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FIXED POINT THEOREM ON PARTIAL METRIC SPACE USING WEAKLY COMPATIBLE AND RECIPROCAL CONTINUOUS MAPPINGS

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Abstract: Common fixed point theorem for four self-maps on partial metric spaces by using reciprocally continuous mappings along with compatible and weakly compatible mappings is proved in this paper. We justify the result with an appropriate example.

Keywords: fixed point; weakly compatible; compatible; reciprocal continuous.

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1. INTRODUCTION

Matthews [1] introduced the concept of partial metric space (shortly PMS) [1] is an extension metric space. In PMS the condition $d(a,a)$ is not necessarily zero and assumption $d(a,a)=0$ is restored by the condition $d(a,a) \leq d(a,b)$. Several results are proved in partial metric spaces like [1],

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[3] and [4].

Various studies on the possible generalizations of existing metric fixed point results to the partial metric spaces were done recently. Concept of compatible mappings which are weaker than the weakly commuting mappings is introduced by Jungck [5] in the year 1986. Additionally, Jungck and Rhoades [6] established the concept of weakly compatible mappings which are rather weaker than the compatible mappings. A new notion of continuity known as reciprocal continuity for the pair of self maps is initiated by R.P. Pant [7]. In this paper we generate a common fixed point theorem for four self maps in which one couple is considered to be reciprocally continuous and compatible and the other couple is weakly compatible mapping.

2. PRELIMINARIES

Definition 2.1: Let V be a nonempty set and let $p : V \times V \to [0,\infty]$ satisfy

(P1) $\alpha = \beta \Leftrightarrow p(\alpha,\alpha) = p(\beta,\beta) = p(\alpha,\beta)$

(P2) $p(\alpha,\alpha) \le p(\alpha,\beta)$,

(P3) $p(\alpha,\beta) = p(\beta,\alpha)$,

(P4) $p(\alpha,\beta) \le p(\alpha,\chi) + p(\chi,\beta) - p(\chi,\chi)$.

For all α,β and $\chi \in V$ .Then (V, p) is said to be a partial metric space and p is said to be a partial metric on V.

Definition 2.2: In a PMS (V, p), a sequence $\{a_\theta\}$

(a) converges to $a \in V$ if and only if $p(a,a) = p(a,a_\theta)$ as $\theta \to \infty$

(b) is known to be cauchy sequence if and only if $p(a_\theta, a_m)$ as $\theta, m \to \infty$ exists.

(c) is known to be complete if every Cauchy sequence $\{a_\theta\}$ converges in it.

Remark 2.3: The following relations hold in PMS.

(i) If $p(\alpha,\beta) = 0,$ then $\alpha = \beta$.

(ii) If $\alpha \ne \beta$, then $p(\alpha,\beta) > 0$.

Definition 2.4: Mappings M_0 and N_0 of a PMS defined as reciprocally continuous if

$\{M_0N_0a_\theta\} = M_0\omega$ and $\{N_0M_0a_\theta\} = N_0\omega$ as $\theta \to \infty$ whenever $\{\alpha_\theta\}$ is a sequence in V such that $\{M_0a_\theta\},\{N_0a_\theta\}$ converge to ω as $\theta \to \infty$ for some of $\omega \in V$.

Definition 2.5: The mappings M_0& N_0 of a PMS defined as compatible if $\{M_0N_0a_\theta\} = \{N_0M_0a_\theta\}$ as $\theta \to \infty$ whenever $\{a_\theta\}$ is a sequence in V such that $\{M_0a_\theta\},\{N_0a_\theta\}$ converge to ω as $\theta \to \infty$ for some of $\omega \in V$.

Definition 2.6: The mappings M_0 and N_0 of a PMS defined as weakly compatible mappings if they commuting at all coincidence points. This means that the mappings satisfy $M_0u = N_0u$

Then $M_0N_0u = N_0M_0u$, for some $u \in V$.

Now we proceed for our main result which generalizes and extend the existing theorem proved in [4].

