, v_8\}$, is shown in Figure 5.
Where, $\gamma_{ed}[D_3(C_9)] = 6 = \left\lceil \dfrac{18}{3} \right\rceil = \left\lceil \dfrac{2(9)}{3} \right\rceil$

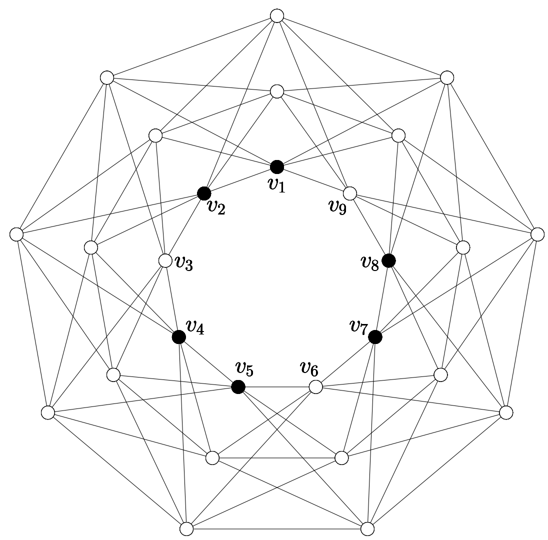

FIGURE 5. Minimal eccentric dominating set of $D_3(C_9)$

$$\textbf{Theorem 4.11. } \gamma_{ed}[Spl_m(C_n)] = \begin{cases} n & ; n = 3 \text{ } or \text{ } n \text{ is an even number with } n \geq 4 \\ 5 & ; n = 5 \text{ } and \text{ } m = 1 \\ 6 & ; n = 5 \text{ } and \text{ } m \geq 2 \\ \left\lceil \dfrac{2n}{3} \right\rceil & ; n \text{ is an odd number with } n \geq 7 \end{cases}$$

Proof. Let $v_1, v_2, v_3, \cdots, v_n$ is the vertex set of cycle C_n.

Let $v_1^i, v_2^i, v_3^i, \cdots, v_n^i$ are the m copies of the vertex set of cycle $C_n, 1 \leq i \leq m$ in the $m-$splitting graph of C_n.

Let the set D is a minimal eccentric dominating set of the $m-$splitting graph of C_n, i.e. $Spl_m(C_n)$.

Now, in the graph $Spl_m(C_n)$, for $1 \leq i \leq m$, the vertices v_k and v_k^i are adjacent to the vertices v_{k-1} and v_{k+1} of cycle C_n, for all $2 \leq k \leq n-1$.

For $k = 1$, the vertices v_1 and v_1^i are adjacent to the vertices v_2 and v_n, for $1 \leq i \leq m$.

Similarly for $k = n$, the vertices v_n and v_n^i are adjacent to the vertices v_{n-1} and v_1, for $1 \leq i \leq m$.

Case-1: $n = 3$

Then, for the graph $Spl_m(C_3)$, the vertices $v_k, v_k^1, v_k^2, v_k^3, \cdots, v_k^m$ are the eccentric vertices to each other for $1 \leq k \leq 3$. Also, the eccentric vertices of v_1^i are v_2^j and v_3^j, the eccentric vertices of v_2^i are v_1^j and v_3^j and the eccentric vertices of v_3^i are v_1^j and v_2^j, for $1 \leq i, j \leq m$.

So, for each $k, 1 \leq k \leq 3$, we have to choose one vertex from the vertices $v_k, v_k^1, v_k^2, v_k^3, \cdots, v_k^m$ to construct the set D.

Moreover, any two vertices of cycle C_3 will dominate to all remaining vertices of $Spl_m(C_3)$.

For minimum cardinality, we have to take the vertices from cycle C_3 to construct the set D.

Then, $D = \{v_1, v_2, v_3\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_3)$.

Thus, $\gamma_{ed}[Spl_m(C_3)] = 3 = n$

Case-2: $n = 4$

Then, for the graph $Spl_m(C_4)$, the eccentric vertices of the vertex v_k are v_k^i, v_{k+2} and v_{k+2}^i for $1 \leq k \leq 2$ and $1 \leq i \leq m$. And, the eccentric vertices of the vertex v_k are v_k^i, v_{k-2} and v_{k-2}^i for $3 \leq k \leq 4$ and $1 \leq i \leq m$.

The eccentric vertices of the vertices v_1^i and v_3^i are v_2^j and v_4^j for $1 \leq i, j \leq m$ and vice versa.

So, we have to choose one vertex from v_1^i or v_3^i and one vertex from v_2^i or v_4^i to construct the set D for $1 \leq i \leq m$.

