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EXISTENCE OF RENORMALIZED SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEM IN MUSIELAK-ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, we prove the existence of renormalized solutions for some class nonlinear elliptic problem of the type

$$-\operatorname{div} a(x,u,\nabla u) + H(x,u,\nabla u) = \mu - \operatorname{div} \phi(u),$$

in the Musielak-Orlicz-Sobolev spaces $W_0^1 L_\varphi(\Omega)$. No Δ_2-condition is assumed on the Musielak function. We assume that $H(x,s,\xi)$ satisfies has a natural growth with respect to its third argument and satisfies the sign condition. The μ is assumed to belong to $L^1(\Omega) + W^{-1}E_\psi(\Omega)$ and $\phi(\cdot) \in C^0(\mathbb{R},\mathbb{R}^N)$ is a continuous function.

Keywords: Musielak-Orlicz-Sobolev spaces; nonlinear elliptic problem; truncations; renormalized solutions.

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1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$ $(N \geq 2)$. This paper is concerned with the existence of renormalized solutions for some class nonlinear elliptic problem of the form:

$$(1.1) \qquad \begin{cases} Au + H(x,u,\nabla u) = \mu - \operatorname{div} \phi(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where A is the Leray-Lions operator defined as:

$$A(u) = -\text{div } a(x,u,\nabla u)$$

and $H(x,s,\xi)$ presents the nonlinearity of the problem (1.1) and satisfies :

$$\begin{aligned} |H(x,s,\xi)| &\leq b(|s|)(d(x)+\varphi(x,|\xi|)), && \text{(natural growth condition)} \\ H(x,s,\xi).s &\geq 0, && \text{(sign condition)} \end{aligned}$$

where $b(\cdot) : I\!R^+ \longmapsto I\!R^+$ is a continuous and non-decreasing function and the nonnegative function $d(x) \in L^1(\Omega)$, $\mu = f - \text{div } F$ belongs to $L^1(\Omega) + W^{-1}E_\psi(\Omega)$ and $\phi(\cdot) \in C^0(I\!R, I\!R^N)$.

The concept of renormalized solutions was introduced by Diperna and Lions in [18] for the study of the Boltzmann equations, this notion of solutions was then adapted to the study of the problem (1.1) by Boccardo et al. in [15] when the right hand side is in $W^{-1,\vec{p}'}(\Omega)$ and in the case where the nonlinearity g depends only on x and u, this work was then studied by Rakotoson in [24] when the right hand side is in $L^1(\Omega)$ and finally by DalMaso et al. in [16] for the case in which the right hand side is general measure data. Some elliptic boundary value problems with $L^1(\Omega)$ or Radon measure data or involving the p-Laplacian have been studied by R$\tilde{a}$dulescu et al. in [25], [26] and[27]. On Orlicz-Sobolev spaces and in variational case, Benkirane and Bennouna have studied in [10] the problem (1.1) where the nonlinearity g depends only on x and u under the restriction that the N-function satisfies the Δ_2- condition, this work was then extended in [1] by Aharouch, Bennouna and Touzani for N-function not satisfying necessarily the Δ_2- condition. If g depends also on ∇u the problem (1.1) has been solved by Aissaoui Fqayeh, Benkirane, El Moumni and Youssfi in [2] without assuming the Δ_2- condition on the N-function. In the framework of variable exponent Sobolev spaces, Bendahmane and Wittbold have treated in [9], they proved the existence and uniqueness of a renormalized solution in Sobolev space with variable exponents $W_0^{1,\vec{p(x)}}(\Omega)$. In [8] Azroul, Barbara, Benboubker and Ouaro have proved the existence of a renormalized solution for some elliptic problem involving the$p(x)$-Laplacian with Neumann nonhomogeneous boundary conditions in the case where the second member f is in $L^1(\Omega)$ Further works for nonlinear elliptic equations with variable exponent can be found in [28] and [29]. In the variational case of Musielak-Orlicz spaces and in the case where $H = 0$ and $\phi = 0$, an existence result for (1.1) has been proved by Benkirane and Sidi El Vally in [11] and then in [12] when the non-linearity g depends only on x and u If g depends also on ∇u the problem (1.1) has recently been solved by Ait Khellou, Benkirane and Douiri in [3] and then in [5] when the right hand side is in $L^1(\Omega)$. M. AL-Hawmi, E.Azroul, H. Hjiaj and A.Touzani have studied (1.1)in [6] the existence of entropy solutions for some anisotropic quasilinear elliptic unilateral when $H = 0$. AL-Hawmi, A. Benkirane, H. Hjiaj and A. Touzani have studied (1.1) in [7] the existence and uniqueness of Entropy Solutions for some Nonlinear Elliptic Unilateral Problems in Musielak-Orlicz-Sobolev spaces when $H = 0$, $\phi = 0$ and $F = 0$. Our main goal, in this paper, is to prove the existence of a renormalized solutions for the problem (1.1) in Musielak-Orlicz space $W^1L_\varphi(\Omega)$. The paper is organized as follows: In section 2, we give some preliminaries and background. Section 3 is devoted to some auxiliary lemmas which can be used to our result. In Section 4, we state our main result and finally give the prove of an existence of a renormalized solutions in section 5.

2. Preliminaries

In this section, we introduce some definitions and known facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [23].

2.1. Musielak-Orlicz function. Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let $\varphi(x,t)$ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:

- (a)**:** $\varphi(x,\cdot)$ is an N-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$, and :

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad , \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$$

- (b)**:** $\varphi(\cdot,t)$ is a measurable function.

A function $\varphi(x,t)$ which satisfies conditions (a) and (b) is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x,t)$ we set $\varphi_x(t) = \varphi(x,t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to t of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:

(c)**:** If there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\gamma(x,t) \leq \varphi(x,ct) \quad \text{for} \quad t \geq T,$$

we write $\gamma \prec\prec \varphi$, and we say that φ dominate γ globally if $T = 0$, and near infinity if $T > 0$.

(d)**:** For every positive constant c and almost everywhere $x \in \Omega$, if

$$\lim_{t \to 0} (\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}) = 0 \quad or \quad \lim_{t \to \infty} (\sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}) = 0.$$

Remark 2.1. [12] *If $\gamma \prec\prec \varphi$ near infinity, then $\forall\ \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have*

$$\gamma(x,t) \leq k(\varepsilon)\varphi(x,\varepsilon t) \quad \forall\ t \geq 0.$$

Remark 2.2. [12] *Let $\psi(x,t)$ is the Musielak-Orlicz function complementary to (or conjugate) of $\varphi(x,t)$ in the sense of Young with respect to the variable s such that*

$$\psi(x,s) = \sup_{t \geq 0}\{st - \varphi(x,t)\}.$$

Remark 2.3. [12] *The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k > 0$ and a nonnegative function $h(\cdot) \in L^1(\Omega)$, such that*

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \qquad a.e. \quad x \in \Omega,$$

for large values of t, or for all values of t.

