Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7553-7567
https://doi.org/10.28919/jmcs/6636
ISSN: 1927-5307

# A NOTE ON CONE METRIC SPACES AND COMMON FIXED POINT RESULTS FOR SET-VALUED MAPPINGS WITH APPLICATIONS

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**Abstract.** The aim of this paper is to establish the existence and uniqueness of common fixed point for a pair of set - valued mappings satisfying more generalized constructive condition in the cone metric spaces setting with normal constant $M = 1$. An illustrative example is provided to support the result obtained. As an application, prove the well–posedness of the common fixed point problem. The presented results generalize many known results in cone metric spaces.

**Keywords:** common fixed point; set–valued mapping; cone metric space; well – posedness.

**2010 AMS Subject Classification:** 47H10, 54H25.

## 1. Introduction

If $(X,d)$ is a complete metric space and $T\colon X \to X$ is a contraction mapping that is $d(Tx,Ty) \leq \alpha d(x,y)$, for $0 < \alpha < 1$ and $x,y \in X$ then the well known Banach contraction mapping principle [7] says that the mapping $T$ has a unique fixed point in $X$. A great number of generalizations of this famous theorem have been obtained by relaxing or weakening the contracting condition and sometimes by withdrawing the requirement of completeness or even both, see [8, 10, 11, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 37, 39] and references given

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Received August 11, 2021

there in. Later, Nadier Jr. [22] has proved multilvalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Many authors have been using the Hausdroff metric to obtain fixed point results for multivalued maps on metric spaces.

Recently, a very interesting generalization of the concept of metric space was obtained by Branciari [9], by replacing the triangle inequality of a metric space by a more general inequality. Correspondingly, the Banach contraction mapping principle was proved in such generalized metric space. Quite Recently, Huang and Zhang [15] generalized the concept of a metrc space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Later Wardowski [42] introduced the concept of multivalued contractions in cone metric spaces and using the notion of normal cones, obtained fixed point theorems for such mappings. As we know, most of known cones are normal with normal constant $M$=1. Further, Rezapour [35] proved two results about common fixed points of multifunctions on cone metric spaces. For a detailed study, see [1, 2, 3, 6, 12, 13, 16, 17, 18, 32, 35, 36, 38, 40, 41] . Motivated by the above work, in this paper, analyze the existence and uniqueness of common fixed point for a pair of set-valued mappings satisfying more generalized contractive condition in cone metric spaces setting with normal constant $M = 1$. An example is given to justify my results. Further, we prove the well-posedness of common fixed point problem. The presented results generalize many known results in cone metric spaces.

## 2. Preliminaries

In this section, recall the definition of cone metric space and some of their properties. The following notions will be used in order to prove the main results.

**Definition 2.1.** *Let E be a real Banach space. A subset P of E is called a cone if the following conditions are satisfied:*

(i) *$P$ is closed nonempty and $P \neq 0$;*

(ii) *$a,b \in \mathbb{R}$, $a,b \geq 0$ and $x,y \in P$ imply that $ax+by \in P$ ;*

(iii) $P \cap (-P) = \{0\}$.

*Given a cone $P$ of $E$, define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. A cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\ \|y\|$. The least positive number satisfying the above inequality is called the normal constant of $P$.*

**Definition 2.2.** *Let $X$ be a nonempty set and $d\colon X \times X \to E$ be a mapping such that the following conditions hold:*

(i) $0 \leq d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$ *;*

(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$*;*

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$*.*

*Then $d$ is called a cone metric on $X$ and $(X,d)$ is called a cone metric space.*

**Example 2.1.** *Let $X = R$, $E = R^2$, $P = \{(x,y) \in E\colon x, y \geq 0\} \subset R^2$ and $d\colon X \times X \to E$ such that $d(x,y) = (|\,x-y\,|, \delta\,|\,x-y\,|)$, where $\delta \geq 0$ is a constant. Then $(X,d)$ is a cone metric space.*

**Example 2.2.** *Let $E = C^1_{\mathbb{R}}([0,\ 1])$ with $\|f\| = \|f\|_\infty + \|f\|_\infty$. The cone $P = \{f \in E : f \geq 0\}$ is a non-normed cone.*

**Definition 2.3.** *Let $(X,d)$ be a cone metric space. We say that $\{x_n\}$ is*

(i) *a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $N$ such that for all $m, n > N$, $d(x_n, x_m) \ll c$;*