3. MAIN RESULTS

Theorem 3.1: Suppose K_0, M_0, N_0 and L_0 are self maps of a complete PMS (V, p) into itself with

$$M_0(V) \subset K_0(V) \text{ and } N_0(V) \subset L_0(V) \tag{3.1.1}$$

$$p(M_0\alpha, N_0\beta) \le \lambda\, p(\alpha,\beta) \tag{3.1.2 }$$

for any $\alpha,\beta \in V$ and $\lambda \in [0,1[$ where,

$$p(\alpha,\beta) = \max\left\{\begin{array}{l} p(M_0\alpha, L_0\alpha), p(N_0\beta, K_0\beta), p(L_0\alpha, K_0\beta), \\ \frac{1}{2}\left[p(M_0\alpha, K_0\beta) + p(N_0\beta, L_0\alpha)\right] \end{array}\right\}. \tag{3.1.3 }$$

The pair $\{M_0, L_0\}$ is reciprocally continuous and compatible and the pair $\{K_0, N_0\}$ is weakly compatible then M_0, K_0, N_0 and L_0 have a unique common fixed point.

Proof: Let a_0 be arbitrary point in V. As $M_0(V) \subset K_0(V)$ and let $a_1 \in V$ such that $M_0a_0 = K_0a_1$ and also, as $N_0a_1 \in L_0(V)$, there exist $a_2 \in V$ such that $N_0a_1 = L_0a_2$. In general, $a_{2\theta+1} \in V$ is chosen such that $M_0a_{2\theta} = K_0a_{2\theta+1}$ and $a_{\theta+2} \in V$ such that $N_0a_{2\theta+1} = L_0a_{2\theta+2}$, we obtain a sequences $\{b_\theta\}$ in V such that

$b_{2\theta} = M_0a_{2\theta} = K_0a_{2\theta+1}$, $b_{2\theta+1} = N_0a_{2\theta+1} = L_0a_{2\theta+2}$, for $\theta \ge 0$.

Now we can prove that a sequence $\{b_\theta\}$ is Cauchy sequence.

By (3.1.2) and using (3.1.3), we observe

$$\begin{aligned} p(b_{2\theta+1}, b_{2\theta+2}) &= p(K_0 a_{2\theta+1}, L_0 a_{2\theta+2}) \\ &= p(M_0 a_{2\theta}, N_0 a_{2\theta+1}) \\ &\leq \lambda\, p(a_{2\theta}, a_{2\theta+1}) \end{aligned} \tag{3.1.4}$$

where $p(a_{2\theta}, a_{2\theta+1}) = \max \left\{ \begin{array}{l} p(M_0 a_{2\theta}, L_0 a_{2\theta}), p(N_0 a_{2\theta+1}, K_0 a_{2\theta+1}), p(L_0 a_{2\theta}, K_0 a_{2\theta+1}), \\ \frac{1}{2}\left[p(M_0 a_{2\theta}, K_0 a_{2\theta+1}) + p(N_0 a_{2\theta+1}, L_0 a_{2\theta}) \right] \end{array} \right\}$

using (3.1.3), we observe

$$p(a_{2\theta}, a_{2\theta+1}) = \max \left\{ \begin{array}{l} p(M_0 a_{2\theta,}, N_0 a_{2\theta-1}), p(N_0 a_{2\theta+1}, M_0 a_{2\theta}), p(N_0 a_{2\theta-1,} M_0 a_{2\theta}), \\ \frac{1}{2}\left[p(M_0 a_{2\theta}, M_0 a_{2\theta}) + p(N_0 a_{2\theta+1}, N_0 a_{2\theta-1}) \right] \end{array} \right\} \tag{3.1.5}$$

From the definition (P4), we have

$$p(N_0 a_{2\theta-1}, N_0 a_{2\theta+1}) + p(M_0 a_{2\theta}, M_0 a_{2\theta}) \leq p(N_0 a_{2\theta-1}, M_0 a_{2\theta}) + p(N_0 a_{2\theta+1}, M_0 a_{2\theta}). \tag{3.1.6}$$

From (3.1.5) and (3.1.6), we observe

$$p(a_{2\theta}, a_{2\theta+1}) = \max \{ p(M_0 a_{2\theta}, N_0 a_{2\theta-1}), p(N_0 a_{2\theta+1}, M_0 a_{2\theta}) \}\ . \tag{3.1.7}$$

