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NUMERICAL SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS USING HERMITE POLYNOMIALS

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Abstract. This study describes a numerical method for solving fractional integro differential equations (FIDEs) using Hermite polynomials (HP). The fractional derivative is considered in the Caputo sense. This type of problem was reduced to a system of linear algebraic equations using the provided method. To demonstrate the theoretical results, numerical results are shown and contrasted to those obtained by different polynomial methods. The solutions of considered problems are stimulated using PYTHON.

Keywords: fractional integro-differential equations; Hermite polynomials; Caputo derivative.

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1. Introduction

A fractional derivative is a derivative of any arbitrary order, real or complex, in applied mathematics and mathematical analysis. The theory of fractional order derivatives and integrals

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can be traced back to Grunwald, Liouville, Leibniz, Riemann and Letnikov. There are numerous books on fractional calculus and fractional differential equations that are worth reading [7],[5],[10]. FIDEs can be used to simulate many problems in science and engineering, including biomedical engineering, earthquake engineering, and fluid dynamics. It is necessary to acquire the solution to these equations in order to better analyse these systems. Due to the difficulty of solving fractional integro-differential equations analytically, several researchers have implemented various numerical methods to find the solution.

With the help of a constructed orthogonal polynomial basis function, Oyedepo T. et al. [9] proposed a numerical solution of linear FIDE using the standard least square method. To find approximate solution of FIDE, Osama and Sarmad [8] utilized Bernstein piecewise polynomials. In [3], Laguerre polynomials(LP) are used to solve approximate solutions of Fractional Fredholm integro-differential equations. To solve FIDE, D.Sh. Mohammed [4] used the least squares method with the aid of a shifted Chebyshev polynomial. The numerical solution of linear FIDE using the least squares method with the aid of shifted Laguerre polynomial is investigated in the work [1]. For solving FIDE, Mahdy et al. [2] used the least squares method with shifted Chedbyshev polynomials of third kind as the basis function. Nanware et al. [6] focus on the study of Bernstein polynomial(BP) to find the numerical solution of FIDE with Caputo derivative. For the solution of FIDE, Oyedepo et al. [11] provided two methods: the least square method and the homotopy perturbation method, both employing Bernestein polynomials as basis functions.

Consider the following linear fractional integro-differential equation:

$$D_*^{\alpha} y(x) = g(x) + \int_0^1 k(x,t) y(t) dt, \quad 0 \leq x,t \leq 1 \tag{1}$$

with the following initial conditions:

$$y^{(j)}(0) = c_j, j = 0,1,......n-1, n-1 < \alpha \leq n, n \in N \tag{2}$$

where $D_*^\alpha y(x)$ is the Caputo fractional derivative of $y(x)$, $k(x,t)$,$g(x)$ are given functions, x and t are real variables varying in the interval $[0,1]$ and $y(x)$ is the unknown function to be determined.

2. PRELIMINARIES

In this section, will go over some fundamental definitions in fractional calculus and properties that will help us for formulation of method to obtain the numerical solution of given problem.

Definition 2.1: The Caputo fractional derivative operator D^α of order α is defined as[5]:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \quad \alpha > 0$$

where $m-1 < \alpha \leq m, m \in N, x > 0$

We have the following properties:

(1) $J^\alpha J^\nu f = J^{\alpha+\nu} f, \alpha, \nu > 0, f \varepsilon C_\mu, \mu > 0$

(2) $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} x^{\beta+\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0$

(3) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k}, x > 0, m-1 < \alpha \leq m$

(4) $D^\alpha J^\alpha f(x) = f(x), \quad x > 0, \quad m-1 < \alpha \leq m$

(5) D^α *C=0*, C is a constant

(6) $D^\alpha x^\beta = \begin{cases} 0, & \beta \in N_0, \quad \beta < \lceil \alpha \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, and \quad \beta \geq \lceil \alpha \rceil \end{cases}$

where $\lceil \alpha \rceil$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, ...\}$

Definition 2.2: The Hermite polynomials are given by[12]

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

The Hermite polynomials are defined on R and can be determined with the aid of the following recurrence formula $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$,

$H_0(x) = 1, H_1(x) = 2x, n = 1, 2, ...$

3. HERMITE POLYNOMIAL METHOD

In this part, we use Hermite polynomials to construct an approximate solution for the fractional integro-differential equation (1)-(2).