(ii) *a convergent sequence if for every $c \in E$ with $0 \ll c$, there is $N$ such that for all $n > N$, $d(x_n, x) \ll c$, for some $x \in X$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

*A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. The limit of a convergent sequence is unique provided $P$ is a normal cone with normal constant $K$(*[15]*).*

**Lemma 2.1.** *Let $(X,d)$ be a cone metric space and $P$ be a normal cone with normal constant $K$. Let $x_n$ be a sequence in $X$. Then $x_n$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.*

**Lemma 2.2.** *Let $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $M=1$ and $A$ be a compact set in $(X,\tau_c)$. Then for every $x \in X$ there exists $a_0 \in A$ such that $\|d(x,a_0)\|$ = $\inf_{a\in A}\|d(x,a)\|$.*

**Lemma 2.3.** *Let $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $M=1$ and $A,B$ be two compact sets in $(X,\tau_c)$. Then $\sup_{x\in B} D(x,A) < \infty$, where $D(x,A)$ = $\inf_{a\in A}\|d(x,a)\|$, for each $x \in X$.*

**Definition 2.4.** *Let $(X,d)$ be a cone metric space, $P$ be a normal cone with normal constant $M=1$, $H_c$ denote the set of all compact subsets of $(X,\tau_c)$ and $A \in H_c$ . Now by lemma(2.2), define*
*$h_A : H_c(X) \to [0,\infty)$ and $d_H : H(X)\times H_c(X) \to [0,\infty)$ by*
*$h_A(B) = \sup_{x\in A} D(x,B) < \infty$, and $d_H(A,B)$ = $\max\{h_A(B),h_B(A)\}$, respectively.*

**Remark 2.1.** *Let $(X,d)$ be a cone metric space and $P$ be a normal cone with normal constant $M=1$. Define $\rho : X\times X \to [0,\infty)$ by $\rho(x,y) = \|d(x,y)\|$. Then, $(X,\rho)$ is ametric space. This implies that for each $A,B \in H_c$ and $x,y \in X$, we have the following relations:*

(i) $D \leq \|d(x,y)\| + D(y,A)$,
(ii) $D \leq D(x,B) + h_B(A)$, *and*
(iii) $D \leq \|d(x,y)\| + D(y,B) + h_B(A)$,

**Definition 2.5.** *(Implicit Relation) Let $\Phi$ be the class of real valued continuous functions $\phi : \mathbb{R}^3_+ \to R_+$ non-decreasing in the first argument and satisfying the following condition: for $x,y > 0$,*

(i) $x \leq \phi(y, \frac{x+y}{2}, \frac{x+y}{2})$ *or*
(ii) $x \leq \phi(x,0,x)$,

*there exists a real number $0 < h < 1$ such that $x \leq hy$.*

**Example 2.3.** *Let $\phi(r,s,t) = r - \alpha\min(s,t) + (2+\alpha)t$, where $\alpha > 0$.*

**Example 2.4.** *Let $\phi(r,s,t) = r^2 - ar\max(s,t) - bs$, where $a > 0, b > 0$.*

**Example 2.5.** *Let $\phi(r,s,t) = r + c\max(s,t)$, where $c \geq 0$.*

**Definition 2.6.** *A sequence $\{x_n\}$ in a cone metric space in $X$ is said to be asymptotically $T-$regular if* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

## 3. Main Results

**Theorem 3.1.** *Let $(X,d)$ be a complete metric space with normal constant $M = 1$ and $S,T : X \to H_c(X)$ be two set-valued mappings such that*

$$d(Sx,Ty) \le \alpha \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Sx)+d(y,Ty)}{2}\} \tag{3.1}$$

