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PAIRWISE LINDELO PERFECT FUNCTIONS

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Abstract. In this paper we introduce the notions and concepts of the perfect functions in the bitopological spaces, which yield to two types called p-Lindelö perfect and s-Lindelö perfect function. Also we study the images and inverse images of certain bitopological properties under these functions. We derive some related results. Finally some product theorems obtained concerning these concept.

Keywords: bitopological space; $p-$Lindelö perfect function; $S-$Lindelö perfect function.

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1. Introduction

In 1963, Kelly [13] introduced the notion of a bitopological space, i.e. a triple (X,τ_1,τ_2) where X is a set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn$'$s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim [15], Fletcher, Hoyle and

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Patty [8]. In 1969, Fletcher et. al, [8] gave the definitions of $\tau_1\tau_2-$open and $P-$open covers in bitopological spaces. A cover $\tilde{U}$ of the bitopological space (X,τ_1,τ_2) is called $\tau_1\tau_2-$open if $\tilde{U}\subset\tau_1\cup\tau_2$, if in addition, $\tilde{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2, it is called P—open. Also they defined the concept of P—compact space as follows: A bitopological space (X,τ_1,τ_2) is called P—compact if every $P-$open cover of the space (X,τ_1,τ_2) has a finite subcover. While in 1972 Datta [6], defined S—compact space as follows: A bitopological space (X,τ_1,τ_2) is called S—compact if every $\tau_1\tau_2-$open cover of the space (X,τ_1,τ_2) has a finite subcover. In 1969 Birsan [4] gave the following definitions: A bitopological space (X,τ_1,τ_2) is called τ_1—compact with respect to τ_2 if for each τ_1—open cover of X, there is a finite τ_2—open subcover. A bitopological space (X,τ_1,τ_2) is called B—compact if it is τ_1—compact with respect to τ_2 and τ_2—compact with respect to τ_1. In 1975 Cooke and Reilly [5] discussed the relations between these definitions. In 1979 Hdieb [11] introduced important theorem to the theory of $[n\ ,\ m]$ $-$compact, paracompact and normal spaces. In 1983 Fora and Hdieb [9] introduced the definition of $P-$Lindelöf, $S-$Lindelöf, $B-$Lindelöf spaces in analogue manner. They also gave the definitions of certain types of functions as follows : A function $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is called P—continuous ($P-$open, $P-$closed, $P-$homeomorphism, respectively), if both functions $f:(X,\tau_1)\longrightarrow(Y,\sigma_1)$ and $f:(X,\tau_2)\longrightarrow(Y,\sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

We now move , dear reader , to present to you a brief introductory summary of the perfect functions in the single topological spaces and some studies about these conjugations in the bitopological spaces and the important results that have been reached according to these studies.A continous function $f:(X,\tau)\longrightarrow(Y,\sigma)$ is said to be perfect if X is a Hausdorff space, f is closed and the fibers $f^{-1}(y)$ are compact subsets of X. In1952, Vainstein [23] for the first time introduced the class of perfect functions in the realm of metric spaces. Independently, perfect functions were introduced and studied (in the realm of locally compact spaces) by Leray in 1950 and Baurkbaki in1951.Later several mathematicians worked on perfect functions and proved several results concerning it is effect on different topological spaces. For instance Baurkabaki (1961),Panomarev (1966), Michael (1971) and Hdeib(1982). In 2017, Qoqazeh et. al, [19], introduced a new definition of the perfect functions in bitopological spaces as , a function $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is called P—perfect, if the function f is P—continuous, P—closed, and for all $y\in Y$, the set $f^{-1}(y)$ is P—compact.), and gives many properties of them.In 2018 , Atoom and Hdeib [**?**] studied perfect functions in bitopological spaces while preparing their PHD thesis and presented many important results in this field.

Dear reader, many important studies in many areas in bitopological spaces have been brought out in the reference list for reference for adults interested in this field.

2. PRELIMINARIES

In this paper we introduce the notions and concepts of the perfect functions in the bitopological spaces, which yield to two types called $p-$Lindelö perfect, $S-$Lindelö perfect function. Also we study the images and inverse images of certain bitopological properties under these functions.We derive some related results. Finally some product theorems obtained concerning these concept . When (X,τ_1,τ_2) has the property Q this means that both (X,τ_1) and (X,τ_2) have this property. For instance a bitopological space (X,τ_1,τ_2) is called compact, if both (X,τ_1) and (X,τ_2) are compact spaces.