Moreover, any two consecutive vertices of cycle C_4 will dominate to all remaining vertices of $Spl_m(C_4)$.

Then, $D = \{v_1, v_2, v_1^1, v_2^1\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_4)$.
Thus, $\gamma_{ed}[Spl_m(C_4)] = 4 = n$

Case-3: $n = 5$

Then, for the graph $Spl_m(C_5)$, the eccentric vertices of the vertex v_k are $v_k^i, v_{k+2}, v_{k+2}^i, v_{k+3}$ and v_{k+3}^i for $1 \leq k \leq 2$ and $1 \leq i \leq m$, the eccentric vertices of the vertex v_k are $v_k^i, v_{k-2}, v_{k-2}^i, v_{k-3}$ and v_{k-3}^i for $4 \leq k \leq 5$ and $1 \leq i \leq m$. And, the eccentric vertices of the vertex v_3 are v_3^i, v_1, v_1^i, v_5 and v_5^i for $1 \leq i \leq m$.

The eccentric vertices of the vertices v_k^i are v_{k-1}^j and v_{k+1}^j for $2 \leq k \leq 4$ and $1 \leq i, j \leq m$. The eccentric vertices of the vertices v_1^i are v_5^j and v_2^j, for $1 \leq i, j \leq m$. The eccentric vertices of the vertices v_5^i are v_4^j and v_1^j for $1 \leq i, j \leq m$.

Moreover, any three consecutive vertices of cycle C_5 will dominate to all remaining vertices of $Spl_m(C_5)$.

Case-(i): $m = 1$

Then, for the graph $Spl_1(C_5)$, the set $D = \{v_1, v_2, v_3, v_1^1, v_3^1\}$ is a minimal eccentric dominating set.

So, $\gamma_{ed}[Spl_1(C_5)] = 5$

Case-(ii): $m \geq 2$

Then, for the graph $Spl_m(C_5)$, the set $D = \{v_1, v_2, v_3, v_1^1, v_2^1, v_3^1\}$ is a minimal eccentric dominating set.

So, $\gamma_{ed}[Spl_m(C_5)] = 6$

Case-4: $n = 6$

Then, for the graph $Spl_m(C_6)$, the eccentric vertices of the vertices v_k and v_k^i are v_{k+3} and v_{k+3}^j, for $1 \leq k \leq 3$ and $1 \leq i, j \leq m$. And the eccentric vertices of the vertices v_k and v_k^i are v_{k-3} and v_{k-3}^j, for $4 \leq k \leq 6$ and $1 \leq i, j \leq m$.

The vertices v_{k-1}^j and v_{k+1}^j are also eccentric vertices of the vertices v_k^i, for $2 \leq k \leq 5$ and $1 \leq i, j \leq m$. The vertices v_6^j and v_2^j are also eccentric vertices of the vertices v_1^i and the vertices v_5^j and v_1^j are also eccentric vertices of the vertices v_6^i, for $1 \leq i, j \leq m$.

So, for minimal eccentric dominating set, we have to choose the vertices from cycle C_6 to construct the set D. And for each $k, 1 \leq k \leq 6$, we have to choose vertex v_k for eccentric vertices of all remaining vertices of the graph $Spl_m(C_6)$.

Then, $D = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_6)$.

Thus, $\gamma_{ed}[Spl_m(C_6)] = 6 = n$

Case-5: $n = 7$

Then, for the graph $Spl_m(C_7)$, the eccentric vertices of the vertices v_k and v_k^i are $v_{k+3}, v_{k+3}^j, v_{k+4}$ and v_{k+4}^j, for $1 \leq k \leq 3$ and $1 \leq i, j \leq m$. And the eccentric vertices of the vertices v_k and v_k^i are $v_{k-3}, v_{k-3}^j, v_{k-4}$ and v_{k-4}^j, for $5 \leq k \leq 7$ and $1 \leq i, j \leq m$. The eccentric vertices of the vertices v_4 and v_4^i are v_7, v_7^j, v_1 and v_1^j, for $1 \leq i, j \leq m$.

The vertices v_{k-1}^j and v_{k+1}^j are also eccentric vertices of the vertices v_k^i, for $2 \leq k \leq 6$ and $1 \leq i, j \leq m$. The vertices v_7^j and v_2^j are also eccentric vertices of the vertices v_1^i and the vertices v_6^j and v_1^j are also eccentric vertices of the vertices v_7^i, for $1 \leq i, j \leq m$.

So, for minimal eccentric dominating set, we have to choose two consecutive vertices from every three vertices of cycle C_7 to construct the set D. Then, $D = \{v_1, v_2, v_4, v_5, v_7\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_7)$.