*for all $x,y \in X$ where $0 \le \alpha < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be a arbitrary point. Then by lemma 2.2, there exist $x_1 \in Sx_0$ and $x_2 \in Tx_1$ such that $D(x_0,Sx_0)$=$\|d(x_0,x_1)\|$ and $D(x_1,Tx_1)$=$\|d(x_1,x_2)\|$.
Likewise, for $n \in \mathbb{N}$, we define a sequence $x_n$ in $X$ such that $x_{2n-1} \in Sx_{2n-2}$, $x_{2n} \in Tx_{2n-1}$.Therefore, $D(x_{2n-2},Sx_{2n-2})$= $\|d(x_{2n-2},x_{2n-1})\|$ and $D(x_{2n-1},Tx_{2n-1})$= $\|d(x_{2n-1},x_{2n})\|$ for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, we have the following

$$\begin{aligned}
\|d(x_{2n},x_{2n+1})\| &= D(x_{2n},Sx_{2n}) \\
&\le h_{Tx_{2n-1}}(Sx_{2n}) \\
&\le d_H(Tx_{2n-1},Sx_{2n}) \\
&\le \alpha \max\{D(x_{2n},x_{2n-1}), D(x_{2n},Sx_{2n}), D(x_{2n-1},Tx_{2n-1}) \\
&\qquad \frac{D(x_{2n},Sx_{2n})+D(x_{2n-1},Tx_{2n-1})}{2}\} \\
&= \alpha \max\{\|d(x_{2n},x_{2n-1})\|, \|d(x_{2n},x_{2n+1})\|, \|d(x_{2n-1},x_{2n})\| \\
&\qquad \frac{\|d(x_{2n},x_{2n+1})\|+\|d(x_{2n-1},x_{2n})\|}{2}\} \\
&= \alpha \max\{\|d(x_{2n},x_{2n+1})\|, \|d(x_{2n-1},x_{2n})\|\}.
\end{aligned}$$

*Case*(i): If $\max\{\|d(x_{2n},x_{2n+1})\|, \|d(x_{2n-1},x_{2n})\|\} = \|d(x_{2n},x_{2n+1})\|$, then $\|d(x_{2n},x_{2n+1})\| \le \alpha \|d(x_{2n},x_{2n+1})\|$ which implies that $\|d(x_{2n},x_{2n+1})\| \to 0$ as $n \to \infty$, since $0 < \alpha < 1$.
*Case*(ii): If $\max\{\|d(x_{2n},x_{2n+1})\|, \|d(x_{2n-1},x_{2n})\|\} = \|d(x_{2n-1},x_{2n})\|$, then $\|d(x_{2n},x_{2n+1})\| \le \alpha \|d(x_{2n-1},x_{2n})\|$ . Proceeding in this way, we obtain $\|d(x_{2n},x_{2n+1})\| \le \alpha^{2n} \|d(x_0,x_1)\|$, $n \in \mathbb{N}$.

Also for $n > m$, we have

$$\begin{aligned}\|d(x_n,x_m)\| &\leq \|d(x_n,x_{n-1})\| + \|d(x_{n-1},x_{n-2})\| + \ldots\cdots + \|d(x_{m+1},x_m,u)\| \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \ldots\cdots + \alpha^m)\,\|d(x_1,x_0)\| \\ &\leq \frac{\alpha^m}{1-\alpha}\,\|d(x_1,x_0)\|\end{aligned}$$

Thus $\|d(x_n,x_m)\| \to 0$, as $n \to \infty$, since $\frac{k^m}{1-k} \to 0$ as $n \to \infty$. Therefore, in both cases, $\{x_n\}$ is a Cauchy sequence in $X$. Hence there exists a point $z \in X$ such that $x_n \to z$, as $n \to \infty$. Further, by using Remark 2.1, we have

$$\begin{aligned}D(z,Sz) &\leq D(z,Tx_{2n-1}) + h_{Tx_{2n-1}}(Sz) \\ &\leq D(z,Tx_{2n-1}) + d_H(Tx_{2n-1},Sz) \\ &\leq \|d(z,x_{2n})\| + \alpha \max\{D(z,x_{2n-1}), D(z,Sz), D(x_{2n-1},Tx_{2n-1}), \\ &\qquad \frac{D(z,Sz) + D(x_{2n-1},Tx_{2n-1})}{2}\} \\ &= \|d(z,x_{2n})\| + \alpha \max\{D(z,x_{2n-1}), D(z,Sz), D(x_{2n-1},x_{2n}), \\ &\qquad \frac{D(z,Sz) + D(x_{2n-1},Tx_{2n})}{2}\} \\ &= \|d(z,x_{2n})\| + \alpha \max\{D(z,x_{2n-1}), D(z,Sz), D(x_{2n-1},x_{2n})\},\end{aligned}$$

now, letting $n \to \infty$, we get $D(z,Sz) = 0$. Hence by lemma 2.2, $z \in Sz$. Similarly,