2.2. Musielak-Orlicz Lebesgue space. In the following, the measurability of a function $u : \Omega \longmapsto I\!R$ means the Lebesgue measurability. We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_{\Omega} \varphi(x, |u(x)|)\, dx$$

where $u : \Omega \longmapsto I\!R$ is a measurable function. The set

$$K_{\varphi}(\Omega) = \{u : \Omega \longmapsto I\!R \quad \text{measurable} \ / \ \rho_{\varphi,\Omega}(u) < +\infty\}$$

is called the Musielak-Orlicz class (the generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longmapsto I\!R \quad \text{measurable} \quad / \ \rho_{\varphi,\Omega}(\frac{u}{\lambda}) \leq \infty, \quad \text{for some } \lambda > 0 \Big\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\Big\{ \lambda > 0 \ / \ \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\, dx \leq 1 \Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |u(x)v(x)|\, dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent [23]. The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [23].

2.3. Musielak-Orlicz-Sobolev space. We now turn to the Musielak-Orlicz-Sobolev space. $W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all measurable functions u such that u and its distributional derivatives up to order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \rho_{\varphi,\Omega}(D^\alpha u) \quad \textit{and} \quad ||u||_{1,\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

for $u \in W^1L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively, and the pair $\langle W^1L_\varphi(\Omega), ||u||_{1,\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [23]:

$$\text{there exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega} \varphi(x,1) \geq c.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by ΠL_φ, we will use the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. The space $W_0^1E_\varphi(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathrm{D}(\Omega)$ in $W^1E_\varphi(\Omega)$, and the space $W_0^1L_\varphi(\Omega)$ as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathrm{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

2.4. Dual space. Let $W^{-1}L_{\psi}(\Omega)$ (resp. $W^{-1}E_{\psi}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\psi}(\Omega)$ (resp. $E_{\psi}(\Omega)$). It is a Banach space under the usual quotient norm. If $\psi(x,t)$ has the Δ_2-condition, then the space $D(\Omega)$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the topology $\sigma(\Pi L_{\varphi}, \Pi L_{\psi})$ (see corollary 1 of [11]).

3. Some technical Lemmas

We present here some lemmas, which will be used later in order to prove the existence theorem:

Lemma 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in (W_0^1L_{\varphi}(\Omega))^N$, then*

$$\int_{\Omega} div\,(u)\,dx = 0$$

Lemma 3.2. *([13]) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions*

(a): *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$,*

(b): *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \geq 1; \tag{3.1}$$

(c):

$$\int_{\Omega} \varphi(x,1)\,dx < \infty; \tag{3.2}$$

(d): *There exists a constant*

$$C > 0 \quad \textit{such that} \quad \psi(x,1) < C \quad \textit{a.e in} \quad \Omega. \tag{3.3}$$

Under this assumptions, $D(\Omega)$ is dense in $L_{\varphi}(\Omega)$ with respect to the modular topology, $D(\Omega)$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence and $D(\overline{\Omega})$ is dense in $W^1L_{\varphi}(\Omega)$ for the modular convergence.

Lemma 3.3. *([2]) Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying*

$$\int_0^\infty \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt=\infty \quad \text{and} \quad \int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}}\,dt<\infty. \tag{3.4}$$

Define a function

$$\varphi_*^{-1}:\Omega\times[0,\infty)\to[0,\infty) \text{ by } \varphi_*^{-1}(x,s)=\int_0^s \frac{\varphi_x^{-1}(\tau)}{\tau^{\frac{N+1}{N}}}\,d\tau \quad \text{for } x\in\Omega \quad \text{and } s\in[0,\infty).$$

and the conditions of Lemma 3.1*. Then*

$$W_0^1L_\varphi(\Omega)\hookrightarrow L_{\varphi_*}(\Omega),$$

where φ_* is the Sobolev conjugate function of φ. Moreover, if ϕ is any Musielak function increasing essentially more slowly than φ_*. near infinity, then the imbedding

$$W_0^1L_\varphi(\Omega)\hookrightarrow L_\phi(\Omega),$$

is compact.

Lemma 3.4. [2] *(Poincaré inequality) Let Ω be a bounded Lipchitz domain of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying the same conditions of Theorem* 3.3*. Then there exists a constant $C>0$ such that*

$$\|u\|_\varphi\le C\|\nabla u\|_\varphi \quad \forall u\in W_0^1L_\varphi(\Omega).$$

Lemma 3.5. [4] *Let be a bounded Lipschitz domain of $\mathbb{R}^N$ and let φ be a Musielak- Orlicz function satisfying the conditions of* (3.1)*. Assume also that the function φ depends only on $N-1$ coordinates of x. Then there exists a constant $\lambda>0$ depending only on Ω such that*

$$\int_\Omega \varphi(x,|v|)\,dx\le\int_\Omega \varphi(x,\lambda|\nabla v|)\,dx \quad \text{for all} \quad v\in W_0^1L_\varphi(\Omega)$$

Lemma 3.6. [19] *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u\in L^1(\Omega)$ such that*

- **(i):** *$u_n\to u$ a.e. in Ω,*
- **(ii):** *$u_n\ge 0$ and $u\ge 0$ a.e. in Ω,*
- **(iii):** $\displaystyle\int_\Omega u_n\,dx\to\int_\Omega u\,dx,$

then $u_n\to u$ in $L^1(\Omega)$.

Lemma 3.7. [11]. *Let $u \in L_\varphi(\Omega)$ and $u_n \in L_\varphi(\Omega)$ with $\|u_n\|_{\varphi,\Omega} \leq C$.*
If $u_n(x) \to u(x)$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L_\varphi(\Omega)$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.

Lemma 3.8. [12] *Let $F : \mathbb{R} \mapsto \mathbb{R}$ be uniformly Lipschitz function, with $F(0) = 0$. Let $\varphi(x,\cdot)$ be a Musielak-Orlicz function and $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of $F'(\cdot)$ is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \;\; in \quad \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e \;\; in \quad \{x \in \Omega : u(x) \in D\}. \end{cases} \tag{3.5}$$

Lemma 3.9. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let φ, ψ and γ be Musielak functions such that $\gamma \prec\prec \psi$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|) \tag{3.6}$$

where k_1, k_2 are real constants and $c(x) \in E_\gamma(\Omega)$.
Then the Nemytskii operator N_f defined by: $N_f(u)(x) = f(x,u(x))$ is strongly continuous from

$$P(E_\varphi(\Omega), 1/k_2) = \{u \in L_\varphi(\Omega) \;\;:\;\; d(u, E_\varphi(\Omega)) < 1/k_2\} \qquad into \qquad E_\gamma(\Omega).$$

Proof.

Let u_n, $u \in P(E_\varphi(\Omega), 1/k_2)$, we suppose that $u_n \to u$ in $P(E_\varphi(\Omega), 1/k_2)$ and we prove that $N_f(u_n) \to N_f(u)$ in $E_\gamma(\Omega)$.

