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SOME FIXED POINT THEOREMS ON A C^*-ALGEBRA VALUED RECTANGULAR QUASI-METRIC SPACES

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Abstract. In this work, we discuss the existence and uniqueness of fixed points for a self-mapping defined on a C^*-algebra valued rectangular quasi-metric space. Our results extend and supplement several recent results in the literature. Some examples are provided to illustrate our results.

Keywords: fixed point; rectangular quasi-metric spaces; C^*-algebra.

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1. Introduction

One of the most famous metrical fixed point theorem is the Banach contraction principle [1], which is the classical tool for solving several nonlinear problems. Based on the noncomplexity and the usefulness of this principle, it have many extension and generalization into several directions [4, 5, 6, 7].

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In 1930, Wilson [9] introduced the concept of quasi-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the qusi-metric spaces.

In 2000, Branciari [2] introduced the notion of rectangular metric spaces where the triangle inequality of metric spaces was replaced by another inequality, so-called rectangular inequality.

In 2014, Ma et al. [8] established the notion of C^*-algebra valued metric spaces by replacing the range set $\mathbb{R}$ with an unital C^*-algebra, which is more general class than the class of metric spaces and utilized the same to prove some fixed point results is such spaces.

The following lemma will be useful in our main results.

Lemma 1.1. [10] *Suppose that A is a unital C^*-algebra with a unit I.*

(1) *For any $a \in A_+$ we have, $a \preceq I \Leftrightarrow \|a\| \leq 1$.*
(2) *If $a \in A_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$.*
(3) *Suppose that $a, b \in A$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$.*
(4) *Let $a \in A'$, if $b, c \in A$ with $b \succeq c \succeq 0_{\mathbb{A}}$ and $I - a \in A'_+$ is an invertible, then $(I-a)^{-1}b \succeq (I-a)^{-1}c$.*

2. MAIN RESULT

We now introduce the definition of a C^*-algebra-valued rectangular quasi-metric spaces.

Definition 2.1. *Let X be a non empty set. Suppose the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:*

(i) *$d(x,y) = 0_{\mathbb{A}}$ if and only if $x = y$; and $0_{\mathbb{A}} \preceq d(x,y)$ for all $x, y \in X$;*
(ii) *$d(x,y) \preceq d(x,u) + d(u,v) + d(v,y)$ for all $x, u, v, y \in X$ and for all distinct points $u, v \in X\{x,y\}$.*

Then $(X, \mathbb{A}_+, d)$ is called a C^*-algebra valued rectangular quasi-metric space.

Remark 2.2. *The C^*-algebra-valued rectangular quasi-metric space generalise the C^*-algebra-valued metric space, C^*-algebra-valued rectangular metric space. The following example illustrates that, in general, a C^*-algebra-valued rectangular quasi-metric space is not necessarily a C^*-algebra-valued rectangular metric space and is not necessarily a C^*-algebra-valued metric space.*

Example 2.3. *Let $X = A \cup B$, where $A = \{0,2\}$ and $B = \{\frac{1}{n},\ n \in \mathbb{N}^*\}$. Let $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition ,scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$ for $i = 1,2,3,4$*

For any $A \in \mathbb{A}$ we define its norm as , $\|\mathbb{A}\| = max_{1\leq i\leq 4}|a_i|$

Define $d : X \times X \to \mathbb{A}$ by

$$\begin{cases} d(0,2) = d(2,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ d\left(\frac{1}{n},0\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ d\left(0,\frac{1}{n}\right) = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \\ d\left(2,\frac{1}{n}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ d\left(\frac{1}{n},2\right) = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \\ d\left(\frac{1}{n},\frac{1}{m}\right) = d\left(\frac{1}{m},\frac{1}{n}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ . \end{cases}$$

Then $(X,\mathbb{A}_+,d)$ is a C^*-algebra valued rectangular quasi- metric space. However we have the following:

