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PERIODIC SOLUTIONS FOR AN IMPULSIVE PREDATOR-PREY MODEL WITH HOLLING TYPE FUNCTIONAL RESPONSE AND TIME DELAYS

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Abstract. In this paper we propose an impulsive predator-prey model with time delays. By applying the continuation theorem of coincidence degree theory, we establish a better estimation on the difference between the supremum and infimum of a differentiable piecewise continuous periodic function.

Keywords: impulsive Predator-prey; Holling-type functional response; time delay.

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1. Introduction

One of the powerful and effective methods on the existence of periodic solutions to periodic systems is the continuation method, which gives easily verifiable suffient conditions. In [5] Bazykin proposed the following Predator-prey system.

(1)
$$
\begin{aligned}
u'(t) &= u(t)\left(a - \varepsilon u(t) - \frac{bv(t)}{1+\alpha u(t)}\right)\\
v'(t) &= v(t)\left(-c + \frac{du(t)}{1+\alpha u(t)} - \eta v(t)\right)
\end{aligned}
$$

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where $u(t), v(t)$ represents the densities of prey and predator populution respectively where $a, b, c, d, \alpha, \eta, \varepsilon$ are positive parameters. System(1) is called Holling Type II predator system model. It is investigated in [5] for the stability of equilibrium and condimension two bifurcations.The global behavior of system(1) has been discussed by many authors, for example [3], [16]. In [2] the effects of the periodicity of eco-logical and environmental parameters and time delays due to gestation and negative feedbacks on the global dynamics of predator-prey systems with Holling-type-II functional response.

$$
(2) \qquad \begin{aligned}
x_1'(t) &= x_1(t)\Big[r_1(t) - a_{11}(t)x_1(t-\tau_1(t)) - \frac{a_{12}(t)x_2(t)}{1+mx_1(t)}\Big] \\
x_2'(t) &= x_2(t)\Big[-r_2(t) + \frac{a_{21}(t)x_1(t-\tau_2(t))}{1+mx_1(t-\tau_2(t))} - a_{22}(t)x_2(t-\tau_3(t))\Big]
\end{aligned}
$$

where $a_{11}, a_{12}, a_{21}, a_{12}, \tau$ are continous ω-periodic functions $\tau_1, \tau_3 \geq 0$ denote the time delays due to negative feedbacks of the prey and the predator populationτ_2 is a time delay due to gestation that is mature adult predators can only contribute to the reproduction of predator biomass $a_{11}(t), a_{22}(t)$ are the intra-specific rates of the prey and the predator respectively,$a_1$is the capturing rate of predator $a_{22}(t)/a_{12}(t)$ is the conversion rate of nutrients into the reproduction of the predator.Time delays due to gestation is acommon example, the consumption of prey by the predator throughtout its past history governs the present birth rate of predator.[1, 2, 5, 7, 13]. It is well known from the fundamental theory of impulsive differential equations [4, 6, 9, 10, 11, 12] that the system (2) has a unique solution.

In this paper we shall consider (2)with impulsive effects.Precisely, we consider the following delayed impulsive system

$$
(3) \qquad \begin{aligned}
x_1'(t) &= x_1(t)[r_1(t) - a_{11}(t)x_1(t-\tau_1(t)) - \frac{a_{12}(t)x_2(t)}{1+mx_1(t)}] \\
x_2'(t) &= x_2(t)[-r_2(t) + \frac{a_{21}(t)x_1(t-\tau_2(t))}{1+mx_1(t-\tau_2(t))} - a_{22}(t)x_2(t-\tau_3(t))] \\
\Delta x_1(t) &= x_1(t^+) - x_1(t) = d_{1k}x_1(t); t = t_k \\
\Delta x_2(t) &= x_2(t^+) - x_2(t) = d_{2k}x_1(t); t = t_k
\end{aligned}
$$

where the assumptions are the same as in (2),$d_{1k}, d_{2k} \in (-1, 0](k \in N), t_k$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{k\to\infty}$ and assume that $d_{1(k+q)} = d_{1k}$,$d_{2(k+q)} = d_{2k}$,$t_{k+q} = t_k + \omega$ for $k \in N$.

In the next section , by using the continuation theorem of coincidence degree theory, we discuss the existence of positive ω-periodic solutions of system(3)and in section [3] the uniqueness and global stability of the positive ω-periodic solutions of system(3).

2. Existence of Positive Periodic Solutions

In this section, we prove the existence of solutions of periodic solution. For the reader's convenience, we provide some notations and definitions and also we first prepare the functional analytic settings:

Let PC_ω be the space of all functions ϕ such that ϕ left continuous at all points, ϕ is right continuous at $t \neq t_k$, $\lim_{t\to t_k^+}\phi(t)$ exists and $\phi(t+\omega) = \phi(t), PC'_\omega$ the space of all functions $\phi \in PC_\omega$ which are continuously differentiable at $t \neq t_k$,, $\lim_{t\to t_k^-}$ exists and $\lim_{t\to t_k^+}\phi'(t)$ and $\lim_{t\to t_k^-}\phi(t)$ exist, $k \in \mathbb{Z}^+$.

