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HYBRID IDEALS IN ORDERED SEMIGROUPS AND SOME CHARACTERIZATIONS OF THEIR REGULARITIES

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Abstract. The main theme of this present paper is to apply the concept of hybrid structure to study some algebraic properties of ordered semigroups. Indeed, we define the notion of hybrid left (resp., right) ideals in ordered semigroups, and their related properties are investigated. Furthermore, we provide characterizations of regular, intra-regular, and weakly regular ordered semigroups in terms of hybrid left and right ideals.

Keywords: ordered semigroup; hybrid structure; hybrid left ideal; hybrid right ideal; regular; intra-regular; weakly regular.

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1. INTRODUCTION

Fuzzy set theory was first presented by Zadeh [13] in 1965. Kuroki [6, 7] applied the notion of fuzzy sets to study some properties of semigroups. In 2007, Kehayopulu and Tsingelis [5] studied fuzzy ideals in ordered semigroups.

Molodtsov [8] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that are free from the difficulties that have troubled the usual theoretical

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approaches. Song et al. [12] initiated the study of int-soft semigroups, int-soft left (resp., right) ideals, and int-soft products. In [2], Dudek and Jun introduced the concept of soft interior ideals in semigroups. Moreover, they used some soft versions of ideals in semigroups to characterize simple semigroups. Muhiuddin and Mahboob [9] gave the notions of int-soft left (resp., right) ideals, int-soft interior ideals, and int-soft bi-ideals in ordered semigroups.

As a parallel circuit of fuzzy sets and soft sets (or hesitant fuzzy sets), Jun et al. [4] introduced the notion of hybrid structure in a set of parameters over an initial universe set. They applied this concept to BCK/BCI-algebras and linear spaces. Finally, the authors introduced the concepts of hybrid subalgebras, hybrid fields, and hybrid linear spaces.

Anis et al. [1] applied the notion of hybrid structures to semigroups. They introduced hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups and investigated several properties. Using these notions, they gave characterizations of subsemigroups and left (resp., right) ideals. They also introduced the concept of hybrid products and discussed some characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals by using the notion of hybrid products. They provided relations between the hybrid intersections and hybrid products. Elavarasan and Jun [3] introduced the notion of hybrid generalized bi-ideals in semigroups and provided some characterizations of regular and left quasi-regular semigroups in terms of hybrid generalized bi-ideals.

In this paper, we introduce some hybrid versions of ideals in ordered semigroups. Their preliminary attributes are investigated. Finally, we apply such notions to characterize some particular regularities of ordered semigroups: regular, intra-regular, and weakly regular.

We recall some basic terms and definitions of ordered semigroups and hybrid structures in this section.

A *groupoid* is an algebra $\langle S;\cdot\rangle$ consisting of a nonempty set S together with $\cdot$ a (binary) operation on S. A groupoid in which its binary operation satisfies the associative property is called a *semigroup*.

Definition 1.1. *An algebraic structure* $\langle S;\cdot,\leq\rangle$ *is called an* ordered semigroup *if the following conditions are satisfied:*

(1) $\langle S;\cdot\rangle$ *is a semigroup;*

(2) $\langle S;\leq\rangle$ *is a partially ordered set;*

(3) for every $a,b,c\in S$ *if* $a\leq b$, *then* $a\cdot c\leq b\cdot c$ *and* $c\cdot a\leq c\cdot b$.

For simplicity, we denoted an ordered semigroup $\langle S;\cdot,\leq\rangle$ by its carrier set as a bold letter **S**. Moreover, we usually write the product $x\cdot y$ of x and y by xy.

Let **S** be an ordered semigroup. For any nonempty subsets A and B of S, and for any subset K of S, we define

$$AB=\{ab : a\in A \text{ and } b\in B\}$$

and

$$(K] := \{a\in S : a\leq k \text{ for some } k\in K\}.$$

Moreover, for any $a\in S$, we define

$$\mathbf{S}_a := \{(x,y)\in S\times S : a\leq xy\}.$$

A nonempty subset A of S is called a *subsemigroup* of **S** if $\langle A;*,\preceq\rangle$ is an ordered semigroup, where $* := \cdot\,|_{A\times A}$ and $\preceq\, := \,\leq\cap(A\times A)$.

Definition 1.2. *Let* **S** *be an ordered semigroup. A nonempty subset A of S in which* $(A]\subseteq A$ *is said to be*

(1) a left ideal *of* **S** *if* $SA\subseteq A$,

(2) a right ideal *of* **S** *if* $AS\subseteq A$.

A nonempty subset A of S is called an *ideal* of **S** if it is both a left ideal and a right ideal of **S**.

For any $a\in S$, we denote by $\mathrm{L}(a)$ (resp., $\mathrm{R}(a)$) the smallest left (resp., right) ideal of **S** containing a. One can show that $\mathrm{L}(a)=(a\cup Sa]$ and $\mathrm{R}(a)=(a\cup aS]$.

In what follows, let I be the unit interval, E a set of parameters, and $\mathscr{P}(U)$ denotes the set of all subsets of U.