So, $\gamma_{ed}[Spl_m(C_7)] = 5 = \left\lceil \frac{14}{3} \right\rceil = \left\lceil \frac{2(7)}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil$

Case-6: n **is an even number, with** $n \geq 8$**.**

Let $n = 2t$, then the eccentric vertices of the vertices v_k and v_k^i are v_{k+t} and v_{k+t}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, for eccentric vertices, we have to choose one vertex from v_k or v_k^i and one vertex from v_{k+t} or v_{k+t}^j, for $1 \leq i, j \leq m$, to construct the set D.

For minimum cardinality, to dominate all remaining vertices, we have to choose the vertices from the vertex set of cycle C_n, to construct the set D.

Then, $D = \{v_1, v_2, v_3, \cdots, v_n\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_n)$.

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = n$

Case-7: n **is an odd number, with** $n \geq 9$**.**

Let $n = 2t + 1$ then the eccentric vertices of the vertices v_k and v_k^i are $v_{k+t}, v_{k+t}^j, v_{k+t+1}$ and v_{k+t+1}^j and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices v_{t+1} and v_{t+1}^i are v_n, v_n^j, v_1 and v_1^j and vice versa, for $1 \leq i, j \leq m$.

So, for each $k, 1 \leq k \leq t$, we can choose one vertex from v_k or v_k^i and one vertex from v_{k+t} or v_{k+t}^j or v_{k+t+1} or v_{k+t+1}^j, for $1 \leq i, j \leq m$, to construct the set D.

For minimum cardinality, we have to choose the vertices from the vertex set of cycle C_n to construct the set D. And we have to choose two consecutive vertices from every three vertices of cycle C_n.

Without lose of generality, we may choose first two consecutive vertices from every three vertices of cycle C_n.

Case-(i): $t = 3s$.

Then, $n = 2t+1 = 2(3s)+1 = 6s+1$

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s-2}, v_{6s-1}, v_{6s+1}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s-2}\} = \{v_1, v_4, v_7, \cdots, v_{3l-2}\}$ where, $l = 2s$ and so, $|D'| = l = 2s$.

Moreover, $|D| = 2|D'|+1 = 2(2s)+1 = 4s+1 = 4\left(\dfrac{n-1}{6}\right)+1 = \dfrac{2n-2}{3}+1 = \dfrac{2n-2+3}{3}$

$$= \frac{2n+1}{3} = \frac{2n}{3}+\frac{1}{3} = \left\lceil \frac{2n}{3} \right\rceil, \text{ since } \left(\frac{2n}{3}+\frac{1}{3}\right) \text{ is an integer.}$$

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil \dfrac{2n}{3} \right\rceil$.

Case-(ii): $t = 3s+1$.

Then, $n = 2t+1 = 2(3s+1)+1 = 6s+3$

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s+1}, v_{6s+2}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s+1}\} = \{v_1, v_4, v_7, \cdots, v_{3l+1}\}$ where, $l = 2s$ and so, $|D'| = l+1 = 2s+1$.

Moreover, $|D| = 2|D'| = 2(2s+1) = 4s+2 = 4\left(\dfrac{n-3}{6}\right)+2 = \dfrac{2n-6}{3}+2 = \dfrac{2n-6+6}{3} = \dfrac{2n}{3} = \left\lceil \dfrac{2n}{3} \right\rceil$, since $\left(\dfrac{2n}{3}\right)$ is an integer.

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil \dfrac{2n}{3} \right\rceil$.

Case-(iii): $t = 3s+2$.

Then, $n = 2t+1 = 2(3s+2)+1 = 6s+5$

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s+4}, v_{6s+5}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s+4}\} = \{v_1, v_4, v_7, \cdots, v_{3l+4}\}$ where, $l = 2s$ and so, $|D'| = l+2 = 2s+2$.

Moreover, $|D| = 2|D'| = 2(2s+2) = 4s+4 = 4\left(\frac{n-5}{6}\right)+4 = \frac{2n-10}{3}+4 = \frac{2n-10+12}{3}$

$$= \frac{2n+2}{3} = \frac{2n}{3}+\frac{2}{3} = \left\lceil\frac{2n}{3}\right\rceil, \text{ since } \left(\frac{2n}{3}+\frac{2}{3}\right) \text{ is an integer.}$$

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil\frac{2n}{3}\right\rceil$.