1) *$(X,\mathbb{A}_+,d)$ is not a C^*-algebra valued metric space, as $d\left(\frac{1}{n},0\right) \neq d\left(0,\frac{1}{n}\right)$, for all $n \geq 1$.*
2) *$(X,\mathbb{A}_+,d)$ is not a C^*-algebra valued asymmetric metric space, as*
$$d(2,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \succ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = d\left(2,\tfrac{1}{4}\right) + d\left(\tfrac{1}{4},0\right).$$
3) *(X,d) is not a C^*-algebra valued rectangular metric space, as*
$$d\left(\tfrac{1}{n},2\right) = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = d\left(2,\tfrac{1}{n}\right), \text{ for all } n \geq 1.$$

Definition 2.4. *Let $(X,\mathbb{A},d)$ is a C^*-algebra valued rectangular quasi-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then*

(i) *We say that $\{x_n\}_{n\in\mathbb{N}}$ forward converges to x with respect to $\mathbb{A}$ if and only if for given $\varepsilon \succ 0_{\mathbb{A}}$, there is N such that for all $n \geq N$, $d(x,x_n) \preceq \varepsilon$. We denote it by*

$$\lim_{n\to+\infty} d(x,x_n).$$

(ii) *We say that $\{x_n\}_{n\in\mathbb{N}}$ backward converges to x with respect to $\mathbb{A}$ if and only if for given $\varepsilon \succ 0_{\mathbb{A}}$, there is N such that for all $n \geq N$, $d(x_n,x) \preceq \varepsilon$. We denote it by*

$$\lim_{n\to+\infty} d(x_n,x) = 0_{\mathbb{A}}.$$

(iii) *We say that $\{x_n\}_{n\in\mathbb{N}}$ forward Cauchy if*

$$\lim_{n,m\to+\infty} d(x_n,x_m) = 0_{\mathbb{A}}.$$

(iv) *We say that $\{x_n\}_{n\in\mathbb{N}}$backward Cauchy if*

$$\lim_{n,m\to+\infty} d(x_m,x_n) = 0_{\mathbb{A}}.$$

Remark 2.5. [4] *Let (X,d) be as in Example 2.3, $\{\frac{1}{n}\}_{n\in\mathbb{N}^*}$ be a sequence in X. However we have the following:*

i) $\lim_{n\to+\infty} d\left(\frac{1}{n},0\right) = 0_{\mathbb{A}}$, $\lim_{n\to+\infty} d\left(\frac{1}{n},2\right) = 1$ *and* $\lim_{n\to+\infty} d\left(0,\frac{1}{n}\right) = 1$, $\lim_{n\to+\infty} d\left(2,\frac{1}{n}\right) = 0_{\mathbb{A}}$. *Then, the sequence $\{\frac{1}{n}\}$ forward converges to 2 and backward converges to 0, so limit is not unique.*

ii) $\lim_{n\to+\infty} d\left(\frac{1}{m},\frac{1}{n}\right) = \lim_{n\to+\infty} d\left(\frac{1}{m},\frac{1}{n}\right) = 1$. *So, forward (backward) convergence dose not imply forward (backward) Cauchy.*

Lemma 2.6. *Let $(X,\mathbb{A},d)$ a C^*-algebra valued rectangular quasi-metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Proof. Let $\varepsilon \succ 0_{\mathbb{A}}$. First assume that $\{x_n\}_n$ is a forward Cauchy sequence, so there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x_m) \preceq \frac{\varepsilon}{3}$ for all $m \geq n \geq n_0$. Since $\{x_n\}_n$forward converges to x so there exists

$n_1 \in \mathbb{N}$ such that $d(x_n, x) \preceq \frac{\varepsilon}{3}$ for all $n \geq n_1$. Also $\{x_n\}_n$ forward converges to y so there exists $n_2 \in \mathbb{N}$ such that$d(y, x_n) \preceq \frac{\varepsilon}{3s}$ for all $n \geq n_2$. Then for all $N \geq \max\{n_0, n_1, n_2\}$,

$$d(x,y) \preceq d(x,x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y) \preceq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

As $\varepsilon \succ 0_{\mathbb{A}}$ was arbitrary, we deduce that $d(x,y) = 0_{\mathbb{A}}$, which implies $x = y$. When $\{x_n\}_n$ is a backward Cauchy sequence, the proof is similar to an earlier state □

Definition 2.7. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued rectangular quasi-metric space. X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_n$ in X forward (backward) converges to $x \in X$.*

Definition 2.8. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued rectangular quasi-metric space. X is said to be complete if X is forward and backward complete.*

Definition 2.9. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued rectangular quasi-metric space. A mapping $T : X \to X$ is a C^*-valued contractive mapping on X, if there exists an $a \in \mathbb{A}$ with $\| \mathbb{A} \| < 1$ such that*

$$d(Tx, Ty) \preceq a^* d(x,y) a \tag{1}$$

for all $x, y \in \mathbb{A}$.