Definition 1.3 ([4])**.** *A* hybrid structure *in E over U is defined to be a mapping* $\widetilde{f}_\lambda := (\widetilde{f},\lambda)\colon E\to\mathscr{P}(U)\times I$ *defined by* $\widetilde{f}_\lambda(x) := (\widetilde{f}(x),\lambda(x))$, *for any* $x\in E$, *where* $\widetilde{f}\colon E\to\mathscr{P}(U)$ *and* $\lambda\colon E\to I$ *are mappings.*

We denote by $H(E)$ the set of all hybrid structures in E over U. We define an order $\ll$ on $H(E)$ as follows: For all $\widetilde{f}_\lambda, \widetilde{g}_\gamma \in H(E)$,

$$\widetilde{f}_\lambda \ll \widetilde{g}_\gamma \quad \Leftrightarrow \quad \widetilde{f} \sqsubseteq \widetilde{g} \quad \text{and} \quad \lambda \succeq \gamma,$$

where $\widetilde{f} \sqsubseteq \widetilde{g}$ and $\lambda \succeq \gamma$ means that $\widetilde{f}(x) \subseteq \widetilde{g}(x)$ and $\lambda(x) \geq \gamma(x)$, respectively, for all $x \in E$. Moreover, we denote $\widetilde{f}_\lambda = \widetilde{g}_\gamma$ if $\widetilde{f}_\lambda \ll \widetilde{g}_\gamma$ and $\widetilde{g}_\gamma \ll \widetilde{f}_\lambda$.

Definition 1.4 ([4]). *Let $\widetilde{f}_\lambda$ and $\widetilde{g}_\gamma$ be hybrid structures in E over U. Then the* hybrid intersection *of $\widetilde{f}_\lambda$ and $\widetilde{g}_\gamma$, denoted by $\widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$, is defined to be a hybrid structure $\widetilde{f}_\lambda \Cap \widetilde{g}_\gamma \colon E \to \mathscr{P}(U) \times I$ assigning any $x \in E$ to $((\widetilde{f} \cap \widetilde{g})(x), (\lambda \vee \gamma)(x))$, where $(\widetilde{f} \cap \widetilde{g})(x) := \widetilde{f}(x) \cap \widetilde{g}(x)$ and $(\lambda \vee \gamma)(x) := \bigvee\{\lambda(x), \gamma(x)\}$.*

Definition 1.5. *Let $\mathbf{S}$ be an ordered semigroup, $\widetilde{f}_\lambda$ and $\widetilde{g}_\gamma$ be hybrid structures in S over U. Then the* hybrid product *of $\widetilde{f}_\lambda$ and $\widetilde{g}_\gamma$, denoted by $\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma$, is defined to be a hybrid structure*

$$\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma \colon S \to \mathscr{P}(U) \times I, x \mapsto ((\widetilde{f} \odot \widetilde{g})(x), (\lambda \circ \gamma)(x)),$$

where

$$(\widetilde{f} \odot \widetilde{g})(x) = \begin{cases} \bigcup_{(x,y) \in \mathbf{S}_x} (\widetilde{f}(x) \cap \widetilde{f}(y)) & \text{if } \mathbf{S}_x \neq \emptyset, \\ \emptyset & \text{otherwise}, \end{cases}$$

and

$$(\lambda \circ \gamma)(x) = \begin{cases} \bigwedge_{(a,b) \in \mathbf{S}_x} \{\max\{\lambda(a), \gamma(b)\} & \text{if } \mathbf{S}_x \neq \emptyset, \\ 1 & \text{otherwise}, \end{cases}$$

for all $x \in S$.

We denote by $\chi_A(\widetilde{f}_\lambda)$ the *characteristic hybrid structure* of A in E over U and is defined to be a hybrid structure

$$\chi_A(\widetilde{f}_\lambda) \colon E \to \{\emptyset, U\} \times \{0, 1\}, x \mapsto (\chi_A(\widetilde{f})(x), \chi_A(\lambda)(x)),$$

where

$$\chi_A(\widetilde{f})(x) = \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$\chi_A(\lambda)(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in E$.

2. Main Results

In this central section, we study some properties of hybrid left (resp., right) ideals and use them to characterize some regularities of ordered semigroups.

Definition 2.1. *Let* **S** *be an ordered semigroup. A hybrid structure* $\widetilde{f}_\lambda$ *in* S *over* U *is called a* hybrid subsemigroup *in* S *over* U *if, for any* $x, y \in S$*, the following statements are hold:*

(1) $\widetilde{f}(xy) \supseteq \widetilde{f}(x) \cap \widetilde{f}(y)$;

(2) $\lambda(xy) \leq \max\{\lambda(x), \lambda(y)\}$.

Definition 2.2. *Let* **S** *be an ordered semigroup. A hybrid structure* $\widetilde{f}_\lambda$ *in* S *over* U *is called a* hybrid left (resp., right) ideal *in* S *over* U *if, for any* $x, y \in S$*, the following statements are hold:*

(1) $\widetilde{f}(xy) \supseteq \widetilde{f}(y)$ *(resp.,* $\widetilde{f}(xy) \supseteq \widetilde{f}(x)$*);*

(2) $\lambda(xy) \leq \lambda(y)$ *(resp.,* $\lambda(xy) \leq \lambda(x)$*);*

(3) if $x \leq y$*, then* $\widetilde{f}(x) \supseteq \widetilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$.

A hybrid structure in S over U is called a *hybrid ideal* in S over U if it is both a hybrid left ideal and a hybrid right ideal in S over U.

