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COMMON FIXED POINT OF KANNAN AND WEAK CONTRACTIVE TYPE MAPPINGS ON A MODULAR METRIC SPACE ENDOWED WITH A GRAPH

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Abstract. In this paper, we reformulate, extend, and establish certain fixed point findings for Kannan type contraction mappings in a modular metric space with a graph. This paper's result is novel and adds to the previously published result of a graph-endowed metric, modular metric spaces.
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1. INTRODUCTION

The metric fixed point theorem is extremely useful and important in mathematics. Ran and Reurings [18] were the first to address the life of fixed points for single valued mappings in partial ordered metric spaces. Fixed point theorems for monotone single valued mappings in a metric space with partial ordering have been widely investigated. Many new discoveries have lately surfaced that offer sufficient criteria for f to be a PO if (X, d) has a partial ordering. The Banach Contraction Principle and the Knaster-Tarski Principle, two basic and useful fixed-point theory theorems, have been combined to provide these results. Jachymski [9,10] achieved some helpful findings for mappings given on a complete metric space provided with a graph instead of partial

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ordering. Bojor [5] proved fixed point outcomes for Kannan mappings in metric spaces provided with a graph. Fixed point theorems for weakly contractive maps have been shown by Samreen and Kamran [19]. Following that, numerous scholars looked at the weakly contractive state in this direction as well as the graph's connection requirement.

The concept of modular spaces was first presented by Nakano [15], and it was further refined by Koshi and Shimogaki [13], Yamamuro [20], and Musielak and Orlicz [14]. The nature and uniqueness of the Banach and Kannan contraction fixed points defined on modular spaces equipped with a graph were recently explored by Aghanians and Nourozi[3].

The concept of modular metric spaces was developed by Chistyakov [6, 7]. Abdou and Khamsi[2] provided an analogue of the Banach contraction concept in modular metric spaces. Alfuraidan[4] recently extended the Banach contraction principle to a modular metric space with a graph that is Jachymski's modular metric version[9] of fixed point findings for mappings with a graph on a metric space. Pathak et al [17] provided a recent result of fixed point theorems for Kannan contractions and weakly contractive mappings on a modular metric space equipped with a graph.

2. PRELIMINARIES

Let X be a set that is non-empty. Throughout, this paper for a function $\omega : (0,\infty) \times X \times X \to (0,\infty)$ will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $\lambda > 0$ and $x,y \epsilon X$.

Definition 2.1.[6,7]. Let X be a non-empty set. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X if it satisfies the following three axioms:

(i) given $x,y \epsilon X, \omega_\lambda(x,y) = 0 \; forall \lambda > 0 \; if \; and \; only \; if \; x = y$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x) for \; all \; \lambda > 0$ and $x,y \epsilon X$

(iii) $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y) forall \lambda,\mu > 0$ and $x,y,z \epsilon X$.

If instead of (i), we have only the condition

$$\omega_\lambda(x,x) = 0 \; forall \lambda > 0 \text{ and } x \in X,$$

Then ω is said to be a (metric) pseudo modular on X. A modular ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \text{ if and only if } \omega_\lambda(x,y) = 0 \; forsome \; \lambda > 0.$$

Finally ω is said to be convex if $for \;\; \lambda,\mu > 0$ and $x,y,z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) = \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y) .$$

Note that for a pseudo modular ω on a set X and $any\ x, y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is non increasing on $(0,\infty)$. Indeed, $if\ 0 < \mu < \lambda$, then $\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$.

Definition 2.2. Let $X_\omega$be a modular metric space.

(1) The sequence $(x_n)_{n\epsilon N}$ in $X_\omega$is said to be convergent to $x \epsilon X_\omega$if

$$\omega_\lambda(x_n, x) \to 0\ as\ n \to \infty\ for\ all\ \lambda > 0 .$$

(2) The sequence $(x_n)_{n\epsilon N}$ in X_ω is said to be Cauchy if

$$\omega_\lambda(x_m, x_n) \to 0\ as\ m, n \to \infty\ for\ all\ \lambda > 0 .$$

(3) A subset C of X_ω is said to be closed if the limit of the convergent sequence of C always belong to C.