Let X,Z be normed linear spaces, $L : \text{Dom}\, L \subset X \to Z$ be a linear transformation, and $N : X \to Z$ be a continous functionn. The map L is knows as a Fredholm map of index zero if $\dim \text{Ker}\, L = \text{codim Im}\, L < +\infty$ and $\text{Im}\, L$ is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$ such that $\text{Im}\, P = \text{Ker}\, L, \text{Ker}\, Q = \text{Im}\, L = \text{Im}\,(I-Q)$. This implies that the restriction $L|_p$ of L to $\text{Dom}\, L \cap \text{Ker}\, P : (I-P)X \to \text{Im}\, L$ is invertible. The inverse of L_P is denoted by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\text{Im}\, Q$ is isomorphic to $\text{Ker}\, L$, there exists an isomorphism $J : \text{Im}\, A \to \text{Ker}\, L$.

We will make some notations and defintions which will be used in the proof of the main theorem

$$d_1 = \textstyle\sum_{k=1}^n \log(1+d_{1k}) \quad d_2 = \sum_{k=1}^n \log(1+d_{2k})$$

$$\bar{f} = \frac{1}{\omega}\int_0^\omega f(t)dt$$

$$f^L = \sup_{t\in[0,\omega]}|f(t)| \quad f^M = \inf_{t\in[0,\omega]}|f(t)|$$

Definition 2.1. *The set $\mathscr{F} \subset PC_\omega$ is said to be equicontinuous if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $u \in \mathscr{F}, k \in \mathbb{Z}^+, t' andt" \in (t_{k-1}, t_k] \cap [0,\omega]$ and $|t' - t"| < \delta$, then $|u'(t') - u(t")| < \varepsilon$.*

Lemma 2.2. [15] *The set $\mathscr{F} \subset PC_\omega$ is relatively compact if and only if*

(i) *$\mathscr{F}$ is bounded, that is, $\|u\| = \sup_{t\in[0,\omega]} \|u(t)\| \leq M$ for each $u \in \mathscr{F}$;*

(ii) *$\mathscr{F}$ is quasi-equicontinuous.*

Our existence theorem for periodic solution of the equation (2) is proved with the help of the following theorem of Gaines and Mawhin [8]

Theorem 2.3. *Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose that*

(i) *for each $\lambda \in (0,1)$,every solution x of $Lx \neq \lambda Nx$ is such thatx $\notin \Omega$;*

(ii) *for each $\lambda \in \partial\Omega \cap KerL, QNx \neq 0$;*

(iii) $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then $Lx = Nx$ has atleast one solution lying in Dom $L \cap \overline{\Omega}$.

Theorem 2.4. *The system (2) has atleast one positive ω-periodic solution provided that*

(A_1) $(\overline{r_1}+d_1)(\overline{a_{21}}-m(\overline{r_2}+d_2))-(\overline{a_{11}})(\overline{r_2}+d_2)e^{(2(\overline{r_1}\omega+d_1))} > 0$

Proof. Let $x_1(t) = e^{u_1(t)}, x_2(t) = e^{u_2(t)}$. Then we obtain the following equivalent system:

$$
\begin{aligned}
u_1'(t) &= \left[r_1(t) - a_{11}(t)e^{u_1(t-\tau_1(t))} - \frac{a_{12}(t)e^{u_2(t)}}{1+mu_1(t)}\right] \\
u_2'(t) &= \left[-r_2(t) + \frac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}} - a_{22}(t)e^{u_2(t-\tau_3(t))}\right] \\
\Delta u_1(t) &= \log(1+d_{1k}) \\
\Delta u_2(t) &= \log(1+d_{2k})
\end{aligned}
\tag{4}
$$

It is easy to see that if system (4) has one ω-Periodic solution $(u_1^*(t), u_2^*(t))$ then the corresponding $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is a periodic solution of (3). Therefore, to complete proof, it suffices to show that the system (4) has atleast one ω periodic solution.

Let

$$X=\{u=(u_1,u_2)^T\in PC_\omega([0,\omega],\mathbb{R}^2):u_i(t+\omega)=u_i(t),i=1,2\},Z=X\times\mathbb{R}^{2(q+1)}.$$

Let us define

$$\|u\|=\max_{t\in[0,\omega]}|u_1(t)|+\max_{t\in[0,\omega]}|u_2(t)|,$$

and for any $(u,\eta)\in Z$

$$\|(u,\eta)\|=\|u\|+\sum_{j=1}^{2(q+1)}|\eta_j|.$$

Then X and Z are Banach spaces. Set L :Dom $L\cap X\to Z, L(u)=(u'(t),\Delta u(t_k)_{k=1}^{q})$, where

$$\text{Dom } L=\{u=(u_1,u_2)^T\in PC_\omega(\mathbb{R},\mathbb{R}^2):u_i\in PC_\omega,i=1,2\}.$$

and $N:X\to Z$,

$$N\begin{pmatrix}u_1\\u_2\end{pmatrix}=\left(\begin{pmatrix}r_1(t)-a_{11}(t)e^{u_1(t-\tau_1(t))}-\dfrac{a_{12}(t)e^{u_2(t)}}{1+me^{u_1(t)}}\\-r_2(t)+\dfrac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}}-a_{22}e^{u_2(t-\tau_3(t))}\end{pmatrix},\left\{\begin{pmatrix}\log(1+d_{1k})\\\log(1+d_{2k})\end{pmatrix}\right\}_{k=1}^{q}\right).$$