Example 2.1. *Let* $S = \{a, b, c, d\}$*. Then we define an associative binary operation and partial order relation on S as follows:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

It is easy to verify that $\mathbf{S} = \langle S; \cdot, \leq \rangle$ *is an ordered semigroup. Now, we let* $U = \{u_1, u_2, u_3, u_4, u_5\}$. *Then we define* $\widetilde{f}$ *and* λ *as follows:*

S	$\widetilde{f}$	λ
a	U	0.2
b	$\{u_2, u_3, u_4\}$	0.5
c	$\{u_3\}$	0.9
d	$\{u_2, u_3\}$	0.7

We obtain that $\widetilde{f}_\lambda$ *is both a hybrid left ideal and a hybrid right ideal of* S *over* U. *This implied that* $\widetilde{f}_\lambda$ *is a hybrid ideal in* S *over* U.

The following lemma provides a characterization of hybrid subsemigroups by the product of hybrid structures.

Lemma 2.1. *Let* $\mathbf{S}$ *be an ordered semigroup,* $\widetilde{f}_\lambda$ *a hybrid structure in* S *over* U *such that* $x \leq y$ *implies* $\widetilde{f}(x) \supseteq \widetilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$. *Then we have that* $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda \ll \widetilde{f}_\lambda$ *if and only if* $\widetilde{f}_\lambda$ *is a hybrid subsemigroup in* S *over* U.

Proof. Let $x \in S$. If $\mathbf{S}_x = \emptyset$, then

$$(\widetilde{f} \odot \widetilde{f})(x) = \emptyset \subseteq \widetilde{f}(x) \text{ and } (\lambda \circ \lambda)(x) = 1 \geq \lambda(x).$$

If $\mathbf{S}_x \neq \emptyset$, then

$$\begin{aligned}(\tilde{f} \odot \tilde{f})(x) &= \bigcup_{(y,z)\in \mathbf{S}_x} [\tilde{f}(y) \cap \tilde{f}(z)] \\ &\subseteq \bigcup_{(y,z)\in \mathbf{S}_x} \tilde{f}(yz) \\ &\subseteq \bigcup_{(y,z)\in \mathbf{S}_x} \tilde{f}(x) \\ &= \tilde{f}(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda \circ \lambda)(x) &= \bigwedge_{(y,z)\in \mathbf{S}_x} \{\max\{\lambda(y), \lambda(z)\} \\ &\geq \bigwedge_{(y,z)\in \mathbf{S}_x} \lambda(yz) \\ &\geq \bigwedge_{(y,z)\in \mathbf{S}_x} \lambda(x) \\ &= \lambda(x).\end{aligned}$$

This means that $\tilde{f}_\lambda \otimes \tilde{f}_\lambda \ll \tilde{f}_\lambda$.

Conversely, we assume that $\tilde{f}_\lambda \otimes \tilde{f}_\lambda \ll \tilde{f}_\lambda$. Then for all $x, y \in S$, we have

$$\begin{aligned}\tilde{f}(xy) &\supseteq (\tilde{f} \odot \tilde{f})(xy) \\ &= \bigcup_{(a,b)\in \mathbf{S}_{xy}} [\tilde{f}(a) \cap \tilde{f}(b)] \\ &\supseteq \tilde{f}(x) \cap \tilde{f}(y),\end{aligned}$$

and

$$\begin{aligned}\lambda(xy) &\leq (\lambda \circ \lambda)(xy) \\ &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{\lambda(a), \lambda(b)\}\} \\ &\leq \max\{\lambda(x), \lambda(y)\}.\end{aligned}$$

This shows that $\tilde{f}_\lambda$ is a hybrid subsemigroup in S over U. □

We provide some interesting results about hybrid ideals as follows.

Theorem 2.1. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\{\widetilde{f}_{i\lambda_i} : i \in I\}$ *a family of hybrid left (resp., right) ideals in S over U. Then* $\Cap_{i\in I}\widetilde{f}_{i\lambda_i}$ *is a hybrid left (resp., right) ideal in S over U.*

Proof. Let $x, y \in S$. Then, since $\widetilde{f}_{i\lambda_i}$ is a hybrid left ideal in S over U for all $i \in I$, we obtain that for each $i \in I$, $\widetilde{f}_i(xy) \supseteq \widetilde{f}_i(y)$ and $\lambda_i(xy) \leq \lambda_i(y)$. Thus,

$$\bigcap_{i\in I}\widetilde{f}_i(xy) \supseteq \bigcap_{i\in I}\widetilde{f}_i(y) \quad \text{and} \quad \bigvee_{i\in I}\lambda_i(xy) \leq \bigvee_{i\in I}\lambda_i(y).$$

Let $x, y \in S$ be such that $x \leq y$. Then, $\widetilde{f}_i(x) \supseteq \widetilde{f}_i(y)$ and $\lambda_i(x) \leq \lambda_i(y)$ for any $i \in I$. Thus,

$$\bigcap_{i\in I}\widetilde{f}_i(x) \supseteq \bigcap_{i\in I}\widetilde{f}_i(y) \quad \text{and} \quad \bigvee_{i\in I}\lambda_i(x) \leq \bigvee_{i\in I}\lambda_i(y).$$

Therefore, $\Cap_{i\in I}\widetilde{f}_{i\lambda_i}$ is a hybrid left ideal in S over U. By similar arguments, we can prove that if $\{\widetilde{f}_{i\lambda_i} : i \in I\}$ a family of hybrid right ideals in S over U, then $\Cap_{i\in I}\widetilde{f}_{i\lambda_i}$ is a hybrid right ideal in S over U. □

Corollary 2.1. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\{\widetilde{f}_{i\lambda_i} : i \in I\}$ *a family of hybrid ideals of* $\mathbf{S}$ *over U. Then* $\Cap_{i\in I}\widetilde{f}_{i\lambda_i}$ *is a hybrid ideal in S over U.*

Now, we characterize ideals of ordered semigroups by the characteristic hybrid structure.