But if $p(a_{2\theta}, a_{2\theta+1}) = p(N_0 a_{2\theta+1}, M_0 a_{2\theta})$ then by (3.1,4), we get

$$p(N_0 a_{2\theta+1}, M_0 a_{2\theta}) \leq \lambda\, p(N_0 a_{2\theta+1}, M_0 a_{2\theta})\ , \quad \lambda \in [0,1) \tag{3.1.8}$$

this implies that $p(N_0 a_{2\theta+1}, M_0 a_{2\theta}) = 0$ .Thus, $p(a_{2\theta}, a_{2\theta+1}) = p(N_0 a_{2\theta-1}, M_0 a_{2\theta})$ and from (3.1.4), we observe

$$p(N_0 a_{2\theta+1}, M_0 a_{2\theta}) \leq \lambda\, p(N_0 a_{2\theta-1}, M_0 a_{2\theta})\ , \tag{3.1.9}$$

which gives

$p(b_{2\theta+2}, b_{2\theta+1}) \leq \lambda\, p(b_{2\theta+1}, b_{2\theta})$ for all $\lambda \geq 0$.

After simple computation, considering $0 \leq \lambda < 1$, we deduce that $\{b_\theta\}$ as a Cauchy sequence. But (V, p) being complete, this gives $\{b_\theta\}$ converges to some point $\omega \in V$. Hence, the subsequences

$\{M_0 a_{2\theta}\}, \{K_0 a_{2\theta+1}\},\ \{N_0 a_{2\theta+1}\}, \{L_0 a_{2\theta+2}\}$ also converge to $\omega \in V$. (3.1.10)

By (3.1.1), $M_0(V) \subset K_0(V)$ implies $\exists\ u \in V$ such that $\omega = Vu$ (3.1.11) and $N_0(V) \subset L_0(V)$ implies $\exists\ v \in V$ such that $L_0 v = \omega$. (3.112)

Since the couple {M0, L0} is reciprocally continuous and compatible then the sequences $\{M_0 L_0 a_{2\theta}\} \to M_0\omega$, $\{L_0 M_0 a_{2\theta}\} \to L_0\omega$ and $\{L_0 M_0 a_{2\theta}\} = \{M_0 L_0 a_{2\theta}\}$ as $\theta \to \infty$.

Therefore $M_0\omega = L_0\omega$. (3.1,13)

From (3.1.1), on letting $\alpha = a_{2\theta}, \beta = u$, we have, $p(M_0 a_{2\theta}, N_0 u) \le ƛ\, p(a_{2\theta}, u)$ (3.1.14)

where $p(a_{2\theta},u) = \max\left\{\begin{matrix} p(M_0 a_{2\theta}, L_0 a_\theta), p(N_0 u, K_0 u), p(L_0 a_{2\theta}, K_0 u), \\ \frac{1}{2}\left[p(M_0 a_{2\theta}, K_0 u)\right] + p(N_0 u, L_0 a_{2\theta}) \end{matrix}\right\}$

letting $\theta \to \infty$,using (3.1.10)and (3.1.11), we observe

$$\lim_{\theta\to\infty} p(a_{2\theta},u) = \max\left\{\begin{matrix} p(\omega,\omega), p(N_0 u,\omega), p(\omega,\omega), \\ \frac{1}{2}\left[p(\omega,\omega)\right] + p(N_0 u,\omega) \end{matrix}\right\} = p(N_0 u,\omega). \qquad (3.1.15)$$

From (3.1.14) and (3.1.15) together on letting $\theta \to \infty$ gives

$$\lim_{\theta\to\infty} p(M_0 a_{2\theta}, N_0 u) \le ƛ \lim_{\theta\to\infty} p(a_{2\theta}, u)$$

this implies $p(\omega, N_0 u) \le ƛ\, p(N_0 u, \omega)$, which is impossible since $ƛ \in [0,1)$.