(4) A subset C of X_ω is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit in C.

(5) A subset C of $X_\omega$is said to be bounded if for all $\lambda > 0$,

$$\delta_\omega(C) = \sup\{\omega_\lambda(x,y); x, y\epsilon C\} < \infty.$$

In general, if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$,for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$,for all $\lambda > 0$, Therefore, as in modular function spaces, we will say that ω satisfies the Δ_2-condition.

If $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$,implies $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$,for all $\lambda > 0$.

The relation between ω -convergence and metric convergence with regard to the Luxemburg distances will be addressed in[6,7].In particular, we have

$$\lim_{n\to\infty} d_\omega(x_n, x) = 0 \text{ iff } \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0, \text{ for all } \lambda > 0.$$

For any $\{x_n\} \in X_\omega$and $x \in X_\omega$. In particular we have ω −convergence and $d_\omega$convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that $d^*_\omega$and d_ω are equivalent which implies

$$\lim_{n\to\infty} d^*_\omega(x_n, x) = 0 \text{ iff } \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0, \text{ for all } \lambda > 0,$$

for any$\{x_n\} \in X_\omega$and $x \in X_\omega$.

Definition 2.3. [6]. Let(X,ω)be a modular metric space .We will say that ω satisfies the Δ_2- type condition iff for any $\alpha > 0$,there exists a $C > 0$ such that

$\omega_{\frac{\lambda}{\alpha}}(x,y) \leq C\omega_{\lambda}(x,y)$,for any $\lambda > 0$, $x, y \in X_{\omega}$, with $x \neq y$.

Note that if ω satisfies the Δ_2-type condition, then ω satisfies the Δ_2-condition. The above definition will allow us to introduce the growth function in the modular metric spaces as was done in the linear case.

Definition 2.4. [6]. Let (X,ω)be a modular metric space. Define the growth function Ω by

$$\Omega(\alpha) = sup\left\{\frac{\omega_{\frac{\lambda}{\alpha}}(x,y)}{\omega_{\lambda}(x,y)}, \lambda > 0, x, y \in X_{\omega}, x \neq y\right\}, \text{ for any } \alpha > 0.$$

The following lemma is useful for this work.

Lemma 2.1. [2] Let (X,ω) be a modular metric space. Assume that ω is a convex regular modular metric which satisfies the Δ_2-type condition.Let $\{x_n\}$ be a sequence in X_{ω}such that $\omega_1(x_{n+1},x_n) \leq K\alpha^n$, n = 1,2,…, where K is an arbitrary non zero constant and $\alpha \in (0,1)$. Then $\{x_n\}$ is Cauchy for both ω and d_{ω}^*

We will use graph theory's following notations and vocabulary (see[11]) relevant to the rest of our outcome.

Let (X,ω) be a modular metric space and M be a non empty subset of X_{ω}. Let Δ denote the diagonal of the Cartesian product $M \times M$. Consider a directed graph G_{ω} such that the set $V(G_{\omega})$of its vertices coincide with M, and the set $E(G_{\omega})$of its edges contain all loops, i.e. $E(G_{\omega}) \supseteq \Delta$.We assume G_{ω}has no parallel edges (arcs), so we can identify $G_{\omega}$with the pair $(V(G_{\omega})\ E(G_{\omega}))$. Our notation and terminology for graph theory are common and can be used in all graph theory books, such as [11,16]. Moreover, we may treat G_{ω}as a weighted graph (see [10]) by assigning to each edge the distance between its vertices.

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(\mathrm{G}^{-1}) = \{(y,x)|(x,y)\epsilon\, E(G)\}.$$

A diagraph G is called an oriented graph if whenever $(u,v)\epsilon\, E(G)$, then$(v,u) \notin E(G)$.The letter $\tilde{G}$ denotes the undirected graph obtain from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat G as a directed graph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \cup E(\mathrm{G}^{-1})$.