- Firstly, we prove that :

$$\text{for any} \quad u \in P(E_\varphi(\Omega), 1/k_2) \quad \text{we have} \quad N_f(u) \in E_\gamma(\Omega).$$

From (3.6) we have: $|N_f(u)(x)| = |f(x,u(x))| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|u(x)|)$

$$\begin{aligned} \gamma_x(|N_f(u)(x)|) &\leq \gamma_x(c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|u(x)|)) \\ &= \gamma_x(\frac{1}{2}(2c(x)) + \frac{1}{2}(2k_1 \psi_x^{-1} \varphi(x, k_2|u(x)|))) \\ &\leq \frac{1}{2}\gamma_x(2c(x)) + \frac{1}{2}\gamma_x(2k_1 \psi_x^{-1} \varphi(x, k_2|u(x)|)) \end{aligned}$$

since $\gamma \prec\prec \psi$ *i.e.* $\forall\varepsilon > 0$, $\exists\alpha > 0$ such that $\gamma_x(t) \leq \alpha\psi_x(\varepsilon t)$, then:

$$\gamma_x(N_f(u)(x)) \leq \frac{1}{2}\gamma_x(2c(x)) + \frac{\alpha}{2}\psi_x(2\varepsilon k_1 \psi_x^{-1}\varphi(x, k_2|u(x)|))$$

we choice ε as $0 < 2\varepsilon k_1 < 1$, since ψ_x is a convex function, it follows that

$$\begin{aligned}\gamma_x(N_f(u(x))) &\leq \frac{1}{2}\gamma_x(2c(x)) + \frac{\alpha}{2}2\varepsilon k_1 \psi_x\psi_x^{-1}(\varphi(x, k_2|u(x)|)) \\ &\leq \frac{1}{2}\gamma_x(2c(x)) + \alpha\varepsilon k_1 \varphi(x, k_2|u(x)|)\end{aligned}$$

we have $c(x) \in E_\gamma(\Omega)$ and $u \in P(E_\varphi(\Omega), 1/k_2)$ then:

$$\int_\Omega \gamma_x(2c(x))\,dx < \infty \qquad \text{and} \qquad \int_\Omega \varphi(x, k_2|u(x)|)\,dx < \infty$$

and we deduce that: $N_f(u) \in E_\gamma(\Omega)$.

- Secondly, we prove that $N_f(u_n) \to N_f(u)$ in $E_\gamma(\Omega)$:

we have $N_f(u_n)(x) = f(x, u_n(x))$ is a caratheodory function *i.e.* f is continuous for x fixed in Ω. We have supposed that

$$u_n \to u \quad \text{in} \quad P(E_\varphi(\Omega), 1/k_2) \qquad \text{then} \qquad u_n \to u \quad \text{a.e. in} \quad \Omega,$$

then

$$f(x, u_n(x)) \to f(x, u(x)) \qquad \text{a.e. in} \quad \Omega$$

hence

$$\gamma_x(f(x, u_n(x))) \to \gamma_x(f(x, u(x))) \qquad \text{a.e. in} \quad \Omega,$$

and there exists $g \in L^1(\Omega)$ such that $\gamma_x(f(x, u_n(x))) \leq g(x)$ a.e. in Ω, then by using Lebesgue's theorem, we can write:

$$N_f(u_n) \to N_f(u) \qquad \text{in} \quad E_\gamma(\Omega),$$

which achieve the proof of Lemma 3.9.

4. ESSENTIAL ASSUMPTIONS

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, and $\varphi(x,t)$ be a Musielak-Orlicz function. We set $\psi(x,t)$ the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$ and satisfies the condition of Lemma 3.8. Let $\gamma(x,t)$ be a Musielak-Orlicz function such that $\gamma \prec\prec \varphi$. We consider a Leray-Lions operator $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$ given by

$$A(u) = -\text{div } a(x,u,\nabla u)$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$, and continuous with respect to $\xi, \xi^* \in I\!R^N$ for almost every $x \in \Omega$) which satisfies the following conditions

$$|a(x,s,\xi)| \leq k_1(c(x) + \psi_x^{-1}(\gamma(x,k_2|s|)) + \psi_x^{-1}(\varphi(x,k_3|\xi|)), \tag{4.1}$$

$$\big(a(x,s,\xi) - a(x,s,\xi^*)\big) \cdot \big(\xi - \xi^*\big) > 0 \qquad \text{for} \qquad \xi \neq \xi^*, \tag{4.2}$$

$$a(x,s,\xi) \cdot \xi \geq \alpha \cdot \varphi(x, |\ \xi\ |), \tag{4.3}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where $c(x)$ is a nonnegative function lying in $E_\psi(\Omega)$ and $\alpha, \lambda > 0$ and $k_1, k_2, k_3 \geq 0$. The nonlinear terms $H(x,s,\xi)$ is a Carathéodory functions satisfying

$$H(x,s,\xi)s \geq 0, \tag{4.4}$$

$$|H(x,s,\xi)| \leq b(|s|)(d(x) + \varphi\big(x,|\xi|\big)), \tag{4.5}$$

where $b(\cdot) : I\!R^+ \longmapsto I\!R^+$ is a continuous and non-decreasing function and the nonnegative function $d(x) \in L^1(\Omega)$. We consider the problem

$$\begin{cases} Au + H(x,u,\nabla u) = f - \text{ div } F - \text{ div } \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

$$f \in L^1(\Omega), \qquad F \in W^{-1}E_\psi(\Omega) \qquad \text{and} \qquad \phi(\cdot) \in C^0(I\!R, I\!R^N). \tag{4.7}$$

Remark 4.1. *A consequence of (4.3) and the continuity of a with respect to ξ, is that, for almost every x in Ω and s in $I\!R$ such that $a(x,s,0) = 0$.*

5. Main Results

Let $k > 0$, we define the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Definition 5.1. *A measurable function u is called renormalized solutions of the strongly nonlinear problem (4.6) if*

$$(5.1) \qquad \begin{cases} T_k(u) \in W_0^1 L_\varphi(\Omega), \qquad a(x, T_k(u), \nabla T_k(u)) \in \left(L_\psi(\Omega)\right)^N, \\ \displaystyle\int_{\{m \le |u| \le m+1\}} a(x,u,\nabla u)\cdot \nabla u \, dx \to 0 \, as \, m \to 0 \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\theta)\,dx + \int_\Omega H(x,u,\nabla u) h(u)\theta\,dx = \int_\Omega f h(u)\theta\,dx \\ \displaystyle + \int_\Omega \phi(u)\cdot\nabla(h(u)\theta)\,dx + \int_\Omega F\cdot\nabla(h(u)\theta)\,dx, \\ \\ \qquad\qquad \textit{for any} \quad h \in C_c^1(I\!R) \textit{ and for all} \quad \theta \in D(\Omega). \end{cases}$$

Theorem 5.1. *Assuming that* $(4.1)-(4.5)$ *and* (4.7) *holds , then the problem (4.6) has at least one renormalized solution.*

Proof of the Theorem 5.1.

Step 1 : Approximate problems. Let $(f_n)_{n\in I\!N} \in W^{-1}E_\psi(\Omega)$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$ (for example $f_n = T_n(f)$), $\phi_n(s) = \phi(T_n(s))$ and $H_n(x,s,\xi) = T_n(H(x,s,\xi)$. Not that $H_n(x,s,\xi)s \ge 0$, $|H_n(x,s,\xi)| \le |H(x,s,\xi)|$ and $|H_n(x,s,\xi)| \le n$. Since ϕ is continuous, we have $|\phi_n(t)| = |\phi(T_n(t))| \le c_n$. We consider the approximate problem

$$(5.2) \qquad \begin{cases} -\text{div } a(x,u_n,\nabla u_n) + H_n(x,u_n,\nabla u_n) = f_n - \text{ div } F_n - \text{ div } \phi_n(u_n) \qquad \text{in } D'(\Omega), \\ \\ u_n \in W_0^1 L_\varphi(\Omega). \end{cases}$$

There exists at least solution $u_n \in W_0^1 L_\varphi(\Omega)$ of equation (5.2) (see [21], Proposition 1 and [12] Theorem 4).