Theorem 2.2. *Let* $\mathbf{S}$ *be an ordered semigroup and L a nonempty subset of S. Then the following statements are equivalent:*

(1) L is a left ideal of $\mathbf{S}$*;*

(2) $\chi_L(\widetilde{f}_\lambda)$ *is a hybrid left ideal in S over U.*

Proof. (1)⇒(2). Assume that L is a left ideal of $\mathbf{S}$. Let $x, y \in S$. Then if $y \in L$, then $xy \in L$ and, moreover,

$$\chi_L(\widetilde{f})(xy) = U = \chi_L(\widetilde{f})(y) \quad \text{and} \quad \chi_L(\lambda)(xy) = 0 = \chi_L(\lambda)(y).$$

If $y \notin L$, then we obtain

$$\chi_L(\widetilde{f})(y) = \emptyset \subseteq \chi_L(\widetilde{f})(xy) \quad \text{and} \quad \chi_L(\lambda)(y) = 1 \geq \chi_L(\lambda)(xy).$$

Let $x,y \in S$ be such that $x \leq y$. We have that $x \in L$ whenever $y \in L$. Furthermore,

$$\chi_L(\widetilde{f})(x) = U = \chi_L(\widetilde{f})(y) \quad \text{and} \quad \chi_L(\lambda)(x) = 0 = \chi_L(\lambda)(y).$$

Otherwise, we have that

$$\chi_L(\widetilde{f})(y) = \emptyset \subseteq \chi_L(\widetilde{f})(x) \quad \text{and} \quad \chi_L(\lambda)(y) = 1 \geq \chi_L(\lambda)(x).$$

This shows that $\chi_L(\widetilde{f}_\lambda)$ is a hybrid left ideal in S over U.

(2)⇒(1). Assume that $\chi_L(\widetilde{f}_\lambda)$ is a hybrid left ideal in S over U. Let $x \in S$ and $y \in L$. Then $U \supseteq \chi_L(\widetilde{f})(xy) \supseteq \chi_L(\widetilde{f})(y) = U$. This means that $\chi_L(\widetilde{f})(xy) = U$, and $0 \leq \chi_L(\lambda)(xy) \leq \chi_L(\lambda)(y) = 0$. That is, $\chi_L(\lambda)(xy) = 0$, and then $xy \in L$. Now, let $x,y \in S$ be such that $x \leq y$. If $y \in L$, then $U \supseteq \chi_L(\widetilde{f})(x) \supseteq \chi_L(\widetilde{f})(y) = U$. This implies that $\chi_L(\widetilde{f})(x) = U$, and $0 \leq \chi_L(\lambda)(x) \leq \chi_L(\lambda)(y) = 0$. That is, $\chi_L(\lambda)(x) = 0$ and then $x \in L$. Therefore, L is a left ideal of **S**. □

Similar to Theorem 2.2, we obtain the following results.

Theorem 2.3. *Let* **S** *be an ordered semigroup and R a nonempty subset of S. Then the following statements are equivalent:*

(1) R is a right ideal of **S***;*

(2) $\chi_R(\widetilde{f}_\lambda)$ is a hybrid right ideal in S over U.

Corollary 2.2. *Let* **S** *be an ordered semigroup and I a nonempty subset of S. Then the following statements are equivalent:*

(1) I is an ideal of **S***;*

(2) $\chi_I(\widetilde{f}_\lambda)$ is a hybrid ideal in S over U.

The following lemma shows some important properties of the characteristic hybrid structures which will be used in our main results. The proof is routine, so it is omitted.

Lemma 2.2. *Let* **S** *be an ordered semigroup and A,B nonempty subsets of S. Then the following conditions hold:*

(1) $\chi_A(\widetilde{f}_\lambda) \Cap \chi_B(\widetilde{f}_\lambda) = \chi_{A\cap B}(\widetilde{f}_\lambda)$;

(2) $\chi_A(\widetilde{f}_\lambda) \otimes \chi_B(\widetilde{f}_\lambda) = \chi_{(AB]}(\widetilde{f}_\lambda)$.

We denote $\chi_S(\widetilde{f}_\lambda) := S(\widetilde{f}_\lambda)$ and is defined as follows: For all $x \in S$, $S(\widetilde{f})(x) = U$ and $S(\lambda)(x) = 0$.

The following theorem provides a characterization of hybrid ideals in S over U in terms of the operation $\otimes$.

Theorem 2.4. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$ *a hybrid structure in* S *over* U. *Suppose that for any* $x, y \in S$, $x \leq y$ *implies* $\widetilde{f}(x) \supseteq \widetilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$. *Then the following statements are equivalent:*

(1) $\widetilde{f}_\lambda$ *is a hybrid left ideal in* S *over* U;

(2) $S(\widetilde{f}_\lambda) \otimes \widetilde{f}_\lambda \ll \widetilde{f}_\lambda$.

