Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7717-7727
https://doi.org/10.28919/jmcs/6637
ISSN: 1927-5307

HARTWIG'S TRIPLE REVERSE ORDER LAW FOR THE CORE INVERSE IN C^*-ALGEBRAS

D. KRISHNASWAMY, V. VIJAYASELVI*

Department of Mathematics, Annamalai University, Annamalainagar-608 002, Tamilnadu, India

Abstract. In this paper Hartwig's Triple reverse order law for the core inverses in C^*-algebras. Further we have the simply algebraic solution for the core inverse in C^*-algebras.

Keywords: generalized inverse; reverse order law; C^*-algebra; core inverse.

2010 AMS Subject Classification: 16W10, 15A09, 46L05.

1. Introduction

Let $\mathscr{A}$ be a complex unital C^*-algebra. An element $a \in \mathscr{A}$ is said to be regular if there exists $b \in \mathscr{A}$ for which $aba = a$; any such b is called an inner inverse.

The core inverse for a complex matrix were introduced by Baksalary and Trenkler [1]. Let $\mathscr{A} \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the ring of all $n \times n$ complex matrices. A matrix $X \in M_n(\mathbb{C})$ is called core inverse of A, if it satisfies $AX = P_A$ and $R(X) \subseteq R(A)$, where $R(A)$ denotes the column space of A, and P_A is the orthogonal projector onto $R(A)$. And if such a matrix exists, then it is unique and denoted by $A^{\textcircled{\#}}$.

Suppose A, B and C are complex matrices for which ABC can be defined.

*Corresponding author

E-mail addresses: krishna_swamy2004@yahoo.co.in, vijayaselvi2505@gmail.com

Received August 11, 2021

We use notations

$$P = A^{\#}ABCC^{\#}, \qquad Q = CC^{\#}B^{\#}A^{\#}A \tag{1}$$

2. Preliminaries

Definition 2.1. [13] An element a is Hermitian if $a^* = a$, and a is called an idempotent if $a^2 = a$. A Hermitian idempotent is said to be a projection.

Definition 2.2. [1] Let $A \in \mathbb{C}_{n\times n}$. A matrix $A^{\#} \in \mathbb{C}_{n,n}$ satisfying $(i) AA^{\#} = P_A$ and $(ii) R(A^{\#}) \subseteq R(A)$ is called core inverse of A.

Definition 2.3. [1] The core inverse of $a \in \mathscr{A}$ is the element $x \in \mathscr{A}$ which satisfies

$(1)\ axa = a \quad (2)\ xax = x \quad (3)\ (ax)^* = ax \quad (6)\ xa^2 = a \quad (7)\ ax^2 = x$

The element x is unique if it exist and is denoted by $a^{\#}$.

Definition 2.4. [12] Let $\mathscr{A}$ be unital C^*-algebra. The element $a \in \mathscr{A}$ has the core inverse if there exists $x \in \mathscr{A}$ such that

$(1)\ axa = a \qquad (2)\ x\mathscr{A} = a\mathscr{A} \quad$ and $\quad (3)\ \mathscr{A}x = \mathscr{A}a^*$

The unique core inverse will be denoted by $a^{\#}$.

Definition 2.5. [13] An elements a is said to be normal $aa^{\#} = a^{\#}a$.

Definition 2.6. [13] An elements a is said to be invertible if $ab = ba = e$.

Theorem 2.7. [13] For any $a \in \mathscr{A}^{\#}$, the following is satisfied:

(a) $(a^{\#})^{\#} = a$;

(b) $(a^*)^{\#} = (a^{\#})^*$;

(c) $(a^*a)^{\#} = a^{\#}(a^{\#})^*$;

(d) $(aa^*)^{\#} = (a^{\#})^*a^{\#}$;

(e) $a^* = a^{\#}aa^* = a^*aa^{\#}$;

(f) $a^{\#} = (a^*a)^{\#}a^* = a^*(aa^*)^{\#}$;

(g) $(a^*)^{\textcircled{\#}} = a(a^*a)^{\textcircled{\#}} = (aa^*)^{\textcircled{\#}}a$.

3. RESULT

For regular element a, b and c of C^*-algebra $\mathscr{A}$ we use notations

$$p = a^{\textcircled{\#}}abcc^{\textcircled{\#}}, \qquad q = cc^{\textcircled{\#}}b^{\textcircled{\#}}a^{\textcircled{\#}}a.$$

analogously to (1).