We call (V', E') a sub graph of $V' \subseteq V(G), E' \subseteq E(G)$, and for any edge $(x,y) \in E', x, y \in V'$.If x and y are vertices in a graph G, then a (directed) path in G from x *to* y *of length* N is a sequence $(x_i)_{i=1}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$.. A graph G is connected if there is a directed path between any two vertices. G is a weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the sub graph $G_x$consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$,where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule : $y\mathcal{R}z$ if there is a (directed) path in G from y *to* z.Clearly G_x is connected.

Definition 2.5. [4] Let (X, ω) be a modular metric space and M be a non empty subset of X_ω. A mapping $T : M \to M$ is called

(i) G_ω- contraction if T preserve edges of G_ω, i.e.,

$$\forall\, x, y \in M\ ((x,y) \in E(G_\omega) \Rightarrow \big(T(x), T(y)\big) \in E(G_\omega)\big),$$

and if there exists a constant $\alpha \in [0,1)$ such that

$$\omega_1\big(T(x), T(y)\big) \leq \alpha\omega_1(x,y) \text{for any} (x,y) \in E(G_\omega).$$

(ii) $(\varepsilon, \alpha) - G_\omega$-uniformly locally contraction if T preserve edges of $G_\omega$and there exists a constant $\alpha \in [0,1)$ such that for any

$(x,y) \in E(G_\omega) \omega_1(T(x), T(y)) \leq \alpha\omega_1(x,y)$ whenever $\omega_1(x,y) < \varepsilon$.

We recall Kannan [12] presented the Kannan-type mappings as follows:

Definition. [12] Let $(X,\ d)$ be a metric space and T be a mapping on X. We say that T is a Kannan type mapping if there exists $0 \leq k < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)] \text{ for all } x, y \in X.$$

It is well known that Banach's contraction mappings are continuous while Kannan-type mappings are not required continuous. There is a major contrast between these two forms of mappings. Again, it can also be noticed that Banach's contraction does not define metric completeness.

Pathak et al [17] introduce the G_ω Kannan contraction and weakly $G_\omega$contractive mappings in a modular metric space endowed with a graph as follows:

Definition 2.6. Let $(X,\ \omega)$ be a modular metric space with a graph G_ω. A mapping $T : M \to M$ is called

1. G_ω- Kannan contraction if T preserve the edges of G_ω ,

   i.e., for all $x, y \epsilon M \left((x,y) \epsilon E(G_\omega) \Longrightarrow (Tx,Ty) \epsilon E(G_\omega)\right)$

   and if there exists positive number $k \epsilon (0, \frac{1}{2})$ such that

$$\omega_\lambda(Tx,Ty) \leq k(\omega_\lambda(Tx,x) + \omega_\lambda(Ty,y))$$

   for any $x, y \epsilon M \; with \; (x,y) \epsilon \; E(G_\omega)$.

2. weakly G_ω contractive if T preserve the edges of G_ω,

   i.e., for all $x, y \epsilon M \; ((x,y) \; \epsilon \; E(G_\omega) \Longrightarrow (Tx,Ty) \epsilon E(G_\omega))$

   and $\omega_\lambda(Tx,Ty) \leq \omega_\lambda(x,y) - \psi\big(\omega_\lambda(x,y)\big)$,

   whenever Ψ is a family of continuous non decreasing function $\psi: [0,\infty) \to [0,\infty)$ such that ψ is positive on $[0,\infty) and \; \psi(0) = 0, \psi(t) < t \; for \; all \; \psi \; \epsilon \; \Psi$.

We add the property below, as Jachymski [9] did.

We say that the triple $(M, d^*_\omega, G_\omega)$ has property (P) if

(P) For any sequence $\{x_n\}_{n \in N}$ in M, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \epsilon E(G_\omega)$, then $(x_n, x) \epsilon E(G_\omega)$, for all n.