$P:X\to X,\ P((u_1,u_2)^T)=(\overline{u_1},\overline{u_2})^T$ and $Q:Z\to Z$,

$$Q\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},\left\{\begin{pmatrix}m_k\\n_k\end{pmatrix}\right\}_{k=1}^{q}\right)=\left(\begin{pmatrix}\frac{1}{\omega}\int_0^\omega \mathrm{u}_1(t)\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^{q}m_k\\\frac{1}{\omega}\int_0^\omega \mathrm{u}_2(t)\,\mathrm{d}t+\frac{1}{\omega}\sum_{k=1}^{q}n_k\end{pmatrix},\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}_{k=1}^{q}\right).$$

It is easy to see that

$$\text{Ker}L=\left\{u=(u_1,u_2)^T\in X:\exists\, c\in\mathbb{R}^2,\ (u_1(t),u_2(t))=c,\ \text{for } t\in\mathbb{R}\right\}.$$

$$\text{Im}L=\left\{y=(u,\eta_1,\eta_2,\dots\eta_{2q})\in Y:\exists\, u\in\text{Dom}L,\ \int_0^\omega u(s)\mathrm{d}s+\sum_{k=1}^{2q}\eta_k=0\right\}.$$

Since Im L is closed in Y and dimker $L=$ codimIm $L=2, L$ is a Fredholm mapping of index zero. Moreover, the generalized inverse (to L) K_p : Im $L\to$ Ker $P\cap$Dom L is

$$K_P(u)=\int_0^t \mathrm{u}(s)\,\mathrm{d}s+\sum_{0<t_k<t}\eta_k-\frac{1}{\omega}\int_0^\omega\int_0^t \mathrm{u}(s)\,\mathrm{d}s\mathrm{d}t-\sum_{k=1}^{2q}\eta_k.$$

Then direct computation gives us

$$QN\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\omega}\int\limits_0^{\omega}\Big[(r_1(t)-a_{11}(t))e^{u_1(t-\tau_1(t))}-\frac{a_{12}(t)e^{u_2(t)}}{1+me^{u_1(t)}}\Big]\mathrm{d}t+\frac{1}{\omega}\sum\limits_{k=1}^{q}\log(1+d_{1k})\\ \frac{1}{\omega}\int\limits_0^{\omega}\Big[-r_2(t)\frac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}}\Big]\mathrm{d}t+\frac{1}{\omega}\sum\limits_{k=1}^{q}\log(1+d_{2k})\end{pmatrix},\left\{\begin{pmatrix}0\\ 0\end{pmatrix}\right\}_{k=1}^{q}$$

and

$$\begin{aligned}
&K_p(I-Q)N\begin{pmatrix}u_1\\ u_2\end{pmatrix}\\
&=\begin{pmatrix}\int\limits_0^{t}\Big[r_1(s)-a_{11}(s)e^{u_1(s-\tau_1(s))}-\frac{a_{11}(s)e^{u_2(s)}}{1+me^{u_1(s)}}\Big]\mathrm{d}s+\sum\limits_{o<t_k<t}\log(1+d_{1k})\\ \int\limits_0^{t}\Big[-r_2(s)+\frac{a_{12}(s)e^{u_1(s-\tau_2(s))}}{1+me^{u_1(s-\tau_2(s))}}-a_{22}e^{u_2(s-\tau_3(s))}\Big]\mathrm{d}s+\sum\limits_{k=1}^{n}\log(1+d_{2k})\end{pmatrix}\\
&-\begin{pmatrix}\frac{1}{\omega}\int\limits_0^{\omega}\int\limits_0^{t}\Big[r_1(s)-a_{11}(s)e^{u_1(s-\tau_1(s))}-\frac{a_{12}(s)e^{u_2(s)}}{1+me^{u_1(s)}}\Big]\mathrm{d}t+\sum\limits_{k=1}^{n}\log(1+d_{1k})\\ \frac{1}{\omega}\int\limits_0^{\omega}\int\limits_0^{t}\Big[-r_2(s)+\frac{a_{12}(s)e^{u_1(s-\tau_2(s))}}{1+me^{u_1(s-\tau_2(s))}}-a_{22}e^{u_2(s-\tau_3(s))}\Big]\mathrm{d}s\mathrm{d}t+\sum\limits_{k=1}^{n}\log(1+d_{2k})\end{pmatrix}\\
&-\begin{pmatrix}(\frac{t}{\omega}-\frac{1}{2})\int\limits_0^{\omega}\Big[r_1(t)-a_{11}(t)e^{u_1(t-\tau_1(t))}-\frac{a_{11}(t)e^{u_2(t)}}{1+me^{u_1(t)}}\Big]\mathrm{d}t+\sum\limits_{k=1}^{n}\log(1+d_{1k})\\ (\frac{t}{\omega}-\frac{1}{2})\int\limits_0^{\omega}\Big[-r_2(t)+\frac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}}-a_{22}e_2^{u}(t-\tau_3(t))\Big]\mathrm{d}t+\sum\limits_{k=1}^{n}\log(1+d_{2k})\end{pmatrix}.
\end{aligned}$$

Clearly, QN and $K_p(I-Q)N$ are continuous. Furthermore, it follows from Lemma 2.2 that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega\subset X$. Therefore, N is L- compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$. In the following, we consider the operator equation $Lu=\lambda Nu,\ \lambda\in(0,1)$, that is,

$$\begin{aligned}
u_1'(t)&=\lambda\Big[r_1(t)-a_{11}(t)e^{u_1(t-\tau_1(t))}-\frac{a_{12}(t)e^{u_2(t)}}{1+me^{u_1(t)}}\Big],\ t\neq t_k,\\
u_2'(t)&=\lambda\Big[-r_2(t)+\frac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}}-a_{22}(t)e^{u_2(t-\tau_3(t))}\Big],\ t\neq t_k,\\
\Delta u_1(t)&=\lambda[\log(1+d_{1k})],\\
\Delta u_2(t)&=\lambda[\log(1+d_{2k})].
\end{aligned}\tag{5}$$