Theorem 3.1. Let $\mathscr{A}$ be a complex unital C^*-algebra and let $a,b,c \in \mathscr{A}$ be such that a,b,c and abc are regular. Then the following conditions are equivalent:

(i) $(abc)^{\textcircled{\#}} = c^{\textcircled{\#}}b^{\textcircled{\#}}a^{\textcircled{\#}}$;

(ii) $q \in p\{3,6,7\}$ and both of a^*apq and $qpcc^*$ are hermitian;

(iii) $q \in p\{3,6,7\}$ and both of a^*apq and $qpcc^*$ are EP;

(iv) $q \in p\{3,6,7\}$, $a^*ap\mathscr{A} = q^*\mathscr{A}$ and $cc^*p^*\mathscr{A} = q\mathscr{A}$;

(v) $pq = pqpq$, $a^*ap\mathscr{A} = q^*\mathscr{A}$ and $cc^*p^*\mathscr{A} = q\mathscr{A}$;

Proof: $(i) \Leftrightarrow (ii)$: The condition $xyx = x$ is easily seen to be equivalent to $pqp = p$, while $yxy = y$ holds precisely when $qpq = q$. Next, if xy is Hermitian, so is $a^*xya = a^*apq$.
The converse part, since $(a^*)^{\textcircled{\#}}(a^*apq)a^{\textcircled{\#}} = xy$. Lastly, yx is Hermitian, So is $c(yx)c^* = qpcc^*$.
Again the converse relation part from the condition $c^{\textcircled{\#}}(qpcc^*)(c^*)^{\textcircled{\#}} = yx$.

$(ii) \Leftrightarrow (iii)$: We show that a^*apq and $qpcc^*$ are regular.

Then $a^{\textcircled{\#}}(a^{\textcircled{\#}})^* \in (a^*apq)\{3,6,7\}$ and $(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \in (qpcc^*)\{3,6,7\}$.

Let $x = a^*apq$, $y = a^{\textcircled{\#}}(a^{\textcircled{\#}})^*$, we get

$$\begin{aligned}
xyx &= a^*apqa^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*apq \\
&= a^*apqa^{\textcircled{\#}}(aa^{\textcircled{\#}})^*apq \\
&= a^*apqa^{\textcircled{\#}}aa^{\textcircled{\#}}apq && (\text{Since } a^{\textcircled{\#}}aa^{\textcircled{\#}} = a^{\textcircled{\#}}) \\
&= a^*apqa^{\textcircled{\#}}apq \\
&= a^*apqa^{\textcircled{\#}}aa^{\textcircled{\#}}abcc^{\textcircled{\#}}cc^{\textcircled{\#}}b^{\textcircled{\#}}a^{\textcircled{\#}}a \\
&= a^*apqa^{\textcircled{\#}}abcc^{\textcircled{\#}}b^{\textcircled{\#}}a^{\textcircled{\#}}a \\
&= a^*apqa^{\textcircled{\#}}(abc)(abc)^{\textcircled{\#}}a \\
&= a^*apq \\
&= x
\end{aligned}$$

$$
\begin{aligned}
yxy &= (a^{\circledast})(a^{\circledast})^* a^* apqa^{\circledast}(a^{\circledast})^* \\
&= a^{\circledast}(aa^{\circledast})^* apqa^{\circledast}(a^{\circledast})^* && (\text{Since } (aa^{\circledast})^* = aa^{\circledast}) \\
&= a^{\circledast}aa^{\circledast}apqa^{\circledast}(a^{\circledast})^* && (\text{Since } a^{\circledast}aa^{\circledast} = a^{\circledast}) \\
&= a^{\circledast}apqa^{\circledast}(a^{\circledast})^* \\
&= a^{\circledast}aa^{\circledast}abcc^{\circledast}cc^{\circledast}b^{\circledast}a^{\circledast}aa^{\circledast}(a^{\circledast})^* \\
&= a^{\circledast}abcc^{\circledast}b^{\circledast}a^{\circledast}(a^{\circledast})^* \\
&= a^{\circledast}abc(abc)^{\circledast}(a^{\circledast})^* \\
&= a^{\circledast}(a^{\circledast})^* \\
&= y \\
xy &= a^* apqa^{\circledast}(a^{\circledast})^* \\
&= a^* aa^{\circledast}abcc^{\circledast}cc^{\circledast}b^{\circledast}a^{\circledast}aa^{\circledast}(a^{\circledast})^* \\
&= a^* abcc^{\circledast}b^{\circledast}a^{\circledast}(a^{\circledast})^* \\
&= a^*(abc)(abc)^{\circledast}(a^{\circledast})^* \\
&= a^*(a^{\circledast})^* \\
&= a^{\circledast}a \\
(xy)^* &= (a^* apqa^{\circledast}(a^{\circledast})^*)^* \\
&= (a^* aa^{\circledast}abcc^{\circledast}cc^{\circledast}b^{\circledast}a^{\circledast}aa^{\circledast}(a^{\circledast})^*)^* \\
&= (a^* abcc^{\circledast}b^{\circledast}a^{\circledast}aa^{\circledast}(a^{\circledast})^*)^* \\
&= (a^*(abc)(abc)^{\circledast}(a^{\circledast})^*)^* \\
&= (a^*(a^{\circledast})^*)^* \\
&= a^{\circledast}a
\end{aligned}
$$