Proof. (1)⇒(2). Assume that $\widetilde{f}_\lambda$ is a hybrid left ideal in S over U. Let $x \in S$. If $\mathbf{S}_x = \emptyset$, then

$$(S(\widetilde{f}) \odot \widetilde{f})(x) = \emptyset \subseteq \widetilde{f}(x) \quad \text{and} \quad (S(\lambda) \circ \lambda)(x) = 1 \geq \lambda(x).$$

If $\mathbf{S}_x \neq \emptyset$, then

$$\begin{aligned}
(S(\widetilde{f}) \odot \widetilde{f})(x) &= \bigcup_{(a,b)\in \mathbf{S}_x} [S(\widetilde{f})(a) \cap \widetilde{f}(b)] \\
&= \bigcup_{(a,b)\in \mathbf{S}_x} \widetilde{f}(b) \\
&\subseteq \bigcup_{(a,b)\in \mathbf{S}_x} \widetilde{f}(ab) \\
&\subseteq \bigcup_{(a,b)\in \mathbf{S}_x} \widetilde{f}(x) \\
&= \widetilde{f}(x),
\end{aligned}$$

and

$$\begin{aligned}
(S(\lambda) \circ \lambda)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{S(\lambda)(a), \lambda(b)\}\} \\
&= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\lambda(b)\} \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \lambda(ab)
\end{aligned}$$

$$\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \lambda(x)$$
$$= \lambda(x).$$

Altogether, we have that $S(\widetilde{f}_\lambda) \otimes \widetilde{f}_\lambda \ll \widetilde{f}_\lambda$.

(2)⇒(1). Let $x, y \in S$. Then

$$\begin{aligned}\widetilde{f}(xy) &\supseteq (S(\widetilde{f}) \odot \widetilde{f})(xy) \\ &= \bigcup_{(a,b)\in \mathbf{S}_{xy}} [S(\widetilde{f})(a) \cap \widetilde{f}(b)] \\ &\supseteq [S(\widetilde{f})(x) \cap \widetilde{f}(y)] \\ &= \widetilde{f}(y),\end{aligned}$$

and

$$\begin{aligned}\lambda(xy) &\leq (S(\lambda) \circ \lambda)(xy) \\ &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{S(\lambda)(a), \lambda(b)\}\} \\ &\leq \max\{S(\lambda)(x), \lambda(y)\} \\ &= \lambda(y).\end{aligned}$$

This shows that $\widetilde{f}_\lambda$ is a hybrid left ideal in S over U. □

The following results can be proved similarly.

Theorem 2.5. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$ *a hybrid structure in* S *over* U*. Suppose that for any* $x, y \in S$*,* $x \leq y$ *implies* $\widetilde{f}(x) \supseteq \widetilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$*. Then the following statements are equivalent:*

(1) $\widetilde{f}_\lambda$ *is a hybrid right ideal in* S *over* U*;*

(2) $\widetilde{f}_\lambda \otimes S(\widetilde{f}_\lambda) \ll \widetilde{f}_\lambda$*.*

Corollary 2.3. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$ *a hybrid structure in* S *over* U*. Suppose that for any* $x, y \in S$*,* $x \leq y$ *implies* $\widetilde{f}(x) \supseteq \widetilde{f}(y)$ *and* $\lambda(x) \leq \lambda(y)$*. Then the following statements are equivalent:*

(1) $\widetilde{f}_\lambda$ *is a hybrid ideal in* S *over* U*;*

(2) $S(\widetilde{f}_\lambda)\otimes\widetilde{f}_\lambda \ll \widetilde{f}_\lambda$ *and* $\widetilde{f}_\lambda\otimes S(\widetilde{f}_\lambda)\ll\widetilde{f}_\lambda$.

Lemma 2.3. *Let* $\mathbf{S}$ *be an ordered semigroup. If* $\widetilde{f}_\lambda$ *is a hybrid left (resp., right) ideal in* S *over* U*, then* $\widetilde{f}_\lambda\otimes\widetilde{f}_\lambda\ll\widetilde{f}_\lambda$.

Proof. Assume that $\widetilde{f}_\lambda$ is a hybrid left ideal in S over U. Let $x\in S$. Then, if $\mathbf{S}_x=\emptyset$, then we obtain $(\widetilde{f}\odot\widetilde{f})(x)=\emptyset\subseteq\widetilde{f}(x)$ and $(\lambda\circ\lambda)(x)=1\geq\lambda(x)$. If $\mathbf{S}_x\neq\emptyset$, then

$$\begin{aligned}(\widetilde{f}\odot\widetilde{f})(x) &= \bigcup_{(a,b)\in\mathbf{S}_x}[\widetilde{f}(a)\cap\widetilde{f}(b)]\\ &\subseteq \bigcup_{(a,b)\in\mathbf{S}_x}[\widetilde{f}(b)]\\ &\subseteq \bigcup_{(a,b)\in\mathbf{S}_x}[\widetilde{f}(ab)]\\ &\subseteq \bigcup_{(a,b)\in\mathbf{S}_x}[\widetilde{f}(x)]\\ &= \widetilde{f}(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda\circ\lambda)(x) &= \bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\lambda(a),\lambda(b)\}\}\\ &\geq \bigwedge_{(a,b)\in\mathbf{S}_x}\lambda(b)\\ &\geq \bigwedge_{(a,b)\in\mathbf{S}_x}\lambda(ab)\\ &\geq \bigwedge_{(a,b)\in\mathbf{S}_x}\lambda(x)\\ &= \lambda(x).\end{aligned}$$