Integration of both sides of the system (5) from 0 to ω gives

$$\omega(\overline{r_1}) + d_1 = \int\limits_0^{\omega} \left[a_{11} e^{u_1(t-\tau_1(t))} + \frac{a_{12}(t) e^{u_2(t)}}{1 + m e^{u_1(t)}} \right] \mathrm{d}t. \tag{6}$$
$$\omega(\overline{r_2}) + d_2 = \int\limits_0^{\omega} \left[\frac{a_{21}(t) e^{u_1(t-\tau_2(t))}}{1 + m e^{u_1(t-\tau_2(t))}} - a_{22}(t) e^{u_2(t-\tau_3(t))} \right] \mathrm{d}t.$$

It follows from the first equation of (5) that

$$\int_0^{\omega} |u_1'(t)| dt < \int_0^{\omega} \left[r_1(t) + a_{11}(t) e^{u_1(t-\tau_1(t))} + \frac{a_{12}(t) e^{u_2(t)}}{1 + m e^{u_1(t)}} \right] dt$$
$$= 2(\overline{r_1}\omega + d_1)$$
$$\int_0^{\omega} |u_2'(t)| dt < \int_0^{\omega} \left[r_2(t) + \frac{a_{21}(t) e^{u_1(t)}}{1 + m e^{u_1(t-\tau_2(t))}} - a_{22}(t) e^{u_2(t-\tau_3(t))} \right] dt$$

Since $(u_1, u_2)^T \in X$, there exists $\varepsilon_i, \eta_i \in [0, \omega],\ i = 1, 2$ such that

$$u(\xi_1) = \min_{t\in[0,\omega]} u_1(t) \qquad u(\eta_1) = \max_{t\in[0,\omega]} u_1(t) \tag{7}$$
$$v(\xi_2) = \min_{t\in[0,\omega]} u_2(t) \qquad u(\eta_2) = \max_{t\in[0,\omega]} u_2(t)$$

It follow from (5) and (7) that

$$\int_0^{\omega} a_{11}(t) e^{u_1(t-\tau_1(t))} dt \quad < \quad \overline{r_1}\omega + d_1 \tag{8}$$

which gives

$$u_1(\varepsilon_1) < \ln \frac{(\bar{r}_1) + d}{a_{11}} \tag{9}$$
$$u_1(t) \le u_1(\varepsilon_1) + \int_0^{\omega} |u_1'(t)| dt$$
$$< \ln \frac{\bar{r}_1 + d_1}{a_{11}} + 2(\bar{r}_1 \omega + d_1)$$

It follows from the second equation of (5) and (7) that

$$\int_0^{\omega} a_{22}(t) e^{u_2(t-\tau_3(t))} dt \le \int_0^{\omega} a_{21}(t) \frac{e^{u_2(t-\tau_2(t))}}{1 + m e^{u_2(t-\tau_2(t)}} dt \le \frac{a_{21}\omega + d_2}{m}$$

which implies that

$$u_2(t) \le u_2(\varepsilon_2) + \int_0^{\omega} |u_2'(t)| dt \quad < \quad \ln \Big[\frac{\bar{a}_{22}}{m\bar{a}_{23}} + d_2 \Big] + 2 \Big[\frac{\bar{a}_{21}\omega}{d_2} \Big]$$

$$u_2(\varepsilon_2) \leq \ln \frac{(\bar{a}_{21})}{m\bar{a}_{22}} + d_2. \tag{10}$$

On the other hand, (5) yields that

$$\int_0^{\omega} a_{21}(t)e^{u_1(t-\tau_2(t))}dt \geq \bar{r}_2\omega + d_2, \tag{11}$$

implying that

$$u_1(\eta_1) \geq \ln \frac{\bar{r}_2\omega + d_2}{a_{21}}. \tag{12}$$

We derive from (11) and (12) that

$$u_1(t) \geq u_1(\eta_1) - \int_0^{\omega} |u_1'(t)|dt \geq \ln \frac{\bar{r}_2 + d_2}{a_{21}} - 2(\bar{r}_1\omega + d_2) \tag{13}$$

which together with (9) gives

$$\begin{aligned} \max_{t\in[0,\omega]} |u_1(t)| < \max |\ln \frac{\bar{r}_1 + d_1}{a_{11}} + 2(\bar{r}_1\omega + d_1)|, |\ln \frac{\bar{r}_2 + d_1}{a_{11}} + 2(\bar{r}_1\omega + d_2)| : \\ = H_1 \end{aligned} \tag{14}$$

Once again from the second equation of (5)

$$\begin{aligned} \int_0^{\omega} e^{u_2(t-\tau_3(t))}dt &= \int_0^{\omega} \frac{a_{22}(t)e^{u_1(t-\tau_2(t))}}{1+me^{u_1(t-\tau_2(t))}}dt - \int_0^{\omega} r_2(t)dt - d_2 \\ &\geq \frac{\bar{a}_{21}\omega e^{u_1(\varepsilon_1)}}{1+me^{u_1(\varepsilon_1)}} - \bar{r}_2\omega - d_2 \end{aligned} \tag{15}$$

which implies

$$\bar{a}_{22}(1+me^{u_1(\varepsilon_1)})e^{u_2(\eta_2)} \geq (\bar{a}_{21}) - m(\bar{r}_2 - d_2)e^{u_1(\varepsilon_1)} - \bar{r}_2 - d_2. \tag{16}$$