Taking the fractional integration to both sides of the equation (1) we get

$$J^{\alpha}D^{\alpha}y(x) = J^{\alpha}g(x) + J^{\alpha}\left(\int_0^1 k(x,t)y(t)dt\right)$$

$$y(x) = \sum_{k=0}^{m-1} y^k(0^+)\frac{x^k}{k!} + J^{\alpha}g(x) + J^{\alpha}\left(\int_0^1 k(x,t)y(t)dt\right) \tag{3}$$

We employ the Hermite polynomial basis to estimate the solution of (1).

$$y(x) = \sum_{i=0}^{n} a_i H_n(x) \tag{4}$$

Substituting (4) into (3)

$$\sum_{i=0}^{n} a_i H_n(x) = \sum_{k=0}^{m-1} y^k(0^+)\frac{x^k}{k!} + J^{\alpha}g(x) + J^{\alpha}\left(\int_0^1 k(x,t)\sum_{i=0}^{n} a_i H_n(t)dt\right)$$

$$\sum_{i=0}^{n} a_i H_n(x) - J^{\alpha}\left(\sum_{i=0}^{n} a_i \psi(x)\right) = \sum_{k=0}^{m-1} y^k(0^+)\frac{x^k}{k!} + J^{\alpha}g(x)$$

where

$$\psi(x) = \int_0^1 k(x,t)H_n(t)dt.$$

Thus we have,

$$\sum_{i=0}^{n} a_i\left[H_n(x) - J^{\alpha}\psi(x)\right] = \sum_{k=0}^{m-1} y^k(0^+)\frac{x^k}{k!} + J^{\alpha}g(x). \tag{5}$$

Putting $x = x_j, j = 0, ..., n$ into equation (5), and $x_j^{'}s$ are being chosen as suitable distinct points in (a,b), we obtain the linear system of $(n+1)$ equations with $(n+1)$ unknowns. To retrieve the unknown $a_i^{\prime}s$, the $(n+1)$ linear equations are solved using the computer package such as Scilab. Using $a_i(i = 0, 1, 2...., n)$ in equation (4) we obtain the approximate solution to the Fractional Integro-differential equation (1).

In this work we defined absolute error as:

$$Absolute \quad error = |Y(x) - Y_m(x)|, \quad 0 \le x \le 1$$

where $Y(x)$ is the exact solution and $Y_m(x)$ is the approximate solution.

4. Numerical Examples

We used two examples to explain the Proposed approach. Scilab is used to generate all of the results.

Example 1. *Consider the fractional integro-differential equation*

$$D^{\frac{1}{2}}y(x) = \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xty(t)dt, \quad 0 \le x \le 1, \quad y(0) = 0 \tag{6}$$

with exact solution $y(x) = x^2 - x$.

By taking the fractional integration on both sides of the equation (6)*, we get*

$$y(x) = \sum_{k=0}^{m-1} y^k(0^+)\frac{x^k}{k!} + J^\alpha \left\{ \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xty(t)dt \right\} \tag{7}$$

To determine the approximate solution of (6) *we set $y(x) = \sum_{i=0}^{2} a_i H_i(x)$. After substituting in equation* (7)*, we have*

$$\sum_{i=0}^{2} a_i H_i(x) = J^\alpha \left\{ \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt\left[\sum_{i=0}^{2} a_i H_i(t)\right] dt \right\}$$

After simplifying the above equation we get,

$$\begin{aligned} &a_0\left[1 - \frac{x^{1+\alpha}}{2\Gamma(2+\alpha)}\right] + a_1\left[2x - \frac{2x^{1+\alpha}}{3\Gamma(2+\alpha)}\right] + a_2\left[4x^2 - 2\right] \\ &= \frac{8\Gamma(2.5)x^{1.5+\alpha}}{3\sqrt{\pi}\Gamma(2.5+\alpha)} - \frac{2\Gamma(1.5)x^{0.5+\alpha}}{\sqrt{\pi}\Gamma(1.5+\alpha)} + \frac{x^{1+\alpha}}{12\Gamma(2+\alpha)} \end{aligned}$$

When $\alpha = 0.5$

$$a_0\left[1 - \frac{x^{1.5}}{2\Gamma(2.5)}\right] + a_1\left[2x - \frac{2x^{1.5}}{3\Gamma(2.5)}\right] + a_2\left[4x^2 - 2\right] = \frac{8\Gamma(2.5)x^2}{3\sqrt{\pi}\Gamma(3)} - \frac{2\Gamma(1.5)x}{\sqrt{\pi}\Gamma(2)} + \frac{x^{1.5}}{12\Gamma(2.5)}$$

Also substituting $x = 0.1, 0.2,$ *and* 0.3 *in above equation, we get a linear system of equations:*

$$(0.9881058)a_0 + (0.1841411)a_1 + (-1.96)a_2 = -0.0880176$$

$$(0.9663582)a_0 + (0.3551443)a_1 + (-1.84)a_2 = -0.1543930$$

$$(0.9381961)a_0 + (0.5175948)a_1 + (-1.64)a_2 = -0.1996994$$

Solving the above equations we get:

$a_0 = 0.4999971, a_1 = -0.4999994$, and $a_2 = 0.2499986$

The values are then substituted into equation (4) we get the approximate solution of (6).