Therefore, $\widetilde{f}_\lambda\otimes\widetilde{f}_\lambda\ll\widetilde{f}_\lambda$. Similarly, we can prove that if $\widetilde{f}_\lambda$ a hybrid right ideal in S over U, then $\widetilde{f}_\lambda\otimes\widetilde{f}_\lambda\ll\widetilde{f}_\lambda$. □

Corollary 2.4. *Let* $\mathbf{S}$ *be an ordered semigroup. If* $\widetilde{f}_\lambda$ *is a hybrid ideal in* S *over* U*, then* $\widetilde{f}_\lambda\otimes\widetilde{f}_\lambda\ll\widetilde{f}_\lambda$.

We recall some particular classes of ordered semigroups as follows. More information about theses classes can be found in [5]. An ordered semigroup **S** is called

(1) a *regular* ordered semigroup if, for each element $a \in S$, there exists an element $x \in S$ such that $a \leq axa$,

(2) an *intra-regular* ordered semigroup if, for each element $a \in S$, there exist elements $x, y \in S$ such that $a \leq xa^2y$.

Lemma 2.4 ([5]). *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent:*

(1) **S** *is regular;*

(2) $R \cap L = (RL]$ *for every right ideal* R *and every left ideal* L *of* **S**.

Lemma 2.5. *Let* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *be a hybrid right ideal and a hybrid left ideal in* S *over* U, *respectively. Then we have* $\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma \ll \widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$.

Proof. Let $\widetilde{f}_\lambda$ and $\widetilde{g}_\gamma$ be a hybrid right ideal and a hybrid left ideal in S over U, respectively. Let $z \in S$. If $\mathbf{S}_z = \emptyset$, then

$$(\widetilde{f} \odot \widetilde{g})(z) = \emptyset \subseteq \widetilde{f}(z) \quad \text{and} \quad (\widetilde{f} \odot \widetilde{g})(z) = \emptyset \subseteq \widetilde{g}(z).$$

Thus, $(\widetilde{f} \odot \widetilde{g})(z) \subseteq \widetilde{f}(z) \cap \widetilde{g}(z) = (\widetilde{f} \cap \widetilde{g})(z)$. Similarly, we have

$$(\lambda \circ \gamma)(z) = 1 \geq \lambda(z) \quad \text{and} \quad (\lambda \circ \gamma)(z) = 1 \geq \gamma(z).$$

Thus, $(\lambda \circ \gamma)(z) \geq \max\{\lambda(z), \gamma(z)\} = (\lambda \cup \gamma)(z)$. If $\mathbf{S}_z \neq \emptyset$, then

$$\begin{aligned}
(\widetilde{f} \odot \widetilde{g})(z) &= \bigcup_{(x,y) \in \mathbf{S}_z} (\widetilde{f}(x) \cap \widetilde{g}(y)) \\
&\subseteq \bigcup_{(x,y) \in \mathbf{S}_z} (\widetilde{f}(xy) \cap \widetilde{g}(xy)) \\
&\subseteq \bigcup_{(x,y) \in \mathbf{S}_z} (\widetilde{f}(z) \cap \widetilde{g}(z)) \\
&= \widetilde{f}(z) \cap \widetilde{g}(z) \\
&= (\widetilde{f} \cap \widetilde{g})(z),
\end{aligned}$$

and

$$\begin{aligned}(\lambda\circ\gamma)(z) &= \bigwedge_{(x,y)\in\mathbf{S}_z}\{\max\{\lambda(x),\gamma(y)\}\}\\ &\geq \bigwedge_{(x,y)\in\mathbf{S}_z}\{\max\{\lambda(xy),\gamma(xy)\}\}\\ &\geq \bigwedge_{(x,y)\in\mathbf{S}_z}\{\max\{\lambda(z),\gamma(z))\\ &= \max\{\lambda(z),\gamma(z)\}\\ &= (\lambda\cup\gamma)(z).\end{aligned}$$

Altogether, we obtain that $\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma \ll \widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$. □

In [5], the authors characterized regular ordered semigroups in terms of left ideals and right ideals. By this idea, we apply the concepts of hybrid ideals to characterize regular ordered semigroups as shown.

Theorem 2.6. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is regular;*

(2) $\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma = \widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$ *for every hybrid right ideal* $\widetilde{f}_\lambda$ *and every hybrid left ideal* $\widetilde{g}_\gamma$ *in* S *over* U.