Notice that property (P) is precisely the hypothesis of Nieto et al.[16] that relaxes the assumption of continuity as in Theorem1.2 ((2) and (3))of [9,16] rephrased in terms of edges.

Definition 2.7. Let X be a set, and S and T self maps of X . A point $x \; in \; X$ is called a coincidence point of $S \; and \; T$ if and only if $Sx = Tx$. We will call $w = Sx = Tx$ a point of coincidence of $S \; and \; T$.

Definition 2.8. A pair (S,T) of self mappings of a metric space (X,d) is said to be weakly compatible if the mappings commute at all of their coincidence points, i.e. $Sx = Tx \; for \; some \; x \in X \; implies \; S(Tx) = T(Sx)$.

Lemma 2.2. [1]. Let $S \; and \; T$ be weakly compatible self - maps of a set X. If $S \; and \; T$ have a unique point of coincidence w (say), then w is the unique common fixed point of $S \; and \; T$.

3. MAIN RESULTS

Theorem 3.1. Let $(X, \; \omega)$ be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$is a nonempty ω - bounded, ω - complete subset of X_ω and the triple

$(M, d_\omega^*, G_\omega)$ has property (P). Let $S, T : M \to M$ be G_ω Kannan type contraction satisfying $\omega_1(Tx, Ty) \leq k[\omega_1(Tx, Sx) + \omega_1(Ty, Sy)]$ and weakly compatible mappings, $T(X_\omega) \subset S(X_\omega)$ and $M_{S,T} := \{x, y \in M; (x, Sx)(y, Ty) \in E(G_\omega)\}$.

If $(x_0, S(x_0)), (y_0 T(y_0)) \in E(G_\omega)$, then the following statements hold:

(i) For any $x, y \in M_{S,T}$ $S, T|_{[x]_{\widetilde{G_\omega}}}$ has a fixed point.

(ii) If G_ω is weakly connected, then S and T has a fixed point in M.

(iii) If $M' = \cup\{[x, y]_{\widetilde{G_\omega}}: x \in M_S \text{ and } y \in M_T\}$ then $S, T|_{M'}$ has a fixed point in M.

Proof. (i) Since $(x_0, S(x_0)) \in E(G_\omega)$ and $(y_0, T(y_0)) \in E(G_\omega)$ then $x_0, y_0 \in M_{S,T}$. Since $T(X_\omega) \subset S(X_\omega)$, S and T are Kannan type contraction, there exists a constant $k \in (0, \frac{1}{2})$ such that $(T(x_0), T(y_0)) \in E(G_\omega)$ and

$$\omega_1(Tx_0, Ty_0) \leq k[\omega_1(Tx_0, Sx_0) + \omega_1(Ty_0, Sy_0)] \quad (3.1.1)$$

By induction, we can construct a sequence, $\{Sx_n\}$ such that $Tx_n = Sx_{n+1}$ and $(Sx_n, Sx_{n+1}) \in E(G_\omega)$. We now show that $\{Sx_n\}$ is a Cauchy sequence.

For any natural n, and using condition (3.1.1), we get

$$\omega_1(Tx_n, Tx_{n-1}) \leq k[\omega_1(Tx_n, Sx_n) + \omega_1(Tx_{n-1}, Sx_{n-1})]$$

$$\omega_1(Tx_n, Tx_{n-1}) \leq k[\omega_1(Tx_n, Tx_{n-1}) + \omega_1(Tx_{n-1}, Tx_{n-2})]$$

which gives that

$$(1 - k)\, \omega_1(Tx_n, Tx_{n-1}) \leq k\, \omega_1(Tx_{n-1}, Tx_{n-2})$$

$$\omega_1(Tx_n, Tx_{n-1}) \leq \frac{k}{(1 - k)} \omega_1(Tx_{n-1}, Tx_{n-2})$$

$$\omega_1(Tx_n, Tx_{n-1}) \leq \alpha \omega_1(Tx_{n-1}, Tx_{n-2})$$

Or $\omega_1(Sx_{n+1}, Sx_n) \leq \alpha \omega_1(Sx_n, Sx_{n-1})$,

where $\alpha = \frac{k}{(1-k)} < 1$.