Theorem 2.10. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued rectangular quasi-metric space and let $T :$ $X \to X$ is a C^*-valued contractive mapping on X, then there exists a unique fixed point in X.*

Proof. It is clear that if $\mathbb{A} = 0_{\mathbb{A}}$, then T maps the X into a single point. Thus without loss of generality, one can suppose that $\mathbb{A} \neq 0_{\mathbb{A}}$. Choose an $x_0 \in X$ and set $x_{n+1} = Tx_n = T^n x_0$. Notice that in C^*-algebra, if $a, b \in \mathbb{A}^+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$.

Substituting $x = x_{n-1}$ and $y = x_n$, from (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq a^* d(x_{n-1}, x_n) a \\ &\preceq (a^*)^2 d(x_{n-2}, x_{n-1}) a^2 \end{aligned}$$

$$\begin{aligned}
&\preceq \dots \\
&\preceq (a^*)^n d(x_0, x_1) a^n
\end{aligned}$$

Substituting $x = x_{n-1}$ and $y = x_n$, from (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\
&\preceq a^* d(x_{n-1}, x_{n+1}) a \\
&\preceq (a^*)^2 d(x_{n-2}, x_n) a^2 \\
&\preceq \dots \\
&\preceq (a^*)^n d(x_0, x_2) a^n
\end{aligned}$$

<u>Case 1:</u> Assume that $m = 2l + 1$ with $l \geq 1$. By property (ii) of the C^*-algebra valued rectangular quasi-metric space, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &= d(x_n, x_{n+2l+1}) \\
&\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+n+2l+1}) \\
&\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+n+2l+1}) \\
&\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+2l}, x_{n+2l+1}) \\
&\preceq (a^*)^n d(x_0, x_1) a^n + (a^*)^{n+1} d(x_0, x_1) a^{n+1} + \dots + (a^*)^{n+2l} d(x_0, x_1) a^{n2l+1} \\
&= \sum_{i=n}^{i=n+2l} (a^*)^k d(x_0, x_1) a^k \\
&= \sum_{i=n}^{i=n+2l} (a^k d(x_0, x_1)^{\frac{1}{2}})^* d(x_0, x_1)^{\frac{1}{2}} a^k \\
&= \sum_{i=n}^{i=n+2l} |d(x_0, x_1)^{\frac{1}{2}} a^k|^2 \\
&\preceq \| \sum_{i=n}^{i=n+2l} |d(x_0, x_1)^{\frac{1}{2}} a^k|^2 \| I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_1)^{\frac{1}{2}}\|^2 \sum_{i=n}^{i=n+2l} \|a\|^{2k} I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_1)^{\frac{1}{2}}\|^2 \frac{\|a\|^{2n}}{1 - \|a\|^2} I_{\mathbb{A}} \to 0_{\mathbb{A}} (n \to \infty).
\end{aligned}$$