$$\begin{aligned}D(z,Tz) &\leq D(z,Sx_{2n}) + h_{Sx_{2n}}(Tz) \\ &\leq D(z,Sx_{2n}) + d_H(Sx_{2n},Tz) \\ &\leq \|d(z,x_{2n+1})\| + \alpha \max\{D(z,x_{2n}), D(z,Tz), D(x_{2n},Sx_{2n}), \\ &\qquad \frac{D(z,Tz) + D(x_{2n},Sx_{2n})}{2}\} \\ &= \|d(z,x_{2n+1})\| + \alpha \max\{D(z,x_{2n}), D(z,Tz), D(x_{2n},x_{2n+1})\},\end{aligned}$$

now, letting $n \to \infty$, we get $D(z,Tz) = 0$. Hence by lemma 2.2, $z \in Tz$. Therefore, $z \in X$ is a common fixed point of $S$ and $T$.

*Uniqueness*:

Let $\widetilde{z}$ be another common fixed point of $S$ and $T$, that is, $S\widetilde{z} = T\widetilde{z} = \widetilde{z}$. Then

$$\begin{aligned}\|d(z,\widetilde{z})\| &= \|d(Sz,T\widetilde{z})\| \\ &\le \alpha \max\{\|d(z,\widetilde{z})\|, \|d(z,Sz)\|, \|d(\widetilde{z},T\widetilde{z})\|, \\ &\qquad \frac{\|d(z,Sz)\| + \|d(\widetilde{z},T\widetilde{z})\|}{2}\}\end{aligned}$$

□

Which implies that $\|d(z,\widetilde{z})\| = 0$, since $\alpha < 1$ for all $u \in X$. Thus $z$ is a unique common fixed point of $S$ and $T$.

**Remark 3.1.** *Theorem 3.1 generalizes Theorem 3.1 of* [12]*, Also, my result establishes the uniqueness of the common fixed point of $S$ and $T$.*

**Corollary 3.1.** *Let (X, d) be a complete cone metric space with normal constant $M = 1$ and $S,T : X \to H_c(X)$ be two set-valued mappings such that*

$$d(Sx,Ty) \le ad(x,y) + bd(x,Sx) + cd(y,Ty) + e\frac{d(x,Sx)+d(y,Ty)}{2}$$

*for all $x,y \in X$ and $a,b,c,e \ge 0$, where $a+b+c+e < 1$. Then $S$ and $T$ have a unique common fixed point in $X$.*

**Remark 3.2.** *Note that corrollary 3.1 reduces to;*

1. *Theoram 2.3 of Razapour* [35] *when $b = c$ and $a = 0 = e$ in corollary 3.1.*

2. *Nadler's result* [22] *in the case $S = T$ and $b = c = e = 0$.*

3. *Corollary 3.3 of Poonkundran and Dharmalingam* [30] *if we take $e = 0$.*

*Note the following results are consequences of corollary 3.1.*

**Corollary 3.2.** *Let (X, d) be a complete cone metric space with normal constant $M = 1$ and $S,T : X \to H_c(X)$ be two set-valued mappings such that*

$$d(Sx,Ty) \le ad(x,Sx) + bd(y,Ty) + c\frac{d(x,Sx)+d(y,Ty)}{2}$$

*for all* $x, y \in X$ *and* $a, b \geq 0$*, where* $a+b<1$*. Then* $S$ *and* $T$ *have a unique common fixed point in* $X$*.*

**Corollary 3.3.** *Let (X, d) be a complete cone metric space with normal constant* $M = 1$ *and* $S\colon X \to H_c(X)$ *be two set-valued mappings such that*

$$d(Sx,Sy) \leq ad(x,y) + bd(x,Sx) + cd(y,Sy) + e\frac{d(x,Sx)+d(y,Sy)}{2}$$

*for all* $x, y \in X$ *and* $a, b, c \geq 0$*, where* $a+b+c<1$*. Then* $S$ *and* $T$ *have a unique common fixed point in* $X$*.*

**Corollary 3.4.** *Let (X, d) be a complete cone metric space with normal constant* $M = 1$ *and* $S\colon X \to H_c(X)$ *be two set-valued mappings such that*

$$d(Sx,Sy) \leq ad(x,Sx) + bd(y,Sy) + c\frac{d(x,Sx)+d(y,Sy)}{2}$$

*for all* $x, y \in X$ *and* $a, b, c \geq 0$*, where* $a+b<1$*. Then* $S$ *and* $T$ *have a unique common fixed point in* $X$*.*