This gives that $N_0 u = \omega$. (3.1.16)

Therefore $K_0 u = N_0 u = \omega$. (3.1.17)

Since the couple {K0,N0} is weakly compatible then $N_0 u = K_0 u = \omega$ and $K_0 N_0 u = N_0 K_0 u$ gives that

$K_0\omega = N_0\omega$. (3.1.18)

Using (3.1.1), $p(M_0 a_{2\theta}, N_0\omega) \le ƛ\, p(a_{2\theta}, \omega)$ (3.1.19)

$p(a_{2\theta},\omega) = \max\left\{\begin{matrix} p(M_0 a_{2\theta}, L_0 a_{2\theta}), p(N_0\omega, K_0\omega), p(L_0 a_{2\theta}, K_0\omega), \\ \frac{1}{2}\left[p(M_0 a_{2\theta}, K_0\omega) + p(N_0\omega, L_0 a_{2\theta})\right] \end{matrix}\right\}$ and

$$\lim_{\theta\to\infty} p(a_{2\theta},\omega) = \max\left\{\begin{matrix} p(M_0\omega, L_0\omega), p(N_0\omega, K_0\omega), p(L_0\omega, K_0\omega), \\ \frac{1}{2}\left[p(M_0\omega, K_0\omega)\right] + p(N_0\omega,\omega) \end{matrix}\right\} = p(N_0\omega,\omega)$$

By (3.19), $\lim_{\theta\to\infty} p(M_0 a_{2\theta}, N_0\omega) \le ƛ \lim_{\theta\to\infty} \mu(a_{2\theta},\omega)$ implies that $p(\omega, N_0\omega) \le ƛ\, p(\omega, N_0\omega)$,which is not possible since $\lambda \in [0,1[$.

This gives $p(\omega, N_0\omega) = 0$ and implies that $N_0\omega = \omega$.

Therefore $N_0\omega = K_0\omega = \omega$. (3.1.20)

To show $M_0\omega = \omega$, put $\alpha = \omega, \beta = a_{2\theta+1}$ in (1), we obtain

$$p(M_0\omega, N_0 a_{2\theta+1}) \le \lambda\, p(\omega, a_{2\theta+1}) \tag{3.1.21}$$

where

$$p(\omega, a_{2\theta+1}) = \max\left\{\begin{array}{l} p(M_0\omega, L_0\omega),\, p(N_0\alpha_{2\theta+1}, K_0 a_{2\theta+1}),\, p(L_0\omega, K_0 a_{2\theta+1}), \\ \frac{1}{2}\left[p(M_0\omega, K_0 a_{2\theta+1}) + p(N_0 a_{2\theta+1}, L_0\omega)\right] \end{array}\right\}$$

letting $N_0 a_{2\theta+1}, K_0 a_{2\theta+1} \to \omega$ as $\theta \to \infty$ and using (3.1.13), we get

$$\lim_{\theta\to\infty} p(\omega, a_{2\theta+1}) = \max\left\{\begin{array}{l} p(M_0\omega, M_0\omega),\, p(\omega, \omega),\, p(M_0\omega, \omega), \\ \frac{1}{2}\left[p(M_0\omega, \omega) + p(\omega, M_0\omega)\right] \end{array}\right\}$$

$$\lim_{\theta\to\infty} p(\omega, a_{2\theta+1}) = p(M_0\omega, \omega) . \tag{3.1.22}$$

From (3.1.21) and (3.1.22), we observe

$$\lim_{\theta\to\infty} p(M_0\omega, N_0 a_{2\theta+1}) \le \lambda \lim_{\theta\to\infty} p(\omega, a_{2\theta+1})$$

which gives

$p(M_0\omega, \omega) \le \lambda\, p(M_0\omega, \omega)$ which is impossible since $\lambda \in [0,1)$.

This gives $p(M_0\omega, \omega) = 0$ and implies that $M_0\omega = \omega$.

Therefore $L_0\omega = M_0\omega = \omega$. (3.1.23)

From (3.1.20) and (3.1.23), we have $N_0\omega = K_0\omega = M_0\omega = L_0\omega = \omega$.

Hence, ω is a common fixed point of M_0, N_0, L_0 and K_0.