Case 2: If $m = n + 2k$ Similarly to case1 we have

$$\begin{aligned}
d(x_n, x_{n+m}) &= d(x_n, x_{n+2k}) \\
&\preceq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+2}, x_{n+n+2k}) \\
&\preceq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+n+2k}) \\
&\preceq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{n+2k-1}, x_{n+2k}) \\
&\preceq (a^*)^n d(x_0, x_2)\, a^n + (a^*)^{n+2} d(x_0, x_1)\, a^{n+2} + \ldots + (a^*)^{n+2k-1} d(x_0, x_1)\, a^{n+2k-1} \\
&= (a^*)^n d(x_0, x_2)\, a^n + \sum_{i=n+2}^{i=n+2k-1} (a^*)^i d(x_0, x_1)\, a^i \\
&= (a^n d(x_0, x_2)^{\frac{1}{2}})^* d(x_0, x_2)^{\frac{1}{2}}\, a^n + \sum_{i=n+2}^{i=n+2k-1} (a^k d(x_0, x_1)^{\frac{1}{2}})^* d(x_0, x_1)^{\frac{1}{2}}\, a^k \\
&= |d(x_0, x_2)^{\frac{1}{2}}\, a^n|^2 + \sum_{i=n+2}^{i=n+2k-1} |d(x_0, x_1)^{\frac{1}{2}}\, a^i|^2 \\
&\preceq \| |d(x_0, x_2)^{\frac{1}{2}}\, a^n|^2 \| I_{\mathbb{A}} + \| \sum_{i=n}^{i=n+2k-1} |d(x_0, x_1)^{\frac{1}{2}}\, a^i|^2 \| I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_2)^{\frac{1}{2}}\|^2 \|a\|^{2n} I_{\mathbb{A}} + \|d(x_0, x_1)^{\frac{1}{2}}\|^2 \sum_{i=n}^{i=n+2k-1} \|a\|^{2i} I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_2)^{\frac{1}{2}}\|^2 \|a\|^{2n} I_{\mathbb{A}} + \|d(x_0, x_1)^{\frac{1}{2}}\|^2 \frac{\|a\|^{2n}}{1 - \|a\|^2} I_{\mathbb{A}} \to 0_{\mathbb{A}} (n \to \infty).
\end{aligned}$$

Therefore x_n is a forward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $z \in X$ such that

$$\lim_{n \to \infty} d(z, x_n) = 0_{\mathbb{A}}. \tag{2}$$

Substituting $x = x_n$ and $y = x_{n-1}$, from (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\
&\preceq a^* d(x_n, x_{n-1})\, a \\
&\preceq (a^*)^2 d(x_{n-1}, x_{n-2})\, a^2 \\
&\preceq \ldots \\
&\preceq (a^*)^n d(x_1, x_0)\, a^n
\end{aligned}$$

Substituting $x = x_{n+1}$ and $y = x_{n-1}$, from (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(Tx_{n-1}, Tx_{n+1}\right) \\ &\preceq (a^*)^n d\left(x_2, x_0\right) a^n \end{aligned}$$

Case 1: Assume that $m = 2l+1$ with $l \geq 1$. By property (ii) of the C^*-algebra valued rectangular quasi-metric space, we have

$$\begin{aligned} d\left(x_{n+m}, x_n\right) &= d\left(x_{n+2l+1}, x_n\right) \\ &\preceq \|d\left(x_1, x_0\right)^{\frac{1}{2}}\|^2 \sum_{i=n}^{i=n+2l} \|a\|^{2k} I_{\mathbb{A}} \\ &\preceq \|d\left(x_1, x_0\right)^{\frac{1}{2}}\|^2 \frac{\|a\|^{2n}}{1-\|a\|^2} I_{\mathbb{A}} \rightarrow 0_{\mathbb{A}} (n \rightarrow \infty). \end{aligned}$$

Case 2: If $m = n+2k$ Similarly to case 1 we have

$$\begin{aligned} d\left(x_{n+m}, x_n\right) &= d\left(x_{n+2k}, x_n\right) \\ &\preceq d\left(x_{n+2}, x_n\right) + d\left(x_{n+3}, x_{n+2}\right) + d\left(x_{n+n+2k}, x_{n+2}\right) \\ &\preceq |d\left(x_2, x_0\right)^{\frac{1}{2}} a^n|^2 + \sum_{i=n+2}^{i=n+2k-1} |d\left(x_1, x_0\right)^{\frac{1}{2}} a^i|^2 \\ &\preceq \||d\left(x_2, x_0\right)^{\frac{1}{2}} a^n|^2\| I_{\mathbb{A}} + \| \sum_{i=n}^{i=n+2k-1} |d\left(x_1, x_0\right)^{\frac{1}{2}} a^i|^2\| I_{\mathbb{A}} \\ &\preceq \|d\left(x_2, x_0\right)^{\frac{1}{2}}\|^2 \|a\|^{2n} I_{\mathbb{A}} + \|d\left(x_0, x_1\right)^{\frac{1}{2}}\|^2 \sum_{i=n}^{i=n+2k-1} \|a\|^{2i} I_{\mathbb{A}} \\ &\preceq \|d\left(x_2, x_0\right)^{\frac{1}{2}}\|^2 \|a\|^{2n} I_{\mathbb{A}} + \|d\left(x_1, x_0\right)^{\frac{1}{2}}\|^2 \frac{\|a\|^{2n}}{1-\|a\|^2} I_{\mathbb{A}} \rightarrow 0_{\mathbb{A}} (n \rightarrow \infty). \end{aligned}$$