So by induction, we construct a sequence $\{Sx_n\}$ such that $(Sx_{n+1}, Sx_n) \in E(G_\omega)$ and $\omega_1(Sx_{n+1}, Sx_n) \leq \alpha^n \omega_1(Sx_1, Sx_0)$ for any $n \geq 1$. Since M is ω-bounded, we have,

$$\omega_1(Sx_{n+1}, Sx_n) \leq \delta_\omega(M) k^n$$

for any $n \geq 1$. Then by lemma 2.1 $\{Sx_n\}$ is ω -Cauchy. Since M is $\omega -$ Complete, therefore $\{Sx_n\}$ is ω - convergence to some point $x \in M$ .By property (P)

$$(Sx_n, x) \in E(G_\omega) \text{ for all } n. \text{ Thus } (Tx_{n-1}, x) \in E(G_\omega).$$

Since $T(X_\omega) \subset S(X_\omega)$,there exists a point $u \in M$ such that $Su = x$.

To prove $Tu = x$. Suppose on the contrary that $Tu \neq x$.

Using the property of ω, we have

$$\omega_1(Sx_{n+1}, Tu) = \omega_1(Tx_n, Tu) \leq k[\omega_1(Tx_n, Sx_n) + \omega_1(Tu, Su)]$$

taking limit $n \to \infty$ on both sides we get

$$\omega_1(x, Tu) \leq k[\omega_1(x, x) + \omega_1(Tu, x)]$$

Implies $(1-k)\omega_1(x, Tu) \leq 0$, a contradiction.

Hence $Tu = x = Su$. Therefore x is a point of coincidence of S and T.

Uniqueness. To prove the uniqueness, suppose that z be another point of coincidence of S and T. Thus by (3.1.1) we get

$\omega_1(x, z) = \omega_1(Tx, Tz) \leq k[\omega_1(Tx, Sx) + \omega_1(Tz, Sz)] = 0$

for all $\lambda > 0$. Hence $\omega_1(x, z) = 0$.

Thus $x = z$ is a unique point of coincidence of S and T.

Since S and T are weakly compatible, by lemma 2.2, $x = z$ is a unique common fixed point of S and T.

(ii) Since $M_{S,T} \neq \emptyset$, there exists an $x_0, y_0 \epsilon M_{S,T}$ and since $G_\omega$is weakly connected, then $[x_0, y_0]_{\widetilde{G_\omega}} = M$ and by M and by (i), mapping S and T has a fixed point.

(iii) It follows easily from (i) and (ii).

Remark 3.1. By taking the mapping S in Theorem 3.1 as Ix_ω,where $Ix_\omega$is an identity mapping on X_ω, we have following corollary which is main result of Pathak et al[16,Theorem 3.1]

Corollary 3.1. Let (X, ω) be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$is a nonempty ω- bounded, ω- complete subset of X_ω and the triple $(M, d_\omega^*, G_\omega)$ has property (P). Let $T : M \to M$ be G_ω Kannan type contraction mapping and

$$M_T := \{x \epsilon M; (x, Tx) \epsilon E(G_\omega)\}.$$

If $(x_0, T(x_0)) \epsilon E(G_\omega)$, then the following statements hold:

(i) For any $x \epsilon M_T$ $T|_{[x]_{\widetilde{G_\omega}}}$ has a fixed point.

(ii) If $G_\omega$is weakly connected, then T has a fixed point in M.

(iii)If $M' = \cup\{[x]_{\widetilde{G_\omega}} : x \epsilon M_T \text{ and } y \epsilon M_T\}$then $T|_{M'}$has a fixed point in M.

Proof. The proof of the corollary follows immediately from theorem 3.1, by putting the mapping S in Theorem 3.1 as Ix_ω,where $Ix_\omega$is an identity mapping on X_ω.