To show ω is one and only one common fixed point, if possible assume that there is another common fixed point ω^* of M_0, N_0, K_0 and L_0 . Then using (1), on letting $\alpha = \omega, \beta = \omega^*$, we observe

$$p(\omega, \omega^*) = p(M_0\omega, N_0\omega^*) \le \lambda\, p(\omega, \omega^*)$$

where $p(\omega,\omega^*)=\max\left\{\begin{matrix} p(M_0\omega,L\omega),\, p(N_0\omega^*,K_0\omega^*),\, p(L_0\omega,K_0\omega), \\ \frac{1}{2}\left[p(M_0\omega,K_0\omega^*)+p(N_0\omega^*,L_0\omega)\right] \end{matrix}\right\}$

$$=\max\left\{\begin{matrix} p(\omega,\omega),\, p(\omega^*,\omega^*),\, p(\omega,\omega), \\ \frac{1}{2}\left[p(\omega,\omega^*)+p(\omega^*,\omega)\right] \end{matrix}\right\}$$

$$= p(\omega,\omega^*).$$

Thus, $p(\omega,\omega^*)\le \lambda\, p(\omega,\omega^*)$, $\lambda\in[0,1)$ and gives that $\omega=\omega^*$. So, ω is the unique common fixed point of M_0, N_0, K_0 and L_0.

We justify our theorem with the appropriate example.

Example 3.2: Let $V=[0,2]$ and $p: V\times V\to[0,\infty)$ be defined by $p(\alpha,\beta)=\max\{\alpha,\beta\}$, $\alpha\neq 1$ and $p(1,1)=0$,.Then (V , p) is a complete PMS. Define $M_0,N_0,K_0,L_0 : V\to V$ by

$$M_0\alpha=N_0\alpha=\begin{cases}1 & if\ 0<\alpha\le 1\\ \frac{\alpha+1}{3} & if\ 1<\alpha\le 2\end{cases} \text{ and } L_0\alpha=K_0\alpha=\begin{cases}1 & if\ \alpha=0\\ \alpha & if\ 0<\alpha\le 1\\ \frac{\alpha}{2} & if\ 1<\alpha\le 2\end{cases}.$$

Clearly, $L_0(V)=(0,1]=K_0(V)$ and $M_0[V]=(\frac{2}{3},1]=N_0[V]$;

We have $M_0(V)\subset K_0(V)$ and $N_0(V)\subset L_0(V)$.

Let a sequence $\{\alpha_\theta\}$ be defined by $\alpha_\theta=2-\frac{1}{\theta}$, for $\theta\ge 1$.

Now $\lim_{\theta\to\infty} M_0\alpha_\theta=\lim_{\theta\to\infty} M_0\left(2-\frac{1}{\theta}\right)=\lim_{\theta\to\infty}\frac{1}{2}\left(2-\frac{1}{\theta}\right)=1$ and

$\lim_{\theta\to\infty} L_0\alpha_\theta=\lim_{\theta\to\infty} L_0\left(2-\frac{1}{\theta}\right)=\lim_{\theta\to\infty}\frac{1}{3}\left(2-\frac{1}{\theta}+1\right)=1$.

Also $\lim_{\theta\to\infty} M_0L_0\alpha_\theta=\lim_{\theta\to\infty} M_0\left[L_0\left(2-\frac{1}{\theta}\right)\right]=\lim_{\theta\to\infty} M_0\left[\left(1-\frac{1}{3\theta}\right)\right]=1$ and

$\lim_{\theta\to\infty} L_0M_0\alpha_\theta=\lim_{\theta\to\infty} L_0\left[M_0\left(2-\frac{1}{\theta}\right)\right]=\lim_{\theta\to\infty} L_0\left[\left(1-\frac{1}{2\theta}\right)\right]=1$, which shows that the couple $\{M_0, L_0\}$ is compatible.

Again $\lim_{\theta\to\infty} M_0L_0\alpha_\theta=1=M_0(1)$ and $\lim_{\theta\to\infty} L_0M_0\alpha_\theta=1=L_0(1)$ and this gives the couple $\{M_0, L_0\}$ is reciprocally continuous.

We observe that 0 and 1 are coincident points of K_0 and N_0 and $K_0N_0(0) = 1 = N_0K_0(0)$ and $K_0N_0(1) = 1 = N_0K_0(1)$. This shows that the couple (N_0, K_0) is weakly compatible.

Also , contractive condition (1) holds true for the value $\lambda \in [0,1[$ .It can be observed that 1 is unique common fixed point of maps K_0, L_0, M_0 and N_0.

4. CONCLUSION

In this paper, we generated a theorem by assuming one pair as compatible, reciprocally continuous and other pair as weakly compatible mapping. Further, we justified the result with a suitable example.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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