Therefore x_n is a backward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $u \in X$ such that

$$\lim_{n \rightarrow \infty} d(u, x_n) = 0_{\mathbb{A}}.$$

So, from Lemma 2.6, we get $z = u$.

On has

$$0_{\mathbb{A}} \preceq d(z,Tz) \preceq d(z,x_n) + d(x_n,Tx_n) + d(Tx_n,Tz)$$
$$\preceq d(z,x_n) + d(x_n,Tx_n) + a^* d(x_n,z)a \to 0_{\mathbb{A}} (n \to \infty).$$

Since

$$0_{\mathbb{A}} \preceq d(Tz,z) \preceq d(Tz,Tx_n) + d(Tx_n,x_n) + d(x_n,z)$$
$$\preceq a^* d(z,x_n)a + d(Tx_n,x_n) + d(x_n,z) \to 0_{\mathbb{A}} (n \to \infty).$$

Therefore $d(z,Tz) = 0_{\mathbb{A}}$ *or* $d(Tz,z) = 0_{\mathbb{A}}$ which implies $Tz = z$, i.e. z is a fixed point of T.

Uniqueness: Suppose that $u \neq z$ is another fixed point of T. Since

$$\begin{aligned}
0_{\mathbb{A}} \preceq d(z,u) = d(Tz,Tu) &\preceq a^* d(z,u)a \\
&\preceq \|a^* d(z,u)a\| \\
&\preceq \|a^*\|\|d(z,u)\|\|a\| \\
&= \|a\|^2\|d(z,u)\| \\
&< \|d(z,u)\|, \textit{ which is a contradiction.}
\end{aligned}$$

Hence $d(z,u) = \theta$ and $z = u$, which implies that the fixed point is unique. □

Example 2.11. *Let $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition ,scalar multiplication and multiplication.*

Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$ for $i = 1,2,3,4$. For any $A \in \mathbb{A}$ we define its norm as, $\| \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \| = \left[\sum_{i=1}^{i=4} |a_i|^2 \right]^{\frac{1}{2}}$.

Let $X = A \cup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and $B = [1,2]$.

Define $d : X \times X \to [0,+\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ for\ all\ x,y \in B; \\ d(x,y) = 0 \Leftrightarrow y = x \ for\ all\ x,y \in X. \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(0,\frac{1}{2}\right) = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix} \\ d\left(\frac{1}{3},0\right) = d\left(\frac{1}{4},\frac{1}{2}\right) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} \\ d\left(0,\frac{1}{3}\right) = d\left(\frac{1}{2},\frac{1}{4}\right) = \begin{pmatrix} 0.35 & 0 \\ 0 & 0.35 \end{pmatrix} \\ d\left(\frac{1}{3},\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{2}\right) = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix} \\ d(x,y) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix} \text{ otherwise.} \end{cases}$$

Then $(X,\mathbb{A}_+,d)$ is a C^*-algebra valued rectangular quasi-metric space. However we have the following:

1) $(X,\mathbb{A}_+,d)$ *is not a C^*-algebra valued asymmetric metric space, as* $d\left(\frac{1}{3},\frac{1}{2}\right) = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.6 \end{pmatrix} \succ \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix} + \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} = d\left(\frac{1}{3},\frac{1}{4}\right) + d\left(\frac{1}{4},\frac{1}{2}\right).$
2) (X,d) *is not a C^*-algebra valued rectangular metric space, as* $d\left(\frac{1}{2},\frac{1}{4}\right) = \begin{pmatrix} 0.35 & 0 \\ 0 & 0.35 \end{pmatrix} \neq \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} = d\left(\frac{1}{4},\frac{1}{2}\right).$