An analog of the Kannan contraction in modular metric spaces is as follows :

Corollary 3.2. Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. If $X_\omega$is complete modular metric space and $T : X_\omega \to X_\omega$ be a self mapping satisfying the inequality

$$\omega_1(Tx,Ty) \leq k[\omega_1(Tx,x) + \omega_1(Ty,y)] \qquad \text{for all } x,y\epsilon X_\omega, \text{ where } k\epsilon\,[0,1).$$

Suppose that there exist $x\epsilon\ X$ such that $\omega_1(x,Tx) < \infty$ for all λ>0. Then T has a unique fixed point in X_ω. More over, for any $x\ \epsilon X_\omega$, sequence $\{T^n x\}$ converges to x.

Theorem 3.2 Let (X,ω) be a modular metric space with a graph G_ω. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$ is a nonempty $\omega -$ bounded, $\omega -$ complete subset of X_ω and the triple $(M, d_\omega^*, G_\omega)$ has property (P). Let $S,T : M \to M$ be weakly G_ω contractive, weakly compatible mappings satisfying $\omega_1(Tx,Ty) \leq \omega_1(Sx,Sy) - \psi(\omega_1(Sx,Sy))$ and T(X_ω) $\subset$ S(X_ω) and $M_{S,T} := \{x,y \in M; (Sx,Sy)(Tx,Ty) \in E(G_\omega)\}$. If$(Tx_0, Ty_0) \in E(G_\omega)$, then the following statements hold:

(i) For any $x \in M_{S,T}, S,T|_{[x]_{\widetilde{G_\omega}}}$, has a fixed point.

(ii) If G_ω is weakly connected, then S and T has a fixed point in M.

(iii) If $M' = \cup\{[x]_{\widetilde{G_\omega}} : x \in M_{S,T}\}$, then $S,T|_{M'}$ has a fixed point in M.

Proof. Since $(Sx_0, Sy_0)) \in E(G_\omega)$ and $(Tx_0,Ty_0) \in E(G_\omega)$ then $x_0, y_0 \in M_{S,T}$. Since S and T are weakly $G_\omega$contractive and $(Tx_0,Ty_0) \in E(G_\omega)$. Then by definition

$$\omega_1(Tx_0,Ty_0) \leq \omega_1(Sx_0,Sy_0) - \psi(\omega_1(Sx_0,Sy_0))$$

Since T(X_ω) ⊂ S(X_ω), by induction, we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = Sx_{n+1}$ and $(x_n, x_{n+1}) \in E(G_\omega)$.

Consider

$$\begin{aligned}\omega_1(x_{n+1},x_n) &= \omega_1(Tx_n,Tx_{n-1}) \\ &\leq \omega_1(Sx_n,Sx_{n-1}) - \psi(\omega_1(Sx_n,Sx_{n-1})) \\ &< \omega_1(Sx_n,Sx_{n-1})\end{aligned}$$

i.e. $\omega_1(x_{n+1},x_n) < \omega_1(x_n,x_{n-1})$

Similarly,

$$\omega_1(x_{n+2},x_{n+1}) = \omega_1(Tx_{n+1},Tx_n)$$

$$\leq \omega_1(Sx_{n+1}, Sx_n) - \psi(\omega_1(Sx_{n+1}, Sx_n))$$
$$< \omega_1(Sx_{n+1}, Sx_n)$$

i.e. $\omega_1(x_{n+2}, x_{n+1}) < \omega_1(x_{n+1}, x_n)$

again $\omega_1(x_{n+3}, x_{n+2}) = \omega_1(Tx_{n+2}, Tx_{n+1})$
$$\leq \omega_1(Sx_{n+2}, Sx_{n+1}) - \psi(\omega_1(Sx_{n+2}, Sx_{n+1}))$$
$$< \omega_1(Sx_{n+2}, Sx_{n+1})$$

i.e. $\omega_1(x_{n+3}, x_{n+2}) < \omega_1(x_{n+2}, x_{n+1})$

Hence in general,
$$\omega_1(x_{i+1}, x_i) \leq \omega_1(x_i, x_{i-1}) - \psi(\omega_1(x_i, x_{i-1})$$
Or $\omega_1(x_{i+1}, x_i) < \omega_1(x_i, x_{i-1})$; $\forall\ i = 1,2,3 \ldots n$