Therefore

$$\begin{aligned} u_1(\eta_1) &\leq u_1(\varepsilon_1) + \int_0^{\omega} |u_1'(t)|dt \\ &< u_1(\varepsilon_1) + 2(\bar{r}_1\omega + d_1) \\ \bar{r}_1 + \frac{d_1}{\omega} &\leq \bar{a}_{11}e^{u_1(\eta_1)} + \bar{a}_{12}e^{u_2(\eta_2)}. \end{aligned} \tag{17}$$

We derive from (16) that

$$\bar{a}_{22}(1+me^{u_1(\varepsilon_1)})e^{u_2(\eta_2)} \geq \frac{(\bar{a}_{21} - m(\bar{r}_2 - d_2))(\bar{r}_1 + d_1 - \bar{a}_{12}e^{u_2(\eta_2)})}{\bar{a}_{11}e^{2(r_1\omega + d_1)}} \tag{18}$$

which together with (2.9) implies

$$u_2(t) \geq u(\eta_2) \geq \ ln\frac{(\bar{r}_1+d_1)(\bar{a}_{21}-m(\bar{r}_2-d_2))-\bar{a}_{11}(\bar{r}_2+d_2-e^{2(\bar{r}_1\omega+d_1)})}{\bar{a}_{11}\bar{a}_{22}e^{2(\bar{r}_1\omega+d_1)}(1+m(\bar{r}_1+d_1))e^{(2\bar{r}_1\omega+\frac{d_1}{\bar{a}_{11}})}} \tag{19}$$
$$+\bar{a}_{21}-m(\bar{r}_1-d_2)+\bar{a}_{12}(\bar{a}_{21}-m(\bar{r}_2+d_1))$$

Hence, by (19) we obtain the inequality

$$u_2(t) \geq u_2(\eta_2)-\int_0^\omega |u_2'(t)|dt \tag{20}$$
$$> \ln\frac{(\bar{r}_1+d_1)(\bar{a}_{21}-m(\bar{r}_2+d_2))-a_{11}(\bar{r}_2+d_2)e^{2(\bar{r}_1\omega+d_1)}}{\bar{a}_{11}\bar{a}_{22}e^{2(\bar{r}_1\omega+d_1)}(1+(m(\bar{r}_1+d_1)))e^{2(\bar{r}_1\omega+\frac{d_1}{\bar{a}_{11}})}}-(2\frac{\bar{a}_{21}\omega}{m}+d_2)$$

which together with (10) leads to

$$\max_{t\in[0,\omega]}\Big|u_2(t)\Big| < \max\Bigg(\Big|\ln\frac{\bar{a}_{21}}{m\bar{a}_{22}}\Big|+\frac{2\bar{a}_{21}\omega}{m},\Big| \tag{21}$$
$$\ln\frac{(\bar{r}_1+d_1)(\bar{a}_{21}-m(\bar{r}_2+d_1))-\bar{a}_{11}(\bar{r}_2+d_2)e^{2(\bar{r}_1\omega+d_1)}}{\bar{a}_{11}\bar{a}_{22}(e^{2(\bar{r}_1\omega+d_1)})(1+m(\bar{r}_1+d_1))e^{2(\frac{\bar{r}_1\omega+d_1}{\bar{a}_{11}})}+\bar{a}_{22}-m(\bar{r}_2+d_2)}\Big|$$
$$+2\frac{\bar{a}_{21}\omega}{m}+d_2\Bigg) := H_2$$

Clearly, H_1, H_2 in (14) and (21) are independent of λ. Denote $H = H_1+H_2+H_0$. where H_0 is taken sufficiently latge such that each solution (α^*, β^*) of the following algebraic equations

$$\frac{\bar{d}_1}{\omega}+r_1-\bar{a}_{11}e^\alpha-\frac{\bar{a}_{12}e^\beta}{1+me^\alpha}=0 \tag{22}$$
$$-\bar{r}_2+\frac{d_2}{\omega}+\frac{\bar{a}_{21}e^\alpha}{1+me^\alpha}-\bar{a}_{22}e^\beta=0$$

satisfies

$$||(\alpha^*,\beta^*)^T|| \ = \ |\alpha^*|+|\beta^*|<H \tag{23}$$

if it exits and the following

$$\max\Bigg[\Big|\ln\frac{\bar{r}_1+d_1}{\bar{a}_{11}}\Big|,\Big|\ln\frac{\bar{r}_2+d_2}{\bar{a}_{21}}\Big|\Bigg] \tag{24}$$
$$+\max\Big[\Big|\ln\frac{\bar{a}_{21}}{m\bar{a}_{22}}\Big|,\Big|\ln\frac{(\bar{r}_1+d_1)(\bar{a}_{21}-m(\bar{r}_2+d_2)-\bar{a}_{11}(\bar{r}_2+d_2))}{\bar{a}_{11}\bar{a}_{22}(1+m\frac{\bar{r}_1+d_1}{\bar{a}_{11}}+\bar{a}_{12}(\bar{a}_{21}-m(\bar{r}_2+d_2)))}\Big|\Big]$$
$$<H.$$