Define mapping $T : X \to X$ by

$$T(x) = \begin{cases} x^{\frac{1}{4}} \ if \ x \in [1,2] \\ 1 \ if \ x \in A. \end{cases}$$

Evidently, $T(x) \in X$. Consider the following possibilities:

case 1 : $x,y \in [1,2]$ $x \neq y$. Then

$$T(x) = x^{\frac{1}{4}},\ T(y) = y^{\frac{1}{4}},\ d(Tx,Ty) = \begin{pmatrix} x^{\frac{1}{4}} - y^{\frac{1}{4}} & 0 \\ 0 & x^{\frac{1}{4}} - y^{\frac{1}{4}} \end{pmatrix}.$$

On the other hand

$$d(x,y) = \begin{pmatrix} x-y & 0 \\ 0 & x-y \end{pmatrix}.$$

it follows that

$$d(Tx,Ty) \preceq a^* d(x,y) a. \tag{3}$$

Indeed

$$\begin{aligned} d(Tx,Ty) &= \begin{pmatrix} x^{\frac{1}{4}} - y^{\frac{1}{4}} & 0 \\ 0 & x^{\frac{1}{4}} - y^{\frac{1}{4}} \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x-y & 0 \\ 0 & x-y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= a^* d(x,y) a. \end{aligned}$$

where

$$a = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

with verify

$$\|a\| = \frac{\sqrt{2}}{\sqrt{3}} \leq 1.$$

case 2 *:* $x \in [1,2]$, $y \in A$. *Then*

$$T(x) = x^{\frac{1}{4}},\ T(y) = 1,\ D(Tx,Ty) = 2\left(x^{\frac{1}{4}} - 1\right).$$

$$d(Tx,Ty) = \begin{pmatrix} x^{\frac{1}{4}} - 1 & 0 \\ 0 & x^{\frac{1}{4}} - 1 \end{pmatrix}.$$

On the other hand

$$d(x,y) = \begin{pmatrix} x-y & 0 \\ 0 & x-y \end{pmatrix}.$$

it follows that

$$d(Tx,Ty) \preceq a^* d(x,y) a. \tag{4}$$

Indeed

$$\begin{aligned} d(Tx,Ty) &= \begin{pmatrix} x^{\frac{1}{4}}-1 & 0 \\ 0 & x^{\frac{1}{4}}-1 \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x-1 & 0 \\ 0 & x-1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x-y & 0 \\ 0 & x-y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= a^* d(x,y) a. \end{aligned}$$

where

$$a = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

with verify

$$\|a\| = \frac{\sqrt{2}}{\sqrt{3}} \leq 1.$$

Hence, the condition (1) *is satisfied. Therefore,* T *has a unique fixed point* $z = 1$.

Example 2.12. *Let* $\mathbb{A} = M_2(\mathbb{R}_+)$ *of all* 2×2 *matrices with the usual addition ,scalar multiplication and multiplication. Define partial ordering on* $\mathbb{A}$ *as* $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ $\Leftrightarrow a_i \geq b_i$ *for* $i = 1,2,3,4$. *For any* $A \in \mathbb{A}$ *we define its norm as ,*$\| \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \| = \left[\sum_{i=1}^{i=4} |a_i|^2 \right]^{\frac{1}{2}}$.

Define $d : X \times X \to [0, +\infty[$ *as follows:*

$$\begin{cases} d(x,y) = \begin{pmatrix} e^x - e^y & 0 \\ 0 & 0 \end{pmatrix} \; if \; x \geq y \\ d(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-x} - e^{-y} \end{pmatrix} \; if \; x \leq y \end{cases}$$

Then $(X, \mathbb{A}_+, d)$ *is a* C^**-algebra valued rectangular quasi-metric space.*

Define mapping $T : X \to X$ *by*

$$T(x) = \frac{x}{4}.$$

Evidently, $T(x) \in X$. *Then*

$$\begin{cases} d(Tx,Ty) = \begin{pmatrix} e^{\frac{x}{4}} - e^{\frac{y}{4}} & 0 \\ 0 & 0 \end{pmatrix} \; if\; x \geq y \\ d(Tx,Ty) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-\frac{x}{4}} - e^{-\frac{y}{4}} \end{pmatrix} \; if\; x \leq y \end{cases}$$

it follows that

$$d(Tx,Ty) \preceq a^* d(x,y) a. \tag{5}$$

Indeed

$$\begin{cases} d(Tx,Ty) \preceq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{x} - e^{y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \; if\; x \geq y \\ d(Tx,Ty) \preceq \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e^{-x} - e^{-y} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \; if\; x \leq y \end{cases}$$

where

$$a = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

with verify

$$\|a\| = \frac{1}{2} < 1.$$

Hence, the condition (1) *is satisfied. Therefore,* T *has a unique fixed point* $z = 0$.