Approximate solution is

$y(x) = 0.4999971 - 0.4999994(2x) + 0.2499986(4x^2 - 2)$

Following TABLE 1 represent a comparison between the approximate solution when $\alpha = 1/2$ with the exact solution $y = x^2 - x$

x	exact solution	Approximate solution of HP ($n = 2$)	Approximate solution of BP($n = 3$)	Approximate solution of LP($n = 3$)	Absolute error of HP	Absolute error of BP	Absolute error of LP
0.1	-0.09	-0.09	-0.0900005	-0.0900001	0	0.0000005	0.0000001
0.2	-0.16	-0.1600001	-0.1600017	-0.1600002	0.0000001	0.0000017	0.0000002
0.3	-0.21	-0.2100002	-0.2100033	-0.2100004	0.0000002	0.0000033	0.0000004
0.4	-0.24	-0.2400005	-0.2400066	-0.2400008	0.0000005	0.0000066	0.0000008
0.5	-0.25	-0.2500009	-0.250013	-0.2500001	0.0000009	0.000013	0.0000001
0.6	-0.24	-0.2400014	-0.240024	-0.2400019	0.0000014	0.000024	0.0000019
0.7	-0.21	-0.2100020	-0.2100409	-0.2100027	0.000002	0.0000409	0.00000027
0.8	-0.16	-0.1600027	-0.160065	-0.1600036	0.0000027	0.000065	0.0000036
0.9	-0.09	-0.0900036	-0.0900977	-0.0900046	0.0000036	0.0000977	0.0000046

TABLE 1. Numerical results of Example 1

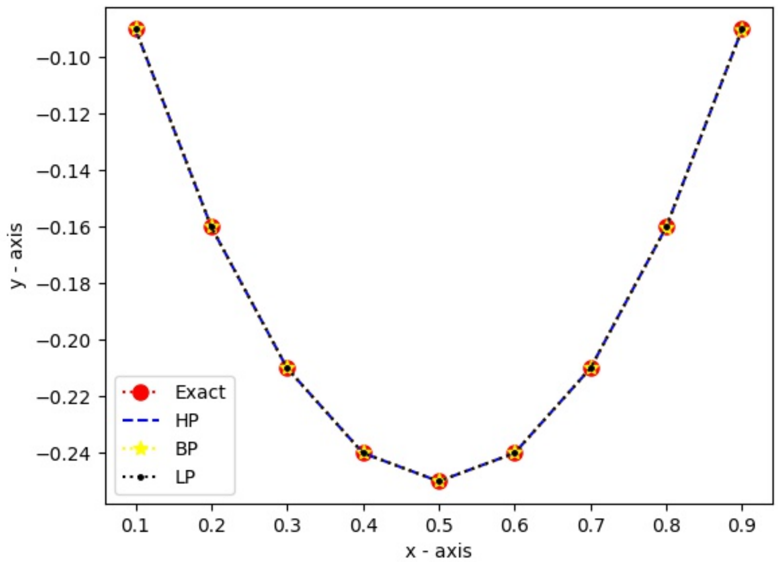

FIGURE 1. Comparison between approximate and exact solution of Example 1

Example 2. *Consider the fractional integro-differential equation*

$$D^{\frac{5}{6}}y(x) = -\frac{3}{91}\frac{x^{1/6}\Gamma(5/6)(-91+216x^2)}{\pi} + (5-2e)x + \int_0^1 xe^t y(t)dt$$

subject to $y(0)=0$*, with exact solution* $y(x) = x - x^3$.

Approximate solution is

$y(x) = 0.0000037 - 0.2500073(2x) + 0.0000018(4x^2-2) - 0.125001(8x^3-2x)$

Following TABLE 2 represent a comparison between the approximate solution with the exact solution $y = x - x^3$

x	exact solution	Approximate solution of HP $(n=3)$	Approximate solution of BP$(n=3)$	Approximate solution of LP$(n=3)$	Absolute error of HP	Absolute error of BP	Absolute error of LP
0.1	0.099	0.0989999	0.0990208	0.0990209	0.0000001	0.0000208	0.0000209
0.2	0.192	0.1919998	0.192092	0.192091	0.0000002	0.000092	0.000091
0.3	0.273	0.2729998	0.2731545	0.2731555	0.0000002	0.0001545	0.0001555
0.4	0.336	0.3359997	0.3360001	0.3362648	0.0000003	0.0000001	0.0002648
0.5	0.375	0.3749996	0.3750002	0.3754694	0.0000004	0.0000002	0.000694
0.6	0.384	0.3839994	0.3840003	0.3848197	0.0000006	0.0000003	0.0008197
0.7	0.357	0.3569991	0.3570005	0.3583661	0.0000009	0.0000005	0.0013661
0.8	0.288	0.2879985	0.2880008	0.2901592	0.0000015	0.0000008	0.0021592
0.9	0.171	0.1709978	0.1710011	0.1742493	0.0000022	0.0000011	0.0032493