Step 2 : A priori estimates. taking $v = T_k(u_n)$ as a test function in (5.2), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx + \int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx$$
$$+\int_\Omega \phi(T_n(u_n))\cdot\nabla T_k(u_n)dx + \int_\Omega F_n\cdot\nabla T_k(u_n)dx. \tag{5.3}$$

Remark that, by Lemma 3.1

$$\int_\Omega \phi(T_n(u_n))\cdot\nabla T_k(u_n)dx = \int_\Omega \text{ div } (\Phi_n(u_n))dx = 0, \tag{5.4}$$

where $\Phi_n(s) = \displaystyle\int_0^{T_k(s)} \phi_n(T_n(\tau))\,d\tau, \Phi_n(u_n) \in W_0^1L_\varphi(\Omega)^N$ by Lemma 3.8, which implies, by using the fact that

$$H_n(x,u_n,\nabla u_n)T_k(u_n) \geq 0, \tag{5.5}$$

On the other hand we have

$$\int_\Omega F_n\cdot\nabla T_k(u_n)dx \leq \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla T_k(u_n)|)dx \tag{5.6}$$

from (5.3), (5.4) , (5.6)and by using the hypothesis (5.5) we get

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \leq Ck.$$

where C is a constant such that $\|f_n)\|_{1,\Omega} \leq C, \forall n.$

Thanks to (4.1) one easily has

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)dx \leq \frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\,dx \leq C_1k. \tag{5.7}$$

On the other hand, by using Lemma 3.5. Taking $v = \frac{1}{\lambda}|T_k(u_n)|$ in (5.7) gives

$$\int_\Omega \varphi(x,\frac{1}{\lambda}|T_k(u_n)|)\,dx \leq \int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \leq kC_1$$

Then, we deduce that,

$$
\begin{aligned}
\text{meas}(\{|u_n|>k\}) &\leq \frac{1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_\Omega \varphi(x,\frac{k}{\lambda})\,dx \\
&\leq \frac{1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_\Omega \varphi(x,\frac{1}{\lambda}|T_k(u_n)|)\,dx \\
&\leq \frac{kC_1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}, \quad \forall n, \quad \forall k>0.
\end{aligned}
$$

For all $\delta>0$, we have

$$
\text{meas}\{|u_n-u_m|>\delta\} \leq \text{meas}\{|u_n|>k\}+\text{meas}\{|u_m|>k\}+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.
$$

$$
\text{meas}\{|u_n-u_m|>\delta\} \leq \frac{2kC_1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}. \tag{5.8}
$$

By using (5.7) and Lemma 3.4, we deduce that $T_k(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$, and then there exists $w_k \in W_0^1L_\varphi(\Omega)$ such that $T_k(u_n) \rightharpoonup w_k$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ strongly in $E_\varphi(\Omega)$ and a.e. in Ω Consequently, we can assume that $T_k(u_n)$ is a cauchy sequence in measure in Ω.

Let $\varepsilon>0$, using (5.8) and the fact that $\frac{2kC_1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})} \to 0$ as$k\to\infty$ there exists some $k=k(\varepsilon)\geq 0$ such that $\text{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon \quad \forall n,m\geq n_0(k(\varepsilon),\delta)$, it follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$
T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad W_0^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi E_\psi)
$$

it follows that

$$
T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad E_\varphi(\Omega) \quad \text{a.e.} \quad in \quad \Omega \tag{5.9}
$$

Now, we shall prove that $a(x,T_k(u_n),\nabla T_k(u_n))_n$ is bounded in $(L_\psi(\Omega))^N$ for all $k>0$, by using the dual norm of $(L_\psi(\Omega))^N$. Let $v_0\in(E_\varphi(\Omega))^N$ such that $\|v_0\|_{\varphi,\Omega}=1$. We have from (4.2)

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\frac{v_0}{k_3}))\cdot(\nabla T_k(u_n)-\frac{v_0}{k_3}))dx\geq 0$$

this implies by (5.7)

$$\begin{aligned}\int_\Omega \frac{1}{k_3}(a(x,T_k(u_n),\nabla T_k(u_n))v_0dx &\leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx\\ &-\int_\Omega a(x,T_k(u_n),\frac{v_0}{k_3}))\cdot(\nabla T_k(u_n)-\frac{v_0}{k_3}))dx\\ &\leq Ck-\int_\Omega a(x,T_k(u_n),\frac{v_0}{k_3}))\cdot\nabla T_k(u_ndx\\ &+\tfrac{1}{k_3}\int_\Omega a(x,T_k(u_n),\frac{v_0}{k_3}))v_0dx\end{aligned}$$

By using Young's inequality in the last two terms of the last side and (5.7) we have

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)v_0dx &\leq Ckk_3+3k_1(1+k_3)\int_\Omega \psi(x,\frac{a(x,T_k(u_n),\frac{v_0}{k_3})}{3k_1})dx\\ &+\int_\Omega \varphi(x,|\nabla T_k(u_n)|)dx+\int_\Omega \varphi(x,|v_0|)dx\\ &\leq Ckk_3+3C_1kk_1k_3+3k_1\\ &+3k_1(1+k_3)\int_\Omega \psi(x,\frac{a(x,T_k(u_n),\frac{v_0}{k_3})}{3k_1})dx\end{aligned}$$

Using (4.1) and the convexity of ψ yields

$$\psi(x,\frac{|a(x,T_k(u_n),\frac{v_0}{k_3})|}{3k_1})\leq\frac{1}{3}\psi(x,c(x))+\gamma(x,k_2T_k(u_n))+\varphi(x,|v_0|)$$

and, since γ grows essentially less rapidly than φ near infinity there exists $\mu(k)>0$ such that $\gamma(x,k_2T_k(u_n))\leq\gamma(x,k_2k)\leq\mu(k)\varphi(x,1)$ Lemma 3.1 then we have by integrating over Ω and using (3.2)

$$\int_\Omega \psi(x,\frac{|a(x,T_k(u_n),\frac{v_0}{k_3})|}{3k_1})\leq\frac{1}{3}(\int_\Omega \psi(x,c(x))+\mu(k)\int_\Omega \varphi(x,1)+\int_\Omega \varphi(x,|v_0|))\leq C_k$$

where C_k is a constant depending on k, we deduce that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))v_0dx\leq C_k\quad \forall v_0\in(E_\varphi(\Omega))^N \text{ with } \|v_0\|\leq 1$$

which shows that $(a(x,T_k(u_n),\nabla T_k(u_n))_n$ is bounded in $(L_\psi(\Omega))^N$.