We now take $\Omega = (u_1(t), u_2(t))^T \in X : ||(u_1,u_2)^T|| < H$. This satifies the condition (a) in Thoerem 2.3. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}^2, (u_1,u_2)^T$ is a constant vector in$\mathbb{R}^2$ with $|u_1|+|u_2| = H$. If (22) has atleast one solution, then

$$QN\Big[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \left\{\begin{pmatrix} m_k \\ n_k \end{pmatrix}\right\}_{k=1}^{n}\Big] = \begin{bmatrix} \frac{\bar{d}_1}{\omega} + r_1 - \bar{a}_{11}e^{u_1} - \frac{\bar{a}_{12}e^{u_2}}{1+me^{u_1}} \\ -\bar{r}_2 + \frac{d_2}{\omega} + \frac{\bar{a}_{21}e^{u_1}}{1+me^{u_1}} - \bar{a}_{22}e^{u_2} \end{bmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}_{k=1}^{n} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If (22) does not have a solution, we can directly derive

$$QN\Big[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Big] \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This proves that condition (ii) in Theorem 2.3 is satisfied. Finally we prove that condition (iii) in Theorem 2.3 holds. Now we define ϕ : Dom $L \times [0,1] \to X$ by

$$\phi(u_i,u_2,\mu) = \Big[\begin{pmatrix} \bar{r}_1 + d_1 - \bar{a}_{11}e^{u_1} \\ \frac{\bar{a}_{21}e^{u_1}}{1+me^{u_1}} - \bar{a}_{22}e^{u_2} \end{pmatrix}\Big] + \mu\Big[\Big(\frac{\bar{a}_{12}e^{u_1}}{1+me^{u_1}} - \bar{r}_2 - d_2\Big)\Big] \tag{25}$$

where $\mu \in [0,1]$ is a parameter. When $(u_1(t), u_2)^T \in \partial\Omega\cap, KerL\ \phi(u_1,u_2,\mu) \neq 0$ otherwise, ther is a constant vector$(u_1,u_2)^T$ with$|u_1+u_2| = H$ satisfing $\phi(u_1,u_2,\mu) = 0$

$$\begin{aligned} \bar{r}_1 + d_1 - \bar{a}_{11}e^{u_1} - \mu\frac{\bar{a}_{21}e^{u_1}}{1+me^{u_1}} &= 0 \\ \frac{\bar{a}_{21}e^{u_1}}{1+me^{u_1}} - \bar{a}_{22}e^{u_2} - \mu(\bar{r}_2 + d_2) &= 0 \end{aligned}$$

A similar argument in (14) and (21) shows that

$$\begin{aligned} |u_1| &< \max\Big\{\Big|\ln\frac{\bar{r}_1+d_2}{\bar{a}_{11}}\Big|, \Big|\ln\frac{\bar{r}_1+d_2}{\bar{a}_{11}}\Big|\Big\} \\ |u_2| &< \max\Big\{\Big|\ln\frac{\bar{a}_{21}}{m\bar{a}_{22}}\Big|, \Big|\ln\frac{(\bar{r}_1+d_1)(\bar{a}_{21}-m(\bar{r}_2+d_2)) - \bar{a}_{11}(\bar{r}_2+d_2)}{\bar{a}_{11}\bar{a}_{22}(1+m\frac{\bar{r}_1+d_1}{\bar{a}_{11}}) + \bar{a}_{12}(\bar{a}_{21}-m(\bar{r}_2+d_2))}\Big|\Big\} \end{aligned}$$

It follows from (24) that $|u_1|+|u_2| < H$ which leads to a contradiction. Using the property of topological degree and taking $J = I : ImQ \to KerL$, we have $(u_1,u_2)^T \to (u_1,u_2)^T$

$$\begin{aligned} \deg(JQN((u_1,u_2)^T, \Omega\cap KerL, (0,0)^T)) &= \deg(\phi(u_1,u_2,1), \Omega\cap KerL, (0,0)^T) \\ &= \deg(\phi(u_1,u_2,0), \Omega\cap KerL, (0,0)^T) \\ &= \deg\Big(((\bar{r}_1+d_1) - \bar{a}_{11}e^{u_1}, \frac{\bar{a}_{21}e^{u_1}}{+me^{u_1}} - \bar{a}_{22}e^{u_2})^T, \partial\cap KerL, (0,0)^T\Big) \end{aligned}$$

The system of algebraic equations

$$\bar{r}_1 + \frac{d_1}{\omega} - \bar{a}_{11}e^{u_1} = 0$$
$$\frac{\bar{a}_{21}e^{u_1}}{1+me^{u_1}} - \bar{a}_{22}e^{u_2} = 0$$

which has a unique solution(u_1^*, u_2^*) which satisfies

$$u_1^* = \ln\frac{\bar{r}_1 + d_1}{\bar{a}_{11}},$$
$$u_2^* = \ln\frac{\bar{a}_{21}(\bar{r}_1 + d_1)}{\bar{a}_{22}(\bar{a}_{11} + m(\bar{r}_1 + d_1))}$$