Definition 2.13. *Let* $(X, \mathbb{A}, d)$ *a* C^**-algebra valued rectangular quasi-metric space. A mapping* $T : X \to X$ *is a* C^**-valued Kannan-type mapping on X, if there exists an* $a \in \mathbb{A}_+^{'}$ *with* $\| a \| < \frac{1}{2}$ *such that*

$$d(Tx,Ty) \preceq a\left[d(x,Tx) + d(y,Ty)\right] \tag{6}$$

for all $x, y \in \mathbb{A}$ *where*

$$\mathbb{A}_+^{'} = \{a \in \mathbb{A}_+ |\; ab = ba \; for\; all\; b \in \mathbb{A}_+\}$$

Theorem 2.14. *Let* $(X, \mathbb{A}, d)$ *a* C^**-algebra valued rectangular quasi-metric space and let* $T : X \to X$ *is a* C^**-valued Kannan-type mapping on X, then there exists a unique fixed point in X.*

Proof. Without loss of generality, one can suppose that $a \neq 0_{\mathbb{A}}$. Notice that $a \in \mathbb{A}_{+}^{'}$, $a\left[d(x,Tx)+d(y,Ty)\right]$ is also a positive element. Choose an $x_0 \in X$ and set $x_{n+1} = Tx_n = T^n x_0$. Substituting $x = x_{n-1}$ and $y = x_n$, from (6), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(Tx_{n-1}, Tx_n\right) \\ &\preceq a\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right]. \end{aligned}$$

Since $a \in \mathbb{A}_{+}^{'}$ with $\| a \| < \frac{1}{2}$, using Lemma 1.1, $I-a$ is invertible and also $\|(I-a)a\| < 1$. Thus

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &\preceq (I-a)^{-1} a\left[d\left(x_{n-1}, x_n\right)\right] \\ &\preceq (I-a)^{-2} a^2 \left[d\left(x_{n-2}, x_{n-1}\right)\right] \\ &\preceq \ldots \\ &\preceq (I-a)^{-n} a^n \left[d\left(x_0, x_1\right)\right] \\ &= h^n \left[d\left(x_0, x_1\right)\right] \end{aligned}$$

with $h^n = (I-a)^{-n} a^n$.

Substituting $x = x_{n-1}$ and $y = x_{n+1}$, from (1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(x_n, x_{n+2}\right) &\preceq (I-a)^{-1} a\left[d\left(x_{n-1}, x_{n+1}\right)\right] \\ &\preceq (I-a)^{-2} a^2 \left[d\left(x_{n-2}, x_n\right)\right] \\ &\preceq \ldots \\ &\preceq (I-a)^{-n} a^n \left[d\left(x_0, x_2\right)\right] \\ &= h^n \left[d\left(x_0, x_2\right)\right] \end{aligned}$$

Case 1: Assume that $m = 2l+1$ with $l \geq 1$. By property (ii) of the C^*-algebra valued rectangular quasi-metric space, we have

$$\begin{aligned} d\left(x_n, x_{n+m}\right) &= d\left(x_n, x_{n+2l+1}\right) \\ &\preceq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + d\left(x_{n+2}, x_{n+n+2l+1}\right) \end{aligned}$$

$$\preceq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})+d(x_{n+4},x_{n+n+2l+1})$$

$$\preceq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...+d(x_{n+2l},x_{n+2l+1})$$

$$\preceq \sum_{i=n}^{i=n+2l} \|h^{\frac{k}{2}}\|^2 \|d(x_0,x_1)^{\frac{1}{2}}\|^2$$

$$\preceq \|d(x_0,x_1)^{\frac{1}{2}}\|^2 \sum_{i=n}^{i=n+2l} \|h^{\frac{k}{2}}\|^2$$

$$\preceq \|d(x_0,x_1)^{\frac{1}{2}}\|^2 \frac{\|h\|^{2n}}{1-\|h\|^2} I_{\mathbb{A}} \to 0_{\mathbb{A}}(n\to\infty).$$