Step 3 : Almost everywhere convergence of the gradients. Let $\eta(t)=t.\exp(\sigma t^2)$,$\sigma>0$ where $\sigma \geq \left(\frac{b(k)}{2\alpha}\right)^2$ one has

$$\eta'(t)-\frac{b(k)}{\alpha}|\eta(t)| \geq \frac{1}{2} \qquad \forall t \in \mathbb{R}. \tag{5.10}$$

Where $k>0$ is a fixed real number which will be used as a level of the truncation.
Let $v_j \in D(\Omega)$ be a sequence which converges to $T_k(u)$ for the modular convergence $W_0^1L_\varphi(\Omega)$ and define the function

$$\rho_m(s)=\begin{cases} 1 & \text{if} \quad |s|\leq m \\ 0 & \text{if} \quad |s|\geq m+1 \\ m+1-|s| & \text{if} \quad m\leq |s|\leq m+1. \end{cases}$$

Where $m>k$.

$$\text{Let } \theta_n^j = T_k(u_n)-T_k(v_j), \qquad \theta^j = T_k(u)-T_k(v_j) \quad \text{and} \quad z_{n,m}^j=\eta(\theta_n^j)\rho_m(u_n)$$

Using in (5.2) the test function $z_{n,m}^j$ gives

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j\,dx+\int_\Omega H_n(x,u_n,\nabla u_n)z_{n,m}^j\,dx \\ &=\int_\Omega f_n z_{n,m}^j\,dx+\int_{m\leq|u_n|\leq m+1}\phi_n(u_n)\cdot\nabla u_n\rho_m'(u_n)\eta(T_k(u_n)-T_k(v_j))dx \\ &+\int_\Omega \phi_n(u_n))\cdot\nabla\eta(T_k(u_n)-T_k(v_j))\rho_m(u_n)dx+\int_\Omega F_n\cdot\nabla z_{n,m}^j dx. \end{aligned} \tag{5.11}$$

In the sequel, we denote by $\varepsilon_i(n,j)$, $i=1,2,\dots$ various real-valued functions of real variables that converge to 0 as $n\to\infty$ and j tends to infinity, i.e. $\lim\limits_{j\to\infty}\lim\limits_{n\to\infty}\varepsilon_i(n,j)=0$.
In view of (5.9), we have $z_{n,m}^j\to\eta(\theta^j)\rho_m(u)$ weakly* in $L^\infty(\Omega)$ as $n\to\infty$ and then

$$\int_\Omega f_n z_{n,m}^j\,dx \to \int_\Omega f\eta(\theta^j)\rho_m(u)\,dx \to 0 \quad \text{as} \quad n\to\infty,$$

and since $\theta^j\to 0$ weakly* in $L^\infty(\Omega)$ we get $\int_\Omega f\eta(\theta^j)\rho_m(u)\,dx\to 0$ as $j\to\infty$, then

$$\int_\Omega f_n z_{n,m}^j\,dx=\varepsilon_0(n,j).$$

By Lemma 3.1, it's easy to see that

$$\int_{m\le|u_n|\le m+1}\phi_n(u_n)\cdot\nabla u_n\rho'_m(u_n)\eta(T_k(u_n)-T_k(v_j))dx=0$$

Concerning the third term in the left-hand side of (5.11) we can write

$$\begin{aligned}\int_\Omega\phi_n(u_n))\cdot\nabla\eta(T_k(u_n)-T_k(v_j))\rho_m(u_n)dx &= \int_\Omega\phi_n(u_n))\cdot\nabla T_k(u_n)\eta'(\theta_n^j)\rho_m(u_n)dx\\ &-\int_\Omega\phi_n(u_n))\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)dx.\end{aligned}$$

By Lemma 3.1, it's easy to see that

$$\int_\Omega\phi_n(u_n))\cdot\nabla T_k(u_n)\eta'(\theta_n^j)\rho_m(u_n)dx=0$$

From (5.9) we have $\phi_n(u_n))\eta'(\theta_n^j)\rho_m(u_n)\to\phi(u))\eta'(\theta^j)\rho_m(u)$ almost everywhere in Ω as $n\to\infty$,furthermore, we can check that

$$\|\phi_n(u_n))\eta'(\theta_n^j)\rho_m(u_n)\|_\psi\le c_mc_1\eta'(2k)|\Omega|$$

Where $c_m=\max\limits_{|t|\le m+1}\phi(t)$ and c_1 is the constant defined in (3.3). Applying [25, Theorem 14.6] we get

$$\lim_{n\to\infty}\int_\Omega\phi_n(u_n))\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)dx=\int_\Omega\phi(u))\cdot\nabla T_k(v_j)\eta'(\theta^j)\rho_m(u)dx$$

and by using the modular convergence of v_j, we obtain

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega\phi_n(u_n))\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)dx=\int_\Omega\phi(u))\cdot\nabla T_k(u)\rho_m(u)dx$$

then, by Lemma 3.1, one has $\int_\Omega\phi(u))\cdot\nabla T_k(u)\rho_m(u)dx=0$.

Hence

$$\int_\Omega\phi_n(u_n))\cdot\nabla\mu(T_k(u_n)-T_k(v_j))\rho_m(u_n)dx=\varepsilon_2(n,j),$$

similarly we have

$$\int_\Omega F_n\cdot\nabla z_{n,m}^j\,dx=\varepsilon_1(n,j).$$

Since $H_n(x,u_n,\nabla u_n)z_{n,m}^j\ge0$ on the subset $\{x\in\Omega:|u_n(x)|>k\}$ and $\rho_m(u_n)=1$ on the subset $\{x\in\Omega:|u_n(x)|\ge k\}$ we have, from (5.11),

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j\,dx+\int_{\{|u_n|\le k\}}H_n(x,u_n,\nabla u_n)\eta(\theta_n^j)\,dx\le\varepsilon_2(n,j). \tag{5.12}$$

For what concerns the first term of the left-hand side of (5.12) we have

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z^j_{n,m}\,dx &= \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j))\eta'(\theta^j_n)\rho_m(u_n)\,dx\\ &-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\rho_m(u_n)\,dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\eta(\theta^j_n)\rho'_m(u_n)\,dx\\ &=\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j))\eta'(\theta^j_n)\,dx\\ &-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\rho_m(u_n)\,dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\eta(\theta^j_n)\rho'_m(u_n)\,dx,\end{aligned}$$

and then

(5.13)
$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z^j_{n,m}\,dx &= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\\ &\times(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j)\eta'(\theta^j_n)\,dx\\ &+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j))\eta'(\theta^j_n)\,dx\\ &-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n)\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\,dx\\ &-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\rho_m(u_n)\,dx\,dx\\ &+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n)\eta(\theta^j_n)\rho'_m(u_n)\,dx,,\end{aligned}$$

where χ^s_j is the characteristic function of the set $\Omega^s_j=\{x\in\Omega : |\nabla T_k(v_j)|\le s\}$.
For the third term, since $(a(x,T_k(u_n),\nabla T_k(u_n))_n$ is bounded in $(L_\psi(\Omega))^N$, we have, for a subsequence, $a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup l_k$ weakly in $(L_\psi(\Omega))^N$ for $\sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega))$ with $l_k\in(L_\psi(\Omega))^N$ and since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^s_j}\in(E_\varphi(\Omega))^N$ we have, by letting $n\to\infty$

$$-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\,dx\to-\int_{\Omega\setminus\Omega^s_j} l_k\cdot\nabla T_k(u))\eta'(\theta^j)\,dx,$$

Using now, the modular convergence of (v_j), we get

$$-\int_{\Omega\setminus\Omega^s_j} l_k\cdot\nabla T_k(v_j)\eta'(\theta^j)\,dx\to-\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx \quad\text{as}\quad j\to\infty,$$

where $\Omega_s=\{x\in\Omega : |\nabla T_k(u)|\le s\}$. We have then proved that

$$-\int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\eta'(\theta^j_n)\,dx\to-\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+\varepsilon_3(n,j). \tag{5.14}$$

Concerning the fourth term, since $\rho_m(u_n)=0$ on the subset $\{|u_n|>m+1\}$, we have

$$-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)\,dx$$
$$=-\int_{\{|u_n|>k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)\,dx$$

and as above

$$\begin{aligned}&-\int_{\{|u_n|>k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_k(v_j)\eta'(\theta_n^j)\rho_m(u_n)\,dx\\&=-\int_{\{|u|>k\}} l_{m+1}\cdot\nabla T_k(u)\rho_m(u)\,dx+\varepsilon_4(n,j)=\varepsilon_5(n,j)\end{aligned}\tag{5.15}$$

where we have used the fact that $\nabla T_k(u)=0$ on the subset $\{x\in\Omega:|u(x)|>k\}$.