TABLE 2. Numerical results of Example 2

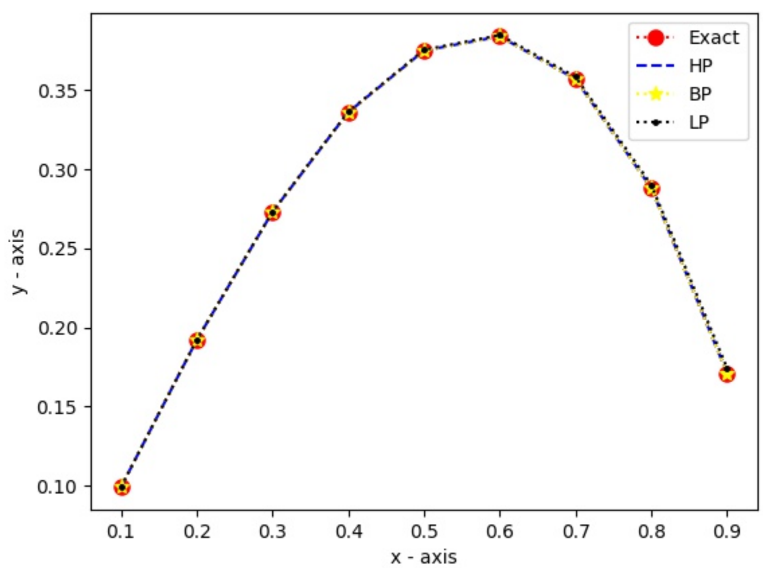

FIGURE 2. Comparison between approximate and exact solution of Example 2

5. Conclusion

The approximate solution of the fractional Integro-differential equations by Hermite polynomial is described in this work, which is a very simple and straight forward method. The approach was illustrated with examples, and the results demonstrate that the method is good and closely agrees with the exact solutions even at lower values of n.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] A.M.S. Mahdy, R.T. Shwayyea, Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method, Int. J. Sci. Eng. Res. 7(4) (2016), 1589-1596.

[2] A.M.S. Mahdy, E.M.H. Mohamed, G.M.A. Marai, Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomials of the third kind method, Theor. Math. Appl. 6(4) (2016), 87-101.

[3] A. Daşcioğlu, D.V. Bayram, Solving Fractional Fredholm Integro-Differential Equations by Laguerre Polynomials, Sains Malays. 48(1) (2019), 251-257.

[4] D.Sh. Mohammed, Numerical solution of Fractional Integro-differential equations by least squares method and shifted chebyshev polynomial, Math. Probl. Eng. 2014 (2014), 431965.

[5] I. Podlubny, Fractional differential equations, Academic Press, San Diego, Calif, USA 198, (1999).

[6] J.A. Nanware, P.M. Goud, T.L. Holambe, Solution of Fractional Integro-differential equations by Bernstein Polynomials, Malaya. J. Mat. 1 (2020), 581-586.

[7] K. Oldham, J. Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order, Elsevier. (1974).

[8] O.H. Mohammed, S.A. Altaie, Approximate solution of Fractional Integro-Differential equations by using Bernstein polynomials, Eng. Technol. J. 30 (8) (2012), 1362-1373.

[9] T. Oyedepo, O.A. Taiwo, Numerical Studies for Solving Linear Fractional Integro-differential Equations Based on Constructed Orthogonal Polynomials, ATBU J. Sci. Technol. Educ. 7(1) (2019), 1-13.

[10] R. Herrmann, Fractional Calculus:An Introduction for Physicists, World Scientific, Singapore, (2014).

[11] T. Oyedepo, O.A. Taiwo, A.F. Adebisi, C.Y. Ishola, O.E. Faniyi, Least Squares Method and Homotopy Perturbation Method for Solving Fractional Integro-Differential Equations, Pac. J. Sci. Technol. 20(1) (2019), 86-95.

[12] W.W. Bell, Special functions for Scientists and Engineers, London D. Van Norstrand Company, (1968).