Thus a direct calculations shows that

$$\begin{aligned}\deg(JQN(u_1,u_2)^T, \Omega \cap KerL, (0,0)^T) &= \left| sgn \begin{pmatrix} -\bar{a}_{11}e^{u_1} & o \\ \frac{\bar{a}_{21}e^{u_1^*}}{(1+me^{u_1})^2} & -\bar{a}_{22}e^{u_2^*} \end{pmatrix} \right| \\ &= sgn\{\bar{a}_{11}\bar{a}_{22}e^{u_1^*+u_2^*}\} \\ &= 1\end{aligned}$$

Finally, it is easy to show that the set $k_p(I-Q)Nx|x \in \bar{\Omega}$ is equicotinous and uniformly bounded.By using the Arzela-Ascoli theorem, we see that $k_p(I-Q)N : \bar{\Omega} \to X$is compact. Consequently L is compact. By now we have proved that Ω satiesfies the Lemma(1.1). Hence (4) has has at least one ω periodic solution.As a consequence,system (2) has atleast one positive $\omega-$periodic solution. □

3. Stability of Positive Periodic Solutions

We consider the nonimpulsive delay differential equation

$$(26) \quad \begin{bmatrix} y_1\prime(t) & = & y_1(t)\big(r_1(t) - a_{11}(t)y_1(t-\tau_1(t)) - \frac{a_{12}(t)y_2(t)}{D_1(t)+my_1(t)}\big) \\ y_2\prime(t) & = & y_2(t)\big(-r_2(t) + a_{21}(t)\frac{y_1(t-\tau_2(t))}{D_2(t)+my_1(t-\tau_2(t))} - a_{22}(t)y_2(t-\tau_3(t))\big) \end{bmatrix}$$

with the initial conditions

$$y_i(s) = f_i(s), f_i(0) > 0, f_i \in\subseteq ([-\tau, 0], R_+), i = 1,2$$

$$(27) \qquad \tau = \max_{t\in[0,\omega]} \tau_1(t), \tau_2(t), \tau_3(t)$$

where

$$a_{11}(t) = \Pi_{0<t_k,t} a_{11}(1+d_1k)$$

$$D_1(t) = \Pi_{0<t_k,t}(1+d_1k)(1+d_2k)^{-1}$$

$$D_2(t) = \Pi_{0<t_k,t}(1+d_1k)^{-1}(1+d_2k)$$

Lemma 3.1. *Let $x(t) = (x_1(t), x_2(t))^T$ is a solution of (4) with initial conditions. Then ther eexists a $T_1 > 0$ such that $0 < x_i(t) \leq H_i, i = 1,2$ fort $\geq T_1$, where*

(28) $$H_1 = H_2 > H = \max\{\frac{r_1^U}{a_{11}}, \frac{r_2^L}{a_{21}}\}$$

Lemma 3.2. *Let $x(t) = (x_1(t), x_2(t))^T$is a solution of (4) with initial conditions. Then ther exists a $T_2 > 0$ such that $0 < x_i(t) \geq h_i, i = 1,2$ for $t \geq T_2$, where*

(29) $$h_1 = h_2 < H = \max\{\frac{r_1^L - a_{12}^U H_1}{a_{11}^U}, e^{[-(r_2)^U]}\}$$

Theorem 3.3. *Assume that the conditions of Theorem(2.3) hold. In addition, assume*

(30) $$\int_0^{\omega} A(t)dt > 0$$

where

$$A(t) = min\{mina_{11}(t) + a_{11}(t)D_1(t)h - mHa_{22}(t)D_2(t)\},$$
$$min\{a_{11}(t)mh + a_{11}(t)D_1(t)h + mha_{22}(t)D_2(t)\}$$

Proof. Let $x^*(t) = (x_1'(t), x_2'(t))^T$ be a positive ω-periodic solution (2) then $y^*(t) = (y_1'(t), y_2')^T, \left(y^*(t) = \Pi_{0<t_k,t}(1+d_ik)^{-1}x_i^*(t), i = 1,2\right)$ is a positive ω-periodic solution (26) and let $(y_1(t), y_2)^T$ be any positive solution of system (26) with the initial conditions (27). It follows from Lemma 3.1, and Lemma 3.2 that there exists T, H_i and h_i such that $\forall t \geq T$

(31) $$h_i \leq y_i^*(t) \leq H_i, h_i \leq y_i(t) \leq H_i, i = 1,2.$$

Choose Lypnov function as follows

(32) $$V(t) = \sum_{i=1}^{2} |logy_i^*(t) - logy_i(t)|.$$

Since for any impulsive time t_k we have

$$V(t_k^+) \quad = \quad \sum_{i=1}^{2} |logd_{ik}y_i^*(t_k1) - logd_iky_i(t_k)| = V(t_k).$$

V(t) is continuous for all $t \geq 0$

On the other hand,from(2. 3)we can obtain that for any $t \in R^K$and $t \neq t_k$

(33) $$\frac{1}{H}|y_i^*(t) - y_i(t)| \leq |logy_i^*(t) - y_i(t)| \leq \frac{1}{h}|y_i^*(t) - y_i(t)|.$$