Therefore $(xy)^* = xy$

$$
\begin{aligned}
yx^2 &= (a^{\circledast})(a^{\circledast})^*(a^* apq)^2 \\
&= (a^{\circledast})(a^{\circledast})^* a^* apqa^* apq \\
&= (a^{\circledast})(aa^{\circledast})^* apqa^* apq \\
&= (a^{\circledast})aa^{\circledast}apqa^* apq \\
&= a^{\circledast}apqa^* apq \\
&= a^{\circledast}aa^{\circledast}abcc^{\circledast}cc^{\circledast}b^{\circledast}a^{\circledast}aa^* apq && (\text{Since } a^{\circledast}aa^{\circledast} = a^{\circledast}) \\
&= a^{\circledast}abcc^{\circledast}b^{\circledast}a^{\circledast}aa^* apq
\end{aligned}
$$

$$
\begin{aligned}
&= a^{\oplus}(abc)(abc)^{\oplus}a^*apq \\
&= a^{\oplus}aa^*apq \\
&= (a^{\oplus}a)^*a^*apq \\
&= a^*(a^{\oplus})^*a^*apq \\
&= (aa^{\oplus}a)^*apq \\
&= a^*apq \\
&= x \\
xy^2 &= a^*apq(a^{\oplus}(a^{\oplus})^*)^2 \\
&= a^*apqa^{\oplus}(a^{\oplus})^*a^{\oplus}(a^{\oplus})^* \\
&= a^*aa^{\oplus}abcc^{\oplus}cc^{\oplus}b^{\oplus}a^{\oplus}aa^{\oplus}(a^{\oplus})^*a^{\oplus}(a^{\oplus})^* \\
&= a^*abcc^{\oplus}b^{\oplus}a^{\oplus}(a^{\oplus})^*a^{\oplus}(a^{\oplus})^* \\
&= a^*(abc)(abc)^{\oplus}(a^{\oplus})^*a^{\oplus}(a^{\oplus})^* \\
&= a^*(a^{\oplus})^*a^{\oplus}(a^{\oplus})^* \\
&= (a^{\oplus}a)^*a^{\oplus}(a^{\oplus})^* \\
&= a^{\oplus}aa^{\oplus}(a^{\oplus})^* \\
&= (a^{\oplus})(a^{\oplus})^* \\
&= y
\end{aligned}
$$

Let $x = qpcc^*$, $y = (c^{\oplus})^*c^{\oplus}$, we get

$$
\begin{aligned}
xyx &= qpcc^*(c^{\oplus})^*c^{\oplus}qpcc^* \\
&= qpc(c^{\oplus}c)^*c^{\oplus}qpcc^* \\
&= qpcc^{\oplus}cc^{\oplus}qpcc^* && \text{(Since } aa^{\oplus}a = a\text{)} \\
&= qpcc^{\oplus}qpcc^* \\
&= qpqpcc^* \\
&= qpcc^* \\
&= x \\
yxy &= (c^{\oplus})^*c^{\oplus}qpcc^*(c^{\oplus})^*c^{\oplus} \\
&= (c^{\oplus})^*c^{\oplus}qpc(c^{\oplus}c)^*c^{\oplus} \\
&= (c^{\oplus})^*c^{\oplus}qpcc^{\oplus}cc^{\oplus} \\
&= (c^{\oplus})^*c^{\oplus}qpcc^{\oplus}
\end{aligned}
$$