Case 1: If $m=n+2k$ Similarly to case1 we have

$$\begin{aligned}
d(x_n,x_{n+m}) &= d(x_n,x_{n+2k})\\
&\preceq d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+d(x_{n+2},x_{n+n+2k})\\
&\preceq d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})+d(x_{n+4},x_{n+5})+d(x_{n+5},x_{n+n+2k})\\
&\preceq d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+...+d(x_{n+2k-1},x_{n+2k})\\
&\preceq \|h\|^n\|d(x_0,x_2)\|+\sum_{i=n}^{i=n+2l} \|h^{\frac{k}{2}}\|^2\|d(x_0,x_2)^{\frac{1}{2}}\|^2\\
&\preceq \left[\|h\|^n\|d(x_0,x_2)\|+\|d(x_0,x_2)^{\frac{1}{2}}\|^2\sum_{i=n}^{i=n+2l}\|h^{\frac{k}{2}}\|^2\right] I_{\mathbb{A}}\\
&\preceq \left[\|h\|^n\|d(x_0,x_2)\|+\|d(x_0,x_2)^{\frac{1}{2}}\|^2\frac{\|h\|^{2n}}{1-\|h\|^2}\right] I_{\mathbb{A}} \to 0_{\mathbb{A}}(n\to\infty).
\end{aligned}$$

Therefore x_n is a forward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d)$ there exists an $z\in X$ such that

$$\lim_{n\to\infty} d(z,x_n)=0_{\mathbb{A}}. \tag{7}$$

Substituting $x=x_n$ and $y=x_{n-1}$, from (6), for all $n\in\mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1})\\
&\preceq a\left[d(x_n,x_{n-1})+d(x_{n+1},x_n)\right]\\
&\preceq h^n\left[d(x_1,x_0)\right].
\end{aligned}$$

Substituting $x = x_{n+1}$ and $y = x_{n-1}$, from (6), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(x_{n+2}, x_n\right) &= d\left(Tx_{n+1}, Tx_{n-1}\right) \\ &\preceq h^n\left[d\left(x_2, x_0\right)\right]. \end{aligned}$$

Therefore x_n is a backward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0_{\mathbb{A}}. \tag{8}$$

So, from Lemma 2.6, we get $z = u$.

On has

$$\begin{aligned} 0_{\mathbb{A}} \preceq d\left(z, Tz\right) &\preceq d\left(z, x_n\right) + d\left(x_n, Tx_n\right) + d\left(Tx_n, Tz\right) \\ &\preceq d\left(z, x_n\right) + d\left(x_n, Tx_n\right) + a\left(d(x_n, Tx_n) + (d(z, Tz))\right) \to 0_{\mathbb{A}}(n \to \infty). \end{aligned}$$

This is equivalent to

$$d\left(z, Tz\right) \preceq (I-1)^{-1}\left[d\left(x_n, Tx_n\right) + a\left(d(x_n, Tx_n)\right)\right] \to 0_{\mathbb{A}}(n \to \infty).$$

Since

$$d\left(Tz, z\right) \preceq (I-1)^{-1}\left[d\left(Tx_n, x_n\right) + a\left(d(Tx_n, x_n)\right)\right] \to 0_{\mathbb{A}}(n \to \infty).$$

Therefore $d\left(z, Tz\right) = 0_{\mathbb{A}}$ *or* $d\left(Tz, z\right) = 0_{\mathbb{A}}$ which implies $Tz = z$, i.e. z is a fixed point of T.

Uniqueness: Suppose that $u \neq z$ is another fixed point of T. Since

$$0_{\mathbb{A}} \preceq d\left(z, u\right) = d\left(Tz, Tu\right) \preceq a(d(z, Tz) + (d(u, Tu)) = \theta.$$

Hence $d\left(z, u\right) = \theta$ and $z = u$, which implies that the fixed point is unique. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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