Since ψ is non decreasing and this shows that $\{x_n\}_{n=1}^{\infty}$ is a ω-Cauchy sequence. Since M is $\omega-$ Complete, therefore $\{Sx_n\}$ is ω - convergence to some point $x \epsilon\ M$. By property (P), $(Sx_n, x) \epsilon\ E(G_\omega)$ for all n. Thus $(Tx_{n-1}, x) \epsilon\ E(G_\omega)$.

Since T(X_ω) $\subset$ S(X_ω),there exists a point $u \epsilon\ M$ such that $Su = x$.

To prove $Tu = x$. Suppose on the contrary that $Tu \neq x$.

Using the property of ω, we have
$$\omega_1(\,Sx_{n+1}, Tu) = \omega_1(Tx_n, Tu) \leq \omega_1(Sx_n, Su) - \psi(\omega_1(Sx_n, Su))$$
taking limit $n \to \infty$ on both sides, we get
$$\omega_1(x, Tu) \leq \omega_1(x, x) - \psi(\omega_1(x, x))$$
Implies $\omega_1(x, Tu) \leq 0$, a contradiction.

Hence $Tu = x = Su$. Therefore x is a point of coincidence of S and T.

Uniqueness. Let x and z be two fixed point of S and T.

Consider,
$$\omega_1(x, z) = \omega_1(Tx, Tz) \leq \omega_1(Sx, Sz) - \psi(\omega_1(Sx, Sz))$$
This gives $\omega_1(x, z) = 0 \Longrightarrow x = z$. Hence point is unique.

Since S and T are weakly compatible, by lemma 2.2, $x = z$ is a unique common fixed point of S and T.

(ii) Since $M_T \neq \emptyset$, there exists an $x_0 \in M_{S,T}$ and since G_ω is weakly connected, then $[x_0]_{\widetilde{G_\omega}} = M$ and by (i), mapping S and T has a fixed point.

(iii) It follows easily from (i) and (ii).

Remark 3.2. By taking the mapping S in Theorem 3.2 as Ix_{ω},where $Ix_{\omega}$is an identity mapping on X_{ω}, we have following corollary for a weakly contractive mapping.

Corollary 3.3 Let (X, ω) be a modular metric space with a graph G_{ω}. Suppose that ω is a convex regular modular metric which satisfies the Δ_2- type condition. Assume that $M = V(G_{\omega})$ is a nonempty $\omega -$ bounded, $\omega -$ complete subset of X_{ω} and the triple $(M, d_{\omega}^{*}, G_{\omega})$ has property (P). Let $T : M \to M$ be weakly G_{ω} contractive mapping satisfying

$\omega_1(Tx, Ty) \leq \omega_1(x, y) - \psi\big(\omega_1(x, y)\big)$ and

$M_T := \{x, y \in M, (Tx, Ty) \in E(G_{\omega})\}$. If$(Tx_{0}, Ty_0) \in E(G_{\omega})$,

then the following statements hold:

(i) For any $x \in M_T, T|_{[x]_{\widetilde{G_{\omega}}}}$, has a fixed point.

(ii) If G_{ω} is weakly connected, then T has a fixed point in M.

(iii) If $M' = \cup\{[x]_{\widetilde{G_{\omega}}} : x \in M_T\}$, then $T|_{M'}$ has a fixed point in M.

Proof. The proof of the corollary follows immediately from theorem 3.2, by putting the mapping S in Theorem 3.2 as Ix_{ω},where $Ix_{\omega}$is an identity mapping on X_{ω}.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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