$$
\begin{aligned}
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} c c^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} a a^{\textcircled{\#}} abcc^{\textcircled{\#}} cc^{\textcircled{\#}} && (\text{Since } a^{\textcircled{\#}} a a^{\textcircled{\#}} = a^{\textcircled{\#}})\\
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} c c^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} abcc^{\textcircled{\#}}\\
&= (c^{\textcircled{\#}})^* (abc)^{\textcircled{\#}} (abc) c^{\textcircled{\#}}\\
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}}\\
&= y\\
xy &= qpcc^* (c^{\textcircled{\#}})^* c^{\textcircled{\#}}\\
&= qpc(c^{\textcircled{\#}} c)^* c^{\textcircled{\#}}\\
&= qpcc^{\textcircled{\#}} cc^{\textcircled{\#}}\\
&= qpcc^{\textcircled{\#}}\\
&= cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} aa^{\textcircled{\#}} abcc^{\textcircled{\#}}\\
&= cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} abcc^{\textcircled{\#}}\\
&= c(abc)^{\textcircled{\#}} (abc) c^{\textcircled{\#}}\\
&= cc^{\textcircled{\#}}\\
(xy)^* &= (qpcc^* (c^{\textcircled{\#}})^* c^{\textcircled{\#}})^*\\
&= (qpc(c^{\textcircled{\#}} c)^* c^{\textcircled{\#}})^*\\
&= (qpcc^{\textcircled{\#}} cc^{\textcircled{\#}})^*\\
&= (qpcc^{\textcircled{\#}})^*\\
&= (cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} aa^{\textcircled{\#}} abcc^{\textcircled{\#}})^*\\
&= (cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} abcc^{\textcircled{\#}})^*\\
&= (c(abc)^{\textcircled{\#}} (abc) c^{\textcircled{\#}})^*\\
&= (cc^{\textcircled{\#}})^*\\
&= cc^{\textcircled{\#}}\\
\text{Therefore} \quad (xy)^* &= xy\\
yx^2 &= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} (qpcc^*)^2\\
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} qpcc^* qpcc^*\\
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} aa^{\textcircled{\#}} abcc^{\textcircled{\#}} cc^* cc^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} aa^{\textcircled{\#}} abcc^{\textcircled{\#}} cc^*\\
&= (c^{\textcircled{\#}})^* c^{\textcircled{\#}} b^{\textcircled{\#}} a^{\textcircled{\#}} abcc^* qpcc^*\\
&= (c^{\textcircled{\#}})^* (abc)^{\textcircled{\#}} (abc) c^* qpcc^*\\
&= (c^{\textcircled{\#}})^* c^* qpcc^*
\end{aligned}
$$

$$
\begin{aligned}
&= (cc^{\textcircled{\#}})^*qpcc^* \\
&= qpcc^* \\
&= x \\
xy^2 &= qpcc^*((c^{\textcircled{\#}})^*c^{\textcircled{\#}})^2 \\
&= qpcc^*(c^{\textcircled{\#}})^*c^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= qpc(c^{\textcircled{\#}}c)^*c^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= qpcc^{\textcircled{\#}}cc^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= qpcc^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= cc^{\textcircled{\#}}b^{\textcircled{\#}}a^{\textcircled{\#}}aa^{\textcircled{\#}}abcc^{\textcircled{\#}}cc^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= c(abc)^{\textcircled{\#}}(abc)c^{\textcircled{\#}}(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= (ab)^{\textcircled{\#}}(ab)(c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= (c^{\textcircled{\#}})^*c^{\textcircled{\#}} \\
&= y
\end{aligned}
$$