Proof. (1)⇒(2). Let $a \in S$. Since $\mathbf{S}$ is regular, there exists $x \in S$ such that $a \leq axa \leq (axa)xa$. This implies that $\mathbf{S}_a \neq \emptyset$. Then

$$\begin{aligned}(\widetilde{f}\odot\widetilde{g})(a) &= \bigcup_{(u,v)\in\mathbf{S}_a}\left(\widetilde{f}(u)\cap\widetilde{g}(v)\right)\\ &\supseteq \widetilde{f}(axa)\cap\widetilde{g}(xa)\\ &\supseteq \widetilde{f}(a)\cap\widetilde{g}(a)\\ &= (\widetilde{f}\cap\widetilde{g})(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda \circ \gamma)(a) &= \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{\lambda(u), \gamma(v)\}\} \\ &\leq \max\{\lambda(axa), \gamma(xa)\} \\ &\leq \max\{\lambda(a), \gamma(a)\} \\ &= (\lambda \cup \gamma)(a).\end{aligned}$$

This means that $\widetilde{f}_\lambda \Cap \widetilde{g}_\gamma \ll \widetilde{f}_\lambda \otimes \widetilde{g}_\gamma$ and by Lemma 2.5, we obtain $\widetilde{f}_\lambda \otimes \widetilde{g}_\gamma = \widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$.

(2)⇒(1). Let R and L be a right ideal and a left ideal of $\mathbf{S}$, respectively. Then $\chi_R(\widetilde{f}_\lambda)$ and $\chi_L(\widetilde{f})_\lambda$ is a hybrid right ideal and a hybrid left ideal in S over U, respectively. By hypothesis, we obtain $\chi_R(\widetilde{f}_\lambda) \otimes \chi_L(\widetilde{f}_\lambda = \chi_R(\widetilde{f}_\lambda) \Cap \chi_L(\widetilde{f}_\lambda)$. Let $a \in (RL]$. Then

$$\begin{aligned}\chi_{R\cap L}(\widetilde{f})(a) &= (\chi_R(\widetilde{f}) \cap \chi_L(\widetilde{f}))(a) \\ &= (\chi_R(\widetilde{f}) \odot \chi_L(\widetilde{f}))(a) \\ &= \chi_{(RL]}(\widetilde{f})(a) \\ &= U,\end{aligned}$$

and

$$\begin{aligned}\chi_{R\cap L}(\lambda)(a) &= (\chi_R(\lambda) \cap \chi_L(\lambda))(a) \\ &= (\chi_R(\lambda) \circ \chi_L(\lambda))(a) \\ &= \chi_{(RL]}(\lambda)(a) \\ &= 0.\end{aligned}$$

This means that $a \in R \cap L$. Thus, we have $(RL] \subseteq R \cap L$. It is easy to see that $R \cap L \subseteq (RL]$, so $R \cap L = (RL]$. By Lemma 2.4, $\mathbf{S}$ is regular. □

Lemma 2.6 ([5])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following conditions are equivalent:*

(1) $\mathbf{S}$ *is intra-regular;*

(2) $R \cap L \subseteq (LR]$ *for every right ideal* R *and every left ideal* L *of* $\mathbf{S}$*.*

Theorem 2.7. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is intra-regular;*

(2) $\widetilde{g}_\gamma \Cap \widetilde{f}_\lambda \ll \widetilde{g}_\gamma \otimes \widetilde{f}_\lambda$ *for every hybrid right ideal* $\widetilde{f}_\lambda$ *in* S *over* U *and every hybrid left ideal* $\widetilde{g}_\gamma$ *of* S *over* U.

Proof. (1)⇒(2). Let $a \in S$. Since $\mathbf{S}$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y = (xa)(ax)$. This implies that $\mathbf{S}_a \neq \emptyset$. Then

$$\begin{aligned}(\widetilde{g} \odot \widetilde{f})(a) &= \bigcup_{(u,v)\in \mathbf{S}_a} (\widetilde{g}(u) \cap \widetilde{f}(v)) \\ &\supseteq \widetilde{g}(xa) \cap \widetilde{f}(ay) \\ &\supseteq \widetilde{g}(a) \cap \widetilde{f}(a) \\ &= (\widetilde{g} \cap \widetilde{f})(a),\end{aligned}$$

and

$$\begin{aligned}(\gamma \circ \lambda)(a) &= \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{\gamma(u), \lambda(v)\}\} \\ &\leq \max\{\gamma(xa), \lambda(ay)\} \\ &\leq \max\{\gamma(a), \lambda(a)\} \\ &= (\gamma \cup \lambda)(a).\end{aligned}$$

This means that $\widetilde{g}_\gamma \Cap \widetilde{f}_\lambda \ll \widetilde{g}_\gamma \otimes \widetilde{f}_\lambda$.

(2)⇒(1). Let R and L be a right ideal and a left ideal of $\mathbf{S}$ over U, respectively. Then $\chi_R(\widetilde{f}_\lambda)$ and $\chi_L(\widetilde{f}_\lambda)$ is a hybrid right ideal and a hybrid left ideal in S over U, respectively. Let $a \in R \cap L$. By hypothesis, we obtain $\chi_R(\widetilde{f}_\lambda) \Cap \chi_L(\widetilde{f}_\lambda) \ll \chi_L(\widetilde{f}_\lambda) \otimes \chi_R(\widetilde{f}_\lambda)$. Then

$$\begin{aligned}U &= \chi_{R\cap L}(\widetilde{f})(a) \\ &= (\chi_R(\widetilde{f}) \cap \chi_L(\widetilde{f}))(a) \\ &\subseteq (\chi_L(\widetilde{f}) \odot \chi_R(\widetilde{f}))(a) \\ &= \chi_{(LR]}(\widetilde{f})(a) \subseteq U.\end{aligned}$$