Thus, for all odd $n \geq 9$, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil\frac{2n}{3}\right\rceil$.

$$\text{Hence, } \gamma_{ed}[Spl_m(C_n)] = \begin{cases} n & ; n = 3 \text{ or } n \text{ is an even number with } n \geq 4 \\ 5 & ; n = 5 \text{ and } m = 1 \\ 6 & ; n = 5 \text{ and } m \geq 2 \\ \left\lceil\frac{2n}{3}\right\rceil & ; n \text{ is an odd number with } n \geq 7 \end{cases}$$

□

Illustration 4.12. Minimal eccentric dominating set of 1-splitting graph of $C_5 = \{v_1, v_2, v_3, v_1^1, v_3^1\}$, is shown in Figure 6 and minimal eccentric dominating set of 2-splitting graph of $C_8 = \{v_1, v_2, v_3,$

$v_4, v_5, v_6, v_7, v_8\}$, is shown in Figure 7.

Where, $\gamma_{ed}[Spl_1(C_5)] = 5$ and $\gamma_{ed}[Spl_2(C_8)] = 8$

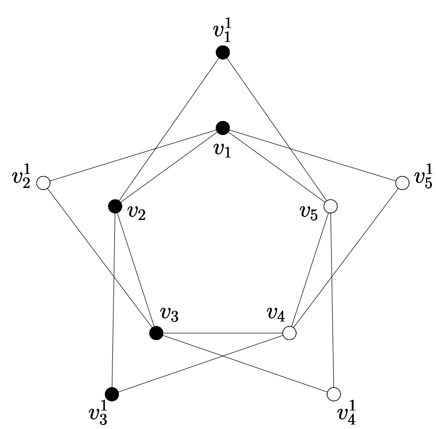

FIGURE 6. Minimal eccentric dominating set of $Spl_1(C_5)$

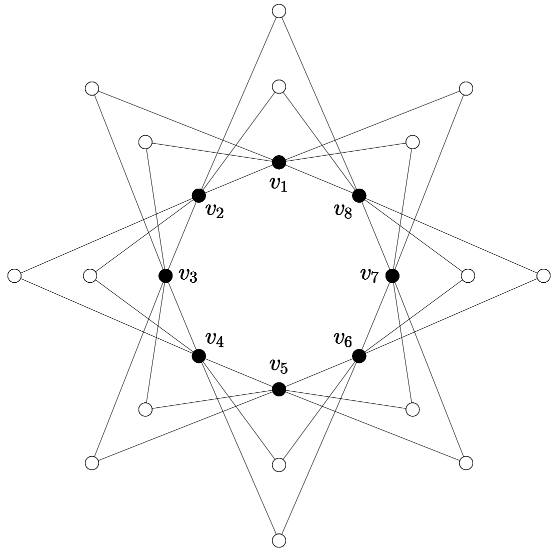

FIGURE 7. Minimal eccentric dominating set of $Spl_2(C_8)$

5. CONCLUSION

The concept of eccentric domination relates a dominating set with the eccentricity of a vertex. We have investigated eccentric domination number of cycle as well as some cycle related graphs including $m-$shadow, extended $m-$shadow, $m-$splitting, extended $m-$splitting and square of cycle.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] J. Clark and D. A. Holton, A First Look at Graph Theory, Allied Publishers Ltd., (1995).

[2] [1]E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, Networks. 10 (1980), 211–219.

[3] S. Ao, E. J. Cockayne, G. MacGillivray and C. M. Mynhardt, Domination Critical Graphs with Higher Independent Domination Numbers, J. Graph Theory, (22) (1996), 9-14.

[4] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., (1998).

[5] T. N. Janakiraman, M. Bhanumathi and S. Muthammai, Eccentric Domination in Graphs, Int. J. Eng. Sci. Adv. Comput. Bio Technol. 1(2) (2010), 1-16.

[6] V. Swaminathan and K. M. Dharmalingam, Degree Equitable Dominations on Graphs, Kragujevac J. Math. 35(1) (2011), 191-197.

[7] J. A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Math. (10) (1997), 529-550.

[8] M. E. Abdel-Aal, New Classes of Odd Graceful Graphs, Int. J. Appl. Graph Theory Wireless Ad hoc Networks Sensor Networks (GRAPH-HOC), 5(2) (2013), 1-12.

[9] S. K. Vaidya and D. M. Vyas, Eccentric Domination Number of Some Path Related Graphs, Malaya J. Mat. 8(4) (2020), 1728-1734.

[10] S. K. Vaidya and K. M. Popat, Some Borderenergetic and Equienergetic Graphs, (Accepted for publication).

[11] S. K. Vaidya and R. M. Pandit, Some New Results on Global Dominating Sets, ISRN Discrete Math. 2012 (2012), Article ID 852129.

[12] S. K. Vaidya and R. N. Mehta, Steiner Domination Number of Some Wheel Related Graphs, Int. J. Math. Soft Comput. 5(2) (2015), 15-19.

[13] D. B. West, Introduction to Graph Theory, Prentice - Hall of India, New Delhi, (2003).