For the second term of (5.13), remark that by using Lemma 3.9 and the fact that $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L_\varphi(\Omega))^N$, by (5.9), we have

$$a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\eta'(\theta_n^j)\to a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\eta'(\theta^j)$$

strongly in $(E_\psi(\Omega))^N$ as $n\to\infty$, then

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s))\eta'(\theta_n^j)\,dx$$

$$\to\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s))\eta'(\theta^j)\,dx\quad\text{as}\quad n\to\infty$$

on the other hand, since $\nabla T_k(v_j)\chi_j^s\to\nabla T_k(u)\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j\to\infty$, it is easy to see that

$$\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s))\eta'(\theta^j)\,dx\to 0\quad\text{as}\quad j\to\infty,$$

where χ^s is the characteristic function of the set Ω_s then

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s))\eta'(\theta_n^j)\,dx=\varepsilon_6(n,j).\tag{5.16}$$

The last term of (5.13) reads as

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\eta(\theta_n^j)\rho'(u_n)\,dx=\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\eta(\theta_n^j)\rho'(u_n)\,dx,$$

then

$$|\int_{\Omega} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\eta(\theta_n^j)\rho'(u_n)\,dx| \leq \eta(2k)\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx$$

Taking $T_1(u_n - T_m(u_n))$ as test function in (5.3) yields

$$\begin{aligned}&\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx + \int_{\{|u_n|>m\}} H_n(x,u_n,\nabla u_n)T_1(u_n-T_m(u_n))\,dx\\ &= \int_{\{|u_n|>m\}} f_nT_k(u_n)\,dx + \int_{\{m\leq|u_n|\leq m+1\}} \phi(T_n(u_n))\cdot\nabla T_k(u_n)dx + \int_{\{m\leq|u_n|\leq m+1\}} F_n\cdot\nabla T_k(u_n)dx.\end{aligned}$$

Thanks to Lemma 3.1 we have

$$\int_{\{m\leq|u_n|\leq m+1\}} \phi(T_n(u_n))\cdot\nabla T_k(u_n)dx = 0$$

$$\int_{\{m\leq|u_n|\leq m+1\}} F_n\cdot\nabla T_k(u_n)dx = 0$$

which implies, by using the fact that $H_n(x,u_n,\nabla u_n)T_1(u_n - T_m(u_n)) \geq 0$ on the subset $\{x\in\Omega : |u_n|\geq m\}$

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \leq \int_{\{|u_n|>m\}} |f_n|\,dx. \tag{5.17}$$

consequently

$$|\int_{\Omega} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,\eta(\theta_n^j)\rho_m'(u_n)\,dx| \leq \eta(2k)\int_{\{|u_n|\geq m\}} |f_n|\,dx$$

Combining this inequality with (5.14), (5.15) and (5.16) we obtain

$$\begin{aligned}\int_{\Omega} a(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j\,dx &\geq -\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx - \eta(2k)\int_{\{|u_n|\geq m\}}|f_n|\,dx\\ &+\int_{\Omega}(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\\ &\times[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\eta'(\theta_n^j)\,dx + \varepsilon_7(n,j)\end{aligned} \tag{5.18}$$

Concerning the second term of the left-hand side of (5.12), we have

$$\begin{aligned}|\int_{\{|u_n|\leq k\}} H_n(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j\,dx| &= |\int_{\{|u_n|\leq k\}}(H_n(x,T_k(u_n),\nabla T_k(u_n))\eta'(\theta_n^j)\,dx|\\ &\leq \int_{\Omega} b(k)c\prime|\eta(\theta_n^j|dx + b(k)\int_{\Omega}\varphi(x,|\nabla T_k(u_n)||\eta(\theta_n^j)|dx\\ &\leq \varepsilon_8(n,j) + \tfrac{b(k)}{\alpha}\int_{\Omega} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\eta'(\theta_n^j)|\,dx\end{aligned}$$

We can write the last term of the last side of this inequality as

$$
\begin{aligned}
&\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\\
&\qquad\qquad \times(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)|\eta(\theta_n^j)|\,dx\\
&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s))|\eta(\theta_n^j)|\,dx\\
&-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)\cdot\nabla T_k(v_j)\chi_j^s|\eta(\theta_n^j)|\,dx,
\end{aligned}\tag{5.19}
$$

we argue as above to show that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s))|\eta(\theta_n^j)|\,dx=\varepsilon_8(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)\cdot\nabla T_k(v_j)\chi_j^s|\eta(\theta_n^j)|\,dx=\varepsilon_9(n,j)$$

then

$$
\begin{aligned}
&|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j\,dx|\\
&\le\frac{b(k)}{\alpha}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\\
&\qquad\qquad \times(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)|\eta(\theta_n^j)|\,dx+\varepsilon_{10}(n,j)
\end{aligned}
$$

Combining this with (5.12) and (5.19), we obtain

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\cdot[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)]\\
&\times(\eta'(\theta_n^j)-\tfrac{b(k)}{\alpha}|\eta(\theta_n^j)|)\,dx\le\varepsilon_{11}(n,j)+\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+\eta(2k)\int_{\{|u_n|\ge m\}}|f_n|\,dx,
\end{aligned}
$$

and by using (5.10) we deduce that

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\cdot[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)]\\
&\times(\eta'(\theta_n^j)-\tfrac{b(k)}{\alpha}|\eta(\theta_n^j)|)\,dx\le 2\varepsilon_{11}(n,j)+2\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+2\eta(2k)\int_{\{|u_n|\ge m\}}|f_n|\,dx,
\end{aligned}\tag{5.20}
$$

On the other hand

$$
\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)]\cdot[\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx\\
&= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\cdot[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)]\,dx\\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot[\nabla T_k(v_j)\chi_j^s) - \nabla T_k(u)\chi^s)]\,dx\\
&- \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\cdot[\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx\\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)]\,dx.
\end{aligned}
$$

We shall pass to the limit in n and in j in the last three terms of the right-hand side of the above equality. Similar tools as in (5.13) and (5.19) gives

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot[\nabla T_k(v_j)\chi_j^s) - \nabla T_k(u)\chi^s)]\,dx = \varepsilon_{12}(n,j)$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\cdot[\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx = \varepsilon_{13}(n,j)$$

(5.21)

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)]\,dx = \varepsilon_{14}(n,j),$$