$(iii) \Rightarrow (iv)$: Since $a^*ap\mathscr{A} = q^*\mathscr{A}$ is equivalent that $a^*ap \in q^*\mathscr{A}$ and $q^* \in a^*ap\mathscr{A}$, we have

$$
\begin{aligned}
a^*ap &= a^*apqp \\
&= a^*apq(a^*apq)^{\textcircled{\#}}a^*apqp && \text{(Using } \{1\}\text{-inverse)} \\
&= (a^*apq)^{\textcircled{\#}}a^*apqa^*ap \\
&= q^*p^*a^*a((a^*apq)^{\textcircled{\#}})^*a^*ap \in q^*\mathscr{A}
\end{aligned}
$$

and

$$
\begin{aligned}
q^* &= q^*p^*q^* \\
&= q^*p^*a^{\textcircled{\#}}aq^* \\
&= q^*p^*(a^{\textcircled{\#}}a)^*q^* && \text{(Since } (ax)^* = ax\text{)} \\
&= q^*p^*a^*(a^{\textcircled{\#}})^*q^* \\
&= q^*p^*a^*aa^{\textcircled{\#}}(a^{\textcircled{\#}})^*q^* \\
&= (a^*apq(a^*apq)^{\textcircled{\#}}a^*apq)^*a^{\textcircled{\#}}(a^{\textcircled{\#}})^*q^* && \text{(Using } \{1\}\text{-inverse)} \\
&= a^*apq(a^*apq)^{\textcircled{\#}}q^* \in a^*ap\mathscr{A}
\end{aligned}
$$

Similarly, $cc^*p^*\mathscr{A} = q\mathscr{A}$ is equivalent that cc^*p^* and $q \in cc^*p^*\mathscr{A}$. So, we have

$$
\begin{aligned}
cc^*p^* &= cc^*p^*q^*p^* \\
&= (qpcc^*)^*p^*
\end{aligned}
$$

$$= (qpcc^*(qpcc^*)^{\text{\#}} qpcc^*)^* p^* \qquad \text{(Since } aa^{\text{\#}}a = a\text{)}$$
$$= qpcc^*(qpcc^*)^{\text{\#}} cc^* p^* \in q\mathscr{A}$$

and

$$\begin{aligned} q &= qpq \\ &= qpcc^{\text{\#}} q \\ &= qpcc^*(c^{\text{\#}})^* c^{\text{\#}} q \\ &= qpcc^*(qpcc^*)^{\text{\#}} qpc^*(c^{\text{\#}})^* c^{\text{\#}} q \\ &= cc^* p^* q^*((qpcc^*)^{\text{\#}})^* q \in cc^* p^* \mathscr{A} \end{aligned}$$

$(iv) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (ii)$: We show that pc and $qa^{\text{\#}}$ are regular. Indeed, $pc = a^{\text{\#}}abc$ and

$$a^{\text{\#}}abc(abc)^{\text{\#}} aa^{\text{\#}}abc = a^{\text{\#}}abc$$

Also,

$$\begin{aligned} cc^* p^*((pc)^{\text{\#}})^* c^{\text{\#}} cc^* p^* &= c(pc)^*((pc)^{\text{\#}})^* c^{\text{\#}} cc^* p^* \\ &= cc^{\text{\#}} cc^* p^* \\ &= cc^* p^* \end{aligned}$$

So $cc^* p^*$ is regular and then, since $qa^{\text{\#}} \in q\mathscr{A} = cc^* p^* \mathscr{A}$ and $cc^* p^*(cc^* p^*)^{\text{\#}} \in cc^* p^* \mathscr{A} = q\mathscr{A}$.

We have

$$\begin{aligned} qa^{\text{\#}} &= cc^* p^* x \qquad \text{(Since } x = a^{\text{\#}}\text{)} \\ &= cc^* p^*(cc^* p^*)^{\text{\#}} cc^* p^* x \\ &= qycc^* p^* x \\ &= qyqa^{\text{\#}} \\ &= qa^{\text{\#}} ayqa^{\text{\#}}. \end{aligned}$$

Hence $qa^{\text{\#}}$ is regular.

Now, analogously using $cc^* p^* \mathscr{A} = q\mathscr{A}$, we get

$$\begin{aligned} p &= pcc^{\text{\#}} \\ &= pc(pc)^{\text{\#}} pcc^{\text{\#}} \\ &= pcc^* p^*((pc)^{\text{\#}})^* c^{\text{\#}} \\ &= pqu \end{aligned}$$

and consequently $pqp = pqpqu = pqu = p$. This show that $q \in p\{1\}$ and $qpqp = qp$.