**Remark 3.3.** *We obtain the set-valued Kamman type contracting condition* [19] *when* $c = 0$ *in corollary 3.3 in the setting of cone metric spaces.*

**Theorem 3.2.** *Let (X, d) be a complete cone metric space with normal constant* $M = 1$ *and* $S$ *and* $T\colon X \to H_c(X)$ *be two set-valued mappings such that* $d(Sx,Ty) \leq \alpha\{d(x,y), d(x,Ty), d(y,Sx),$
$\frac{d(x,Ty)+d(y,Sx)}{2}\}$ *for all* $x, y \in X$ *and* $0 \leq \alpha < 1$*. Then* $S$ *and* $T$ *have a unique common fixed point in* $X$*.*

*Proof.* Using the similar argument of the proof of theorem 3.1, we can show that there exists a Cauchy sequence $x_n$ in $X$ such that $x_{2n-1} \in Sx_{2n-2}, x_{2n} \in Tx_{2n-1}$. Therefore, $D(x_{2n-2}, Sx_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|$ and $D(x_{2n-1}, Tx_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|$ for all $n \in \mathbb{N}$. Thus, we can find a element $\widetilde{x} \in X$ such that $x_n \to \widetilde{x}$ as $n \to \infty$.

Applying Remark 2.1, we obtain

$$\begin{aligned}
D(\widetilde{x},S\widetilde{x}) &\le D(\widetilde{x},Tx_{2n-1})+h_{Tx_{2n-1}}(S\widetilde{x})\\
&\le D(\widetilde{x},Tx_{2n-1})+d_H(Tx_{2n-1},S\widetilde{x})\\
&\le \|d(\widetilde{x},x_{2n})\|+\alpha\max\{D(\widetilde{x},x_{2n-1}),D(\widetilde{x},Tx_{2n-1}),\\
&\qquad D(x_{2n-1},S\widetilde{x}),\frac{D(\widetilde{x},Tx_{2n-1})+D(x_{2n-1},S\widetilde{x})}{2}\}\\
&= \|d(\widetilde{x},x_{2n})\|+\alpha\max\{D(\widetilde{x},x_{2n}),D(x_{2n-1},S\widetilde{x}),D(\widetilde{x},x_{2n-1})\},
\end{aligned}$$

now, letting $n\to\infty$, we get $D(\widetilde{x},S\widetilde{x})=0$. Hence, by lemma 2.2, $\widetilde{x}\in S\widetilde{x}$.

$$\begin{aligned}
D(\widetilde{x},T\widetilde{x}) &\le D(\widetilde{x},Sx_{2n})+h_{Sx_{2n}}(T\widetilde{x})\\
&\le D(\widetilde{x},Sx_{2n})+d_H(Sx_{2n},T\widetilde{x})\\
&\le \|d(\widetilde{x},x_{2n+1})\|+\alpha\max\{D(\widetilde{x},x_{2n}),D(\widetilde{x},Sx_{2n}),\\
&\qquad D(x_{2n},T\widetilde{x}),\frac{D(\widetilde{x},Sx_{2n})+D(x_{2n},T\widetilde{x})}{2}\}\\
&= \|d(\widetilde{x},x_{2n+1})\|+\alpha\max\{D(\widetilde{x},x_{2n+1}),D(x_{2n},T\widetilde{x}),D(\widetilde{x},x_{2n})\},
\end{aligned}$$

now, letting $n\to\infty$, we get $D(\widetilde{x},T\widetilde{x})=0$. Hence, by lemma 2.2, $\widetilde{x}\in T\widetilde{x}$. Therefore, $\widetilde{x}\in X$ is a common fixed point of $S$ and $T$. *Uniqueness:*