Which implies that

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)]\cdot[\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx$$

$$= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\cdot[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)]\,dx$$

$$+\varepsilon_{15}(n,j),$$

For $r \le s$, one has

$$0 \le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\,dx$$

$$\le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\,dx$$

$$= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx$$

$$\le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)\chi^s)]\,dx$$

$$= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)] \cdot [\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_j)]\,dx$$

$$+\varepsilon_{15}(n,j) \le \varepsilon_{16}(n,j) + 2\int_{\Omega\setminus\Omega_s} l_k \cdot \nabla T_k(u)\,dx + 2\eta(2k)\int_{\{|u_n|\ge m\}} |f_n|\,dx,$$

This implies that, by passing at first to the limit sup over n and then over j,

$$0 \le \limsup_{n\to\infty} \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\,dx$$

$$\le 2\int_{\Omega\setminus\Omega_s} l_k \cdot \nabla T_k(u)\,dx + 2\eta(2k)\int_{\{|u_n|\ge m\}} |f_n|\,dx$$

Letting s and $m \to 1$ and using the fact that $l_k \cdot \nabla T_k(u) \in L^1(\Omega)$ we get, since $|\Omega\setminus\Omega_s| \to 0$ and $|\{|u_n| \ge m\}| \to 0$

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\,dx \to 0 \text{ as } n \to \infty.$$

As in [14] , we deduce that there exists a subsequence, still denoted by u_n, such that

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ a.e in } \Omega. \tag{5.22}$$

which implies that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,T_k(u),\nabla T_k(u)) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega)),\ \forall k > 0. \tag{5.23}$$

Step 4 : Modular convergence of the truncations. Going back to the equation (5.20), we can write

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot\nabla T_k(u_n)dx \leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]dx$$

$$\leq 2\varepsilon_{11}(n,j)+2\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+2\eta(2k)\int_{\{|u_n|\geq m\}}|f_n|\,dx,$$

then, by using (5.21), we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot\nabla T_k(u_n)dx \leq \varepsilon_{17}(n,j)+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+2\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+2\eta(2k)\int_{\{|u_n|\geq m\}}|f_n|\,dx,$$

Passing to the limit sup over n in both sides of this inequality yields

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot\nabla T_k(u_n)dx \leq \lim_{n\to\infty}\varepsilon_{17}(n,j)+\int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+2\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+2\mu(2k)\int_{\{|u|\geq m\}}|f_n|\,dx,$$

when $j\to\infty$, we obtain

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot\nabla T_k(u_n)dx \leq \int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)\chi^s dx$$

$$+2\int_{\Omega\setminus\Omega_s} l_k\cdot\nabla T_k(u)\,dx+2\mu(2k)\int_{\{|u|\geq m\}}|f|\,dx,$$

Letting s and $m\to\infty$ gives

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx \leq \int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx,$$

then by using Fatou's Lemma we have

$$\int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx \leq \liminf_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx,$$

consequently

$$\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx=\int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx,$$

and, by using Lemma 3.6, we conclude that

$$a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\to a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) \text{ in } L^1(\Omega).,\tag{5.24}$$

The convexity of the Musielak-Orlicz function φ and (9) allow us to get

$$\begin{aligned}\varphi(\frac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2}) &\leq \frac{1}{2\alpha}a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\\ &+\frac{1}{2\alpha}a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u),\end{aligned}$$

and by (5.24) we obtain

$$\lim_{|E|\to 0}\sup_n\int_\Omega \varphi(\frac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2})dx=0$$

which implies, by using Vitali's theorem, that

$$T_k(u_n)\to T_k(u) \text{ in } W_0^1L_\varphi(\Omega) \text{ for the modular convergence } \forall k>0$$

Step 5 : Equi-integrability of the non-linearities. We shall prove that $H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Thanks to (5.22) we have $H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u)$ a.e inΩ , so it suffices to prove that $H_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω.

Let $E\subset\Omega$ be a measurable subset of Ω. We have for any $m>1$,

$$\int_E|H_n(x,u_n,\nabla u_n)|\,dx=\int_{E\cap\{|u_n|\leq m\}}|H_n(x,u_n,\nabla u_n)|\,dx+\int_{E\cap\{|u_n|>m\}}|H_n(x,u_n,\nabla u_n)|\,dx$$

Taking

$$T_1(u_n-T_{m-1}(u_n))=\begin{cases}0 & \text{if} \quad |u_n|\leq m-1\\ sgn(u_n) & \text{if} \quad |u_n|>m\\ u_n-(m-1)sgn(u_n) & \text{if} \quad m-1\leq|u_n|\leq m.\end{cases}$$

as test function in (5.2), gives

$$\int_{\{m-1\le|u_n|\le m\}} a(x,u_n,\nabla u_n)\cdot\nabla_n dx+\int_{\{|u_n|>m-1\}} H_n(x,u_n,\nabla u_n)T_1(u_n-T_{m-1}(u_n))\,dx$$

$$=\int_{\{|u_n|>m-1\}} f_nT_1(u_n-T_{m-1}(u_n))\,dx+\int_{\{m-1\le|u_n|\le m\}}\phi(T_n(u_n))\cdot\nabla u_n dx+\int_{\{m-1\le|u_n|\le m\}} F_n\cdot\nabla u_n dx.$$

consequently

$$\int_{\{|u_n|>m-1\}}|H_n(x,u_n,\nabla u_n)|\,dx\le\int_{\{|u_n|>m-1\}}|f_n|\,dx$$

Let $\varepsilon>0$, there exists $m=m(\varepsilon)>1$ such that

$$\int_{E\cap\{|u_n|>m\}}|H_n(x,u_n,\nabla u_n)|\,dx\le\frac{\varepsilon}{2},\quad\forall n$$

On the other hand

$$\begin{aligned}\int_{E\cap\{|u_n|\le m\}}|H_n(x,u_n,\nabla u_n)|\,dx &\le\int_E|H_n(x,T_m(u_n),\nabla T_m(u_n)|\,dx\\ &\le b(m)\int_E(d(x)+\varphi(x,|\nabla T_m(u_n)|)dx\\ &\le\frac{b(m)}{\alpha}\int_E a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dx\\ &+b(m)\int_E d(x)dx,\end{aligned}$$

By virtue of the strong convergence (5.24) and the fact that $d\in L^1(\Omega)$, there exists $\nu>0$ such that

$$|E|<\nu\text{ implies}\qquad\int_{E\cap\{|u_n|\le m\}}|H_n(x,u_n,\nabla u_n)|\,dx\le\frac{\varepsilon}{2},\quad\forall n$$

So that

$$|E|<\nu\text{ implies}\qquad\int_E|H_n(x,u_n,\nabla u_n)|\,dx\le\varepsilon,\quad\forall n$$

which shows that $H_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω. By Vitali's theorem, we conclude that $H(x,u_n,\nabla u_n)\in L^1(\Omega)$

$$H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u)\text{ in }L^1(\Omega).,\tag{5.25}$$

Step 6 : Passage to the limit. Turning to the inequality (5.17), we have for the first term

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx &= \int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))dx\\ &= \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_{m+1}(u_n))dx\\ &-\int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dx.\end{aligned}$$

then by (5.24) we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n dx &= \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_{m+1}(u_n)dx\\ &-\int_\Omega a(x),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dx\\ &= \int_\Omega a(x,u,\nabla u)\cdot(\nabla T_{m+1}(u)-\nabla T_m(u))dx\\ &= \int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\cdot\nabla u dx.\end{aligned}$$