Calculating the upper -right derivative of V(t) along the solutions of (3.1) it follows that

$$D^+V(t) = \sum_{i=1}^{2} (\frac{y_1^*(t)}{y_1(t)} - \frac{y_i^*(t)}{y_i^*(t)})sgn(y_i^*(t) - y_i(t))$$

$$\leq sgn(y_1^*(t) - y_1(t))\big[-a_{11}[y_1^*(t-\tau_1(t)) - y_1(t-\tau_1(t))]$$

$$-a_{12}(t)\big[\frac{y_2^*(t)}{D_1(t)+my_1^*(t)} - \frac{y_2(t)}{D_1(t)+my_1(t)}\big]$$

$$+sgn(y_2^*(t) - y_2(t))\big[a_{21}(t)\big[\frac{y_2^*(t-\tau_2(t))}{D_2(t)+my_1^*(t-\tau_2(t))} - \frac{y_2(t-\tau_2(t))}{D_2(t)+my_1^*(t-\tau_2(t))}\big]$$

$$-a_{22}(t)[y_2^*(t-\tau_3(t)) - y_2(t-\tau_3(t))]\big]$$

(34) $$D^+V(t) \leq -sgn(y_1^*(t) - y_1(t))[a_{11}(y_1^*(t-\tau_1(t)) - y_1(t-\tau_2(t)))]$$

$$-sgn(y_2^*(t) - y_2(t))\big[a_{22}(t)[y_2^*(t-\tau_3(t)) - y_2(t-\tau_3(t))]\big] + \Delta_1 + \Delta_2$$

where

$$\Delta_1 = -sgn(y_1^*(t) - y_1(t))a_{12}(t)\Big[\frac{y_2^*(t)}{D_1(t)+my_1^*(t)} - \frac{y_2(t)}{D_1(t)+my_1(t)}\Big]$$

$$= -sgn(y_1^* - y_1)a_{12}(t)\Big[\frac{y_2^*(t)(D_1(t)+my_1(t)) - y_2(t)(D_1(t)+my_1^*(t))}{(D_1(t)+my_1^*(t))(D_1(t)+my_1(t))}\Big]$$

$$= -sgn(y_1^* - y_1)a_{12}(t)$$

$$\Big[\frac{y_2^*D_1(t) + m(y_2^*y_1 - y_1^*y_2 - y_2y_1 + y_2y_1) - D_1(t)y_2 + D_1(t)y_2 - y_2D_1(t)}{(D_1(t)+my_1^*(t))(D_1(t)+my_1(t))}\Big]$$

$$\leq -a_{12}mh|y_2^* - y_2| - a_{12}mh|y_1^* - y_1| - a_{12}D_1(t)|y_2^* - y_2|$$

$$\begin{aligned}
\Delta_2 &= -sgn(y_2^*(t)-y_2(t))a_{21}(t)\Big[\frac{y_1^*(t)}{D_2(t)+my_1^*(t)}-\frac{y_1(t)}{D_2(t)+my_1(t)}\Big]\\
&= -sgn(y_2^*-y_2)a_{21}\Big[\frac{D_2(t)y_1^*+my_1y_1^*-D_2(t)y_1-my_1^*y_1}{(D_2(t)+my_1^*)(D_2(t)+my_1)}\Big]\\
&\le a_{21}(t)D_2(t)|y_1^*-y_1|
\end{aligned}$$

It follows from Δ_1 and Δ_2 that

$$\begin{aligned}
D^+V(t) &\le -a_{11}(t)|y_1^*-y_1|-a_{22}(t)|y_2^*-y_2|+a_{21}(t)D_2(t)|y_1^*-y_1|\\
&\quad -a_{12}mh|y_2^*-y_2|-a_{12}(t)mh|y_1^*-y_1|-a_{12}(t)D_1(t)|y_2^*-y_2|\\
&\le |y_1^*-y_1|[-a_{11}(t)+a_{21}(t)D_2(t)-a_{12}mh]\\
&\qquad +|y_2^*-y_2|[-a_{22}(t)-a_{12}(t)mh-a_{12}(t)D_1(t)]\\
&\le -[a_{11}(t)-a_{21}(t)D_2(t)+a_{12}mh]|y_1^*-y_1|\\
&\qquad -[a_{22}(t)+a_{12}(t)mh-a_{12}(t)D_1(t)]|y_2^*-y_2|\\
&\le -A(t)
\end{aligned}$$

From this ,we further have any $t\ge 0; V(t)\le V(0)e^{(-\int_0^t A(u)du)}$.From(26)we can $\int_0^t A(u)du\to\infty$ as $t\to\infty$,Hence, $V(t)\to 0$ as $t\to\infty$.Further from (33) we have

$$\lim_{t\to\infty}|y_i^*-y_i(t)| = \lim_{t\to\infty}\Big[\Pi_{0<t_k,t}(1+d_ik)^{-1}|x_i^*(t)-x_i(t)|\Big] = 0, i=1,2$$

Therefore $\lim_{t\to\infty}|x_i^*(t)-x_i(t)| = 0, i=1,2$ □

4. ILLUSTRATING EXAMPLE

Example 4.1. *To illustrate the result obtained, we consider the system*

$$\begin{aligned}
x_1'(t) &= x_1(t)\left[(1+0.1\sin t)-0.1x_1(t-0.1)-\frac{x_2(t)}{1+x_1(t)}\right]\\
x_2'(t) &= x_2(t)\left[-\frac{1}{10^5}(2+\sin t)+\frac{9x_1(t-0.3)}{1+x_1(t-0.3)}-x_2(t-0.1)\right]\\
\Delta x_1(t) &= (\frac{1}{e}-1)x_1(t), t=t_k\\
\Delta x_2(t) &= (\frac{1}{e}-1)x_2(t), t=t_k
\end{aligned}$$

It is very simple to verify the conditions of the Theorem 2.4 and the system has atleast one positive periodic solutions.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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