Let $\widetilde{z}$ be another common fixed point of $S$ and $T$, that is $S\widetilde{z}=T\widetilde{z}=\widetilde{z}$. Then

$$\begin{aligned}
\|d(\widetilde{x},\widetilde{z})\| &= \|d(S\widetilde{x},T\widetilde{z})\|\\
&\le \alpha\max\{\|d(\widetilde{x},\widetilde{z})\|,\|d(\widetilde{x},T\widetilde{z})\|,\|d(\widetilde{z},S\widetilde{x})\|,\\
&\qquad \frac{\|d(x,T\widetilde{z})\|+\|d(\widetilde{z},S\widetilde{x})\|}{2}\}
\end{aligned}$$

which implies that $\|d(\widetilde{x},\widetilde{z})\|=0$, since $\alpha<1$ for all $u\in X$. Thus $\widetilde{x}$ is a unique common fixed point of $S$ and $T$. □

**Remark 3.4.** *Theorem 3.2 generalizes Theorem 3.2 of* [12]. *Further, it establishes the uniqueness of the common fixed point of $S$ and $T$.*

**Corollary 3.5.** *Let (X, d) be a complete cone metric space with normal constant* $M = 1$ *and* $S,T: X \to H_c(X)$ *be two set-valued mappings such that* $d(Sx,Ty) \leq ad(x,y) + bd(x,Ty) + cd(y,Sx) + e\frac{d(x,Ty)+cd(y,Sx)}{2}$ *for all* $x,y \in X$ *and* $a,b,c \geq 0$*, where* $a+b+c < 1$*. Then* $S$ *and* $T$ *have a unique common fixed point in* $X$*.*

*The following example supports my results.*

**Example 3.1.** *Consider the metric defined i Example 2.1. Now define* $S,T: X \to H_c(X)$ *such that* $Sx = \{0\}$ *and* $Tx = \{x^2\}$*, for all* $x \in X$*.*

$$d(Sx,Ty) = (|y|^2, \delta|y^2|) \tag{3.2}$$

*and*

$$\begin{aligned} &\max\{d(x,y), d(x,Sx), D(y,Ty), \frac{d(x,Sx)+d(y,Ty)}{2}\} \\ &= \max\{(|x-y|, \delta|x-y|), (|x|, \delta|x|), (|y-y^2|, \delta|y-y^2|), \\ &\qquad \frac{(|x|, \delta|x|) + (|y-y^2|, \delta|y-y^2|)}{2}\} \end{aligned} \tag{3.3}$$

*From Equations (3.2) and (3.3), it can be easily viewed that all the conditions of theorem 3.1 and 3.2 are satisfied. Hence* $S$ *and* $T$ *have a unique common fixed point 0.*

## 4. Applications

The concept of well-posedness of a fixed point problem has generated much interest to several mathematicians, for example [4, 5, 21, 34]. Here, we study well-posedness of a common fixed point problem of mappings in theorems 3.1 and 3.2.

**Definition 4.1.** *Let* $(X,d)$ *be a complete cone metric space and* $f$ *be a set mapping. Then the fixed point problem of* $f$ *is said to be well-posed if*

(i) $f$ *has a unique fixed point* $x_0 \in X$,

(ii) *for any sequence* $\{x_n\} \subset X$ *and* $\lim_{n\to\infty} D(x_n, fx_n) = 0$ *we have* $\lim_{n\to\infty} D(x_n, x_0) = 0$.

*Let* $CFP(T,f,X)$ *denote a common fixed point problem of set mappings* $T$ *and* $f$ *on* $X$ *and* $CF(T,f)$ *denote the set of all common fixed points of* $T$ *and* $f$*.*

**Definition 4.2.** *A $CFP(T,f,X)$ is called well-posed if $CF(T,f)$ is singleton and for any sequence $\{x_n\}$ in $X$ with $\widetilde{x} \in CF(T,f)$ and*
$\lim_{n\to\infty} D(x_n, fx_n) = \lim_{n\to\infty} D(x_n, Tx_n) = 0$ *implies* $\widetilde{x} = \lim_{n\to\infty} x_n$.