Also, using $a^*ap\mathscr{A} = q^*\mathscr{A}$, we get

$$\begin{aligned} q &= qa^{\#}a \\ &= qa^{\#}(qa^{\#})^{\#}qa^{\#}a \\ &= qa^{\#}(a^{\#})^*q^*((qa^{\#})^{\#})^*a \\ &= qa^{\#}(a^{\#})^*a^*apv \\ &= qa^{\#}(aa^{\#})^*apv \\ &= qa^{\#}aa^{\#}aapv && (\text{Since } aa^{\#}a = a) \\ &= qa^{\#}apv \\ &= qpv \end{aligned}$$

$qpq = qpqpv = qpv = q.$

Since, $a^*ap\mathscr{A} = q^*\mathscr{A}$ and $cc^*p^*\mathscr{A} = q\mathscr{A}$,

$$q^*p^*a^*apq = q^*p^*q^*t = q^*t = a^*apq$$

and

$$qpcc^*p^*q^* = qpqz = cc^*p^*q^*$$

Hence a^*apq and $qpcc^*$ are hermitian.

Remark 3.2. Let us mention for some special cases when triple reverse order low for the core inverse of products of three regular elements a, b and c of C^*algebra $\mathscr{A}$ holds.

If a is unitary we get that

$$(abc)^{\#} = c^{\#}b^{\#}a^{\#} \Leftrightarrow (bc)^{\#} = c^{\#}b^{\#}.$$

Similarly, if c is unitary

$$(abc)^{\#} = c^{\#}b^{\#}a^{\#} \Leftrightarrow (ab)^{\#} = b^{\#}a^{\#}.$$

Theorem 3.3. Let $\mathscr{A}$ be complex unital C^*-algebra, let $a, b, c \in \mathscr{A}$ be regular elements and let b be unitary. Then the following statements are equivalent:

(i) abc is regular and $(abc)^{\#} = c^{\#}b^{\#}a^{\#}$,

(ii) $[bcc^{\#}b^{\#}, a^*a] = 0$ and $[b^{\#}a^{\#}ab, cc^*] = 0$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] O. M. Baksalary, Core inverse of matrices, linear Multilinear Algebra 58 (6)(2010), 681 - 697.

[2] A. Ben-Israel and T. N. E. Greville, Generalized inverses: Theory and Applications, 2nd ed., Springer, New York, 2003.

[3] E. Boasso, On the Moore-Penrose inverse, EP Banach algebra elements, J. Math. Anal. Appl. 339 (2008), 1003-1014.

[4] N. C. Dincic and D. S. Djordjevic, Hartwig's triple reverse order law revisited, Linear Multilinear Algebra 62 (7)(2014), 918-924.

[5] G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413 - 416.

[6] T. N. E. Greville, Note on the generalized inverse of a matrix product, SIAM Rev. 8 (1966), 518 - 521.

[7] R. E. Hartwig, The reverse order law revisited, Linear Algebra Appl., 76 (1986), 241 - 246.

[8] R. E. Harte and M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103 (1992), 71 - 77.

[9] J. J. Koliha. et. al., Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426 (2007), 371 - 381.

[10] T. T. Liu and J. L. Chen, Characterizations of core and dual core inverse in rings with involution, Linear Multilinear Algebra 66 (2018), 717 - 730.

[11] X. Liu and Shuxia Wu. et. al., New results on reverse order law for $\{1,2,3\}$ and $\{1,2,4\}$ inverse of bounded operators, Math. Comput. 82 (283)(2013), 1597-1607.

[12] N. Mihajlovic, Group inverse and core inverse in Banach and C^*-algebras, Commun. Algebra 48 (4)(2020), 1803 - 1819.

[13] D. Mosic and D. S. Djordjevic, Moore-Penrose invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (5-7)(2009), 732 - 745.

[14] D. S. Rakic and N. C. Dincic. et. al., Core inverse and core partial order of Hilbert space operator, Appl. Math. Comput 244 (2014), 283 -302.

[15] D. S. Rakic and N. C. Dincic. et. al., Group, Moore-Penrose, Core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014), 115 - 133.

[16] W. Sun and Y. Wei, Triple reverse order law for weihted generalized inverses, Appl. Math. Comput. 125 (2-3)(2002), 221 - 229.

[17] Y. Tian, Reverse order law for the weighted Moore-Penrose inverse of a triple matrix product with applications, Int. Math. J. 3 (1)(2003), 107-117.

[18] S. Z. Xu, J. L. Chen, D. Mosic, New characterizations for core and dual core inverses in rings with involution, Front. Math. China 12 (1)(2017), 231 - 246.