Consequently, by letting n to infinity in (5.17) we get

$$\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\cdot\nabla u dx \leq \int_{\{|u|\geq m\}}|f|dx.$$

we take $m\to\infty$, we obtain

$$\lim_{m\to\infty}\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\cdot\nabla u dx = 0. \tag{5.26}$$

Now, from (5.24) and Lemma 3.6 we deduce that

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \to a(x,u,\nabla u)\cdot\nabla u \text{ in } L^1(\Omega).. \tag{5.27}$$

Let $h\in C_c^1(I\!R)$ and $\theta\in D(\Omega)$. Taking $h(u_n)\theta$ as test function in (5.2), we get

(5.28)

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n h'(u_n)\theta\,dx+\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla h(u_n)\theta\,dx+\int_\Omega H_n(x,u_n,\nabla u_n)h(u_n)\theta\,dx\\ &=\int_\Omega f_n h(u_n)\theta\,dx+\int_\Omega \phi_n(u_n)\cdot\nabla(h(u_n)\theta)\,dx+\int_\Omega F\cdot\nabla(h(u_n)\theta)\,dx\end{aligned}$$

Since h and h' have compact support in $I\!R$ there exists ε such that $supp h\subset[-\varepsilon,\varepsilon]$ and $supp h'\subset[-\varepsilon,\varepsilon]$ then for $n>\varepsilon$ we can write

$$\phi_n(t)h(t)) = \phi(T_n(t))h(t) = \phi(T_\varepsilon(t))h(t)$$

$$\phi_n(t)h'(t)) = \phi(T_n(t))h'(t) = \phi(T_\varepsilon(t))h'(t)$$

Moreover, the functions ϕh and $\phi h'$ belong to $(C^0(\mathbb{R})\cap L^\infty(\mathbb{R}))^N$ Since $u_n \in W_0^1L_\varphi(\Omega)$ there exists two positive constants μ_1, μ_2 such that

$$\int_\Omega \varphi(x, \frac{|\nabla u_n|}{\mu_1})dx \leq \mu_2$$

Let β be a positive constant such that $\|h(u_n)\nabla\theta\|_\infty \leq \beta$ and $\|h'(u_n)\theta\|_\infty \leq \beta$ For δ large enough, we have

$$\begin{aligned}
\int_\Omega \varphi(x, \frac{|\nabla(h(u_n)\theta)|}{\delta})dx &\leq \int_\Omega \varphi(x, \frac{|h(u_n)\nabla\theta| + |h'(u_n)\theta||\nabla u_n|}{\delta})dx \\
&\leq \int_\Omega \varphi(x, \frac{\beta + \frac{\beta\mu_1|\nabla u_n|}{\mu_1}}{\delta})dx \\
&\leq \int_\Omega \varphi(x, \frac{\beta}{\delta})dx + \frac{\beta\mu_1}{\delta}\int_\Omega \varphi(x, \frac{\nabla u_n|}{\mu_1})dx \\
&\leq \int_\Omega \varphi(x,1)dx + \frac{\beta\mu_1\mu_2}{\delta} \leq C
\end{aligned}$$

which implies that $h(u_n)\theta)$ is bounded in $W_0^1L_\varphi(\Omega)$ and then we deduce that

$$h(u_n)\theta \rightharpoonup h(u)\theta \text{ weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)). \tag{5.29}$$

On the other hand, for any measurable subset E of Ω we have

$$\begin{aligned}
\|\phi(T_\varepsilon(u_n)\chi_E\|_\psi &= \sup_{\|v\|_\varphi\leq 1} |\int_E \phi(T_\varepsilon(u_n))vdx| \\
&\leq c_\varepsilon \sup_{\|v\|_\varphi\leq 1} \|\chi_E\|_\psi\|v\|_\varphi dx \\
&\leq c_\varepsilon \frac{1}{M^{-1}\frac{1}{|E|}}dx
\end{aligned}$$

where $c_\varepsilon = \max_{|t|\leq\varepsilon} \phi(t)$ and M is the N-function defined by $M = \sup_{x\in\Omega} \psi(x,t)$ then

$$\lim_{|E|\to\infty} \sup_n \|\phi(T_\varepsilon(u_n)\chi_E\|_\psi = 0$$

consequently from (5.9) and by using [[22], Lemma 11.2] we obtain

$$\phi(T_\varepsilon(u_n)) \to \phi(T_\varepsilon(u)) \text{ strongly in } \ (E_\psi(\Omega))^N \tag{5.30}$$

It follows that by (5.29) and (5.30)

$$\int_\Omega \phi_n(u_n) \cdot \nabla(h(u_n)\theta)\,dx \to \int_\Omega \phi(u) \cdot \nabla(h(u)\theta)\,dx \text{ as } n \to \infty$$

and

$$\int_\Omega F_n \cdot \nabla(h(u_n)\theta)\,dx \to \int_\Omega F \cdot \nabla(h(u)\theta)\,dx \text{ as } n \to \infty$$

For the first term of (5.28), we have

$$|a(x,u_n,\nabla u_n) \cdot \nabla u_n h'(u_n)\theta| \le \beta a(x,u_n,\nabla u_n) \cdot \nabla u_n$$

So, by using Vitali's theorem and (5.27) we get

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla u_n\, h'(u_n)\theta\,dx \to \int_\Omega a(x,u,\nabla u) \cdot \nabla u\, h'(u)\theta\,dx$$

Concerning the second term of (5.28), we have

$$h(u_n)\nabla\theta \to h(u)\nabla\theta \text{ strongly in } \ (E_\varphi(\Omega))^N$$

and

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L_\psi(\Omega))^N \text{ for } \sigma(\Pi L_\psi(\Omega), \Pi E_\varphi(\Omega)).$$

then

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla\theta\, h(u_n)\,dx \to \int_\Omega a(x,u,\nabla u) \cdot \nabla\theta\, h(u)\,dx$$

Since $h(u_n)\theta \to h(u)\theta$ weakly in $L^\infty(\Omega))^N$ for $\sigma^*(L^\infty(\Omega), L^1(\Omega))$ and by using (5.24), we have

$$\int_\Omega H_n(x,u_n,\nabla u_n)\, h(u_n)\theta\,dx \to \int_\Omega H(x,u,\nabla u)\, h(u)\theta\,dx$$

and

$$\int_\Omega f_n\, h(u_n)\theta\,dx \to \int_\Omega f\, h(u)\theta\,dx$$

Finally, we can easily pass to the limit in each term of (5.28) and obtain

$$\int_\Omega a(x,u,\nabla u)\cdot[h'(u)\theta\nabla u+h(u)\nabla\theta]\,dx+\int_\Omega H(x,u,\nabla u)h(u)\theta\,dx=\int_\Omega fh(u)\theta\,dx$$
$$+\int_\Omega \phi(u)\cdot[h'(u)\theta\nabla u+h(u)\nabla\theta]\,dx+\int_\Omega F\cdot[h'(u)\theta\nabla u+h(u)\nabla\theta]\,dx,$$

which completes the proof of the Theorem 5.1.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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