**Theorem 4.1.** *Let $(X,d)$ be a complete cone metric space and $T,f$ be set mappings on $X$ as in Theorem 3.1. Then the common fixed point problem of $f$ and $T$ is well posed.*

*Proof.* From Theorem 3.1, the mappings $f$ and $T$ have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in $X$ and $\lim_{n\to\infty} D(fx_n, x_n) = \lim_{n\to\infty} D(Tx_n, x_n) = 0$.
Without loss of generality, assume that $v \neq x_n$ for any non-negative integer $n$. Using (3.1) and $v \in fv = Tv$, we get

$$\begin{aligned} D(v,x_n) &\leq D(Tv,Tx_n) + D(Tx_n,x_n) \\ &= D(fv,Tx_n) + D(Tx_n,x_n) \\ &\leq D(Tx_n,x_n) + \alpha \max(D(v,x_n), D(v,fv), \\ &\qquad D(x_n,Tx_n), \frac{d(v,fv)+d(x_n,Tx_n)}{2}) \end{aligned}$$

As $n \to \infty$ we obtain $(1-\alpha)D(v,x_n) \leq 0$ which is a contraction since $\alpha < 1$. Hence we obtain $d(v,x_n) \to 0$ as $n \to \infty$. This completes the proof. □

**Corollary 4.1.** *Let (X, d) be a complete cone metric space with normal constant $M = 1$ and $S,T : X \to H_c(X)$ be two set-valued mappings such that*
$d(Sx,Ty) \leq ad(x,y) + bd(x,Sx) + cd(y,Ty) + e\frac{d(x,Sx)+d(y,Ty)}{2}$ *for all $x,y \in X$ and $a,b,c,e \geq 0$, where $a+b+c+e < 1$. Then the common fixed point problem of $S$ and $T$ is well-posed.*

**Corollary 4.2.** *Let (X, d) be a complex cone metric space with normal constant $M = 1$ and $S,T : X \to H_c(X)$ be two set-valued mappings such that*
$d(Sx,Ty) \leq ad(x,Sx) + bd(y,Ty) + c\frac{d(x,Sx)+d(y,Ty)}{2}$ *for all $x,y \in X$ and $a,b \geq 0$, where $a+b < 1$. Then the common fixed point problem of $S$ and $T$ is well-posed.*

**Corollary 4.3.** *Let (X, d) be a complete cone metric space with normal constant $M = 1$ and $S : X \to H_c(X)$ be a set-valued mappings such that*

*$d(Sx,Sy) \leq ad(x,y)+bd(x,Sx)+cd(y,Sy)+e\frac{d(x,Sx)+d(y,Sy)}{2}$ for all $x,y \in X$ and $a,b,c \geq 0$, where $a+b+c<1$. Then the common fixed point problem of $S$ and $T$ is well-posed.*

**Corollary 4.4.** *Let (X, d) be a complete cone metric space with normal constant $M=1$ and $S\colon X \to H_c(X)$ be a set-valued mappings such that*
*$d(Sx,Sy) \leq ad(x,Sx)+bd(y,Sy)+c\frac{d(x,Sx)+d(y,Sy)}{2}$ for all $x,y \in X$ and $a,b \geq 0$, where $a+b<1$. Then the common fixed point problem of $S$ and $T$ is well-posed.*

**Theorem 4.2.** *Let $(X,d)$ be a complete cone metric space with normal constant $M=1$ and $S,T: X \to H_C(X)$ be two set-valued mappings such that*
*$d(Sx,Ty) \leq \alpha \max\{d(x,y),d(x,Ty),d(y,Sx),\frac{d(x,Ty)+d(y,Sx)}{2}\}$ for all $x,y \in X$ and $0 \leq \alpha < 1$. Then the common fixed point problem of $S$ and $T$ is well-posed.*

*Proof.* Note that the mappings $f$ and $T$ have a unique common fixed point by Theorem 3.2, say $v \in X$. Let $\{x_n\}$ be a sequence in $X$ and $\lim_{n\to\infty} D(fx_n,x_n) = \lim_{n\to\infty} D(Tx_n,x_n) = 0$.
Without loss of generality, assume that $v \neq x_n$ for any non-negative integer $n$. Using (3.1) and $v \in fv = Tv$, we get

$$\begin{aligned}
D(v,x_n) &\leq D(Tv,Tx_n)+D(Tx_n,x_n)\\
&= D(fv,Tx_n)+D(Tx_n,x_n)\\
&\leq D(Tx_n,x_n)+\alpha \max(D(v,x_n),D(x_n,fx_n),\\
&\qquad D(x_n,Tv),\frac{d(v,fx_n)+d(x_n,Tv)}{2})
\end{aligned}$$

As $n \to \infty$ we obtain $(1-\alpha)D(v,x_n) \leq 0$ which implies that $D(v,x_n) \to 0$ as $n \to \infty$. Therefore the common fixed point problem of $S$ and $T$ is well-posed. □

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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