This implies that $\chi_{(LR]}(\widetilde{f})(a) = U$, and

$$\begin{aligned} 0 &= \chi_{R\cap L}(\lambda)(a) \\ &= (\chi_R(\lambda) \cap \chi_L(\lambda))(a) \\ &\geq (\chi_L(\lambda) \circ \chi_R(\lambda))(a) \\ &= \chi_{(LR]}(\lambda)(a) \\ &\geq 0. \end{aligned}$$

That is, $\chi_{(LR]}(\lambda)(a) = 0$. This means that $a \in (LR]$. Thus, we have $R \cap L \subseteq (LR]$. By Lemma 2.6, **S** is intra-regular. □

An ordered semigroup **S** is *left* (*resp., right*) *weakly regular* [11] if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xaya$ (resp., $a \leq axay$). If **S** is left weakly regular and right weakly regular, then it is called a *weakly regular ordered semigroup*.

Lemma 2.7 ([11]). *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent:*

(1) **S** *is left* (*resp., right*) *weakly regular;*
(2) $L = (L^2]$ (*resp.,* $R = (R^2]$) *for every left ideal* L (*resp., right ideal* R) *of* **S***;*
(3) $L(a) = ((L(a))^2]$ (*resp.,* $R(a) = ((R(a))^2]$) *for every* $a \in S$.

Theorem 2.8. *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent:*

(1) **S** *is left weakly regular;*
(2) $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda$ *for every hybrid left ideal* $\widetilde{f}_\lambda$ *in* S *over* U.

Proof. (1)⇒(2). Assume that $\widetilde{f}_\lambda$ is a hybrid left ideal in S over U. By Lemma 2.3, we have that $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda \ll \widetilde{f}_\lambda$. On the other inclusion, since **S** is left weakly regular, there exist $x, y \in S$ such that $a \leq xaya = (xa)(ya)$. This implies that $\mathbf{S}_a \neq \emptyset$. Then

$$\begin{aligned}(\widetilde{f} \odot \widetilde{f})(a) &= \bigcup_{(u,v)\in \mathbf{S}_a} (\widetilde{f}(u) \cap \widetilde{f}(v)) \\ &\supseteq \widetilde{f}(xa) \cap \widetilde{f}(ya) \\ &\supseteq \widetilde{f}(a) \cap \widetilde{f}(a) \\ &= \widetilde{f}(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda \circ \lambda)(a) &= \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{\lambda(u), \lambda(v)\}\} \\ &\leq \max\{\lambda(xa), \lambda(ya)\} \\ &\leq \max\{\lambda(a), \lambda(a)\} \\ &= \lambda(a).\end{aligned}$$

This means that $\widetilde{f}_\lambda \ll \widetilde{f}_\lambda \otimes \widetilde{f}_\lambda$. Altogether, we have that $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda$.

(2)⇒(1). Let $a \in S$ and $b \in L(a)$. Then since $\mathrm{L}(a)$ is a left ideal of $\mathbf{S}$, we have that $\chi_{L(a)}(\widetilde{f}_\lambda)$ is a hybrid left ideal in S over U. By hypothesis, we have that

$$\chi_{\mathrm{L}(a)}(\widetilde{f}_\lambda) \otimes \chi_{\mathrm{L}(a)}(\widetilde{f}_\lambda) = \chi_{\mathrm{L}(a)}(\widetilde{f}_\lambda).$$

Then

$$\begin{aligned}\chi_{((\mathrm{L}(a))^2]}(\widetilde{f})(b) &= (\chi_{\mathrm{L}(a)}(\widetilde{f}) \odot \chi_{\mathrm{L}(a)}(\widetilde{f}))(b) \\ &= \chi_{\mathrm{L}(a)}(\widetilde{f})(b) \\ &= U,\end{aligned}$$

and

$$\begin{aligned}\chi_{((\mathrm{L}(a))^2]}(\lambda)(b) &= (\chi_{\mathrm{L}(a)}(\lambda) \circ \chi_{\mathrm{L}(a)}(\lambda))(b) \\ &= \chi_{\mathrm{L}(a)}(\lambda)(b) \\ &= 0.\end{aligned}$$

This means that $b \in ((\mathrm{L}(a))^2]$. Hence, $\mathrm{L}(a) \subseteq ((\mathrm{L}(a))^2]$. On the other hand $((\mathrm{L}(a))^2] \subseteq (S\mathrm{L}(a)] \subseteq (\mathrm{L}(a)] = \mathrm{L}(a)$. Altogether, we obtain $\mathrm{L}(a) = ((\mathrm{L}(a))^2]$. By Lemma 2.7, **S** is left weakly regular. □

Similarly, we can prove the following theorem.

Theorem 2.9. *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent:*

(1) **S** *is right weakly regular;*

(2) $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda$ *for every hybrid right ideal* $\widetilde{f}_\lambda$ *in* S *over* U.

Corollary 2.5. *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent:*

(1) **S** *is weakly regular;*

(2) $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda$ *for every hybrid ideal* $\widetilde{f}_\lambda$ *in* S *over* U.

3. Conclusion

In this paper, we present the concept of left (resp., right) hybrid ideals in ordered semigroups. We study some important properties. Lastly, we apply such concepts to characterize some instance regularities of ordered semigroups; regular, intra-regular and left (resp., right) weakly regular.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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