We will use the letters $P-$, $S-$ to denote the pairwise and semi, respectively, e.g. P—compact stands for pairwise compact, S—compact stands for semi compact.

τ_i-closure, τ_i-interior of a set A will be denoted by $CL_{\tau_i}(A)$, $Int_{\tau_i}(A)$ respectively. The product of τ_1 and τ_2 will be denoted by $\tau_1\times\tau_2$.

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the set of all real numbers, integer numbers, natural numbers, and rational numbers respectively. Let τ_{dis}, τ_{ind}, τ_u, τ_s, τ_{coc}, τ_{cof}, τ_l, τ_r denote the discrete, the indiscrete usual, Sorgenfrey line, cocountable, cofinite, left-ray, and right-ray topologies respectively. Let ω_0 and ω_1 denote the cardinal numbers of $\mathbb{Z}$ and $\mathbb{R}$ respectively.

3. Main Results In Pairwise Lindelö Perfect Functions

In this section, we will introduce the concept of Lindelö Perfect functions in bitopological spaces, and introduce some of their properties, and relate it to other spaces.

Let us recall known definitions which will be used in the sequel.

Definition 3.1. [9] *A bitopological space (X,τ_1,τ_2) is called P—Lindelöf if every $P-$open cover of the space (X,τ_1,τ_2) has a countable subcover.*

A bitopological space (X,τ_1,τ_2) is called S—Lindelöf if every $\tau_1\tau_2-$open cover of the space (X,τ_1,τ_2) has a countable subcover.

Definition 3.2. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $p-$Lindelö perfect, if f is $p-$continuous, $p-$closed, and for each $y\in Y$, $f^{-1}(y)$ is $p-$Lindelöf.*

Corollary 3.2.1. *In above definition if $f^{-1}(y)$ is countable then f is $p-$Lindelö perfect function.*

Example 3.3. *Let $f:(R,\tau_u,\tau_{ind})\longrightarrow(R,\tau_u,\tau_{ind})$ be the identy function ,where τ_u and τ_{ind} are the usual and indiscrete topoligies, respectively. then f is $p-$Lindelö perfect function.*

Since f is $P-$continuous , $P-$closed and for each $y\in Y$ any $P-$open cover $\tilde{U}$ of $f^{-1}(y)$ must be contains X because the only non empty open set in $(R\ ,\tau_{ind})$ is X. Hence $\{X\}$ is a countable subcover of$\tilde{U}$. Hence $f^{-1}(y)$ is $p-$Lindelöf .

Example 3.4. *Let $X=\{0\ .1\}$ and $\tau_1=\{\phi\ ,\ X\ ,\{0\}\}$, $\tau_2=\{\phi\ ,\ X\ ,\{1\}\}$. Define $f:(X,\tau_1,\tau_2)\longrightarrow(R,\tau_1,\tau_2)$ by $f(x)=0$. Then f is $p-$Lindelö perfect function.*

Theorem 3.5. *If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is a $p-$Lindelö perfect function, then for every $p-$Lindelöf subset $Z\subseteq Y$, the inverse image $f^{-1}(Z)$ is a $p-$Lindelöf.*

Proof. Let $\tilde{U}=\{U_\alpha{:}\alpha\in\Lambda\}$ be a $p-$open cover of (X,τ_1,τ_2) , since f is a $p-$Lindelö perfect function, then $\forall y\in Y$, $f^{-1}(y)$ is $p-$Lindelöf subset of X. So there exists a countable subsets Λ_y , Λ_y^* of Λ ,s.t $f^{-1}(y)\subseteq\left(\bigcup\limits_{\alpha\in\Lambda_y}\{V_\alpha:\alpha\in\Lambda_y\}\right)\bigcup\left(\bigcup\limits_{\alpha\in\ \Lambda_y^*}\{W_\alpha:\alpha\in\overset{*}{\Lambda_y}\}\right)$, where $\{V_\alpha : \alpha \in \Lambda_y\}$ is τ_1-open subsets of X and $\{W_\alpha : \alpha \in \overset{*}{\Lambda}_y\}$ is τ_2-open subsets of X. Now , let $O_y = Y - f\ (X - \bigcup_{\alpha\in\Lambda_y} V_\alpha)$ is a σ_1-open subset of Y containing y and $O_y^* = Y - f\ (X - \bigcup_{\alpha\in\overset{*}{\Lambda}_y} W_\alpha)$ is also a σ_2-open subset of Y containing y.Then $y \in O_y \cup O_y^*$ .Since $f^{-1}(O_y) \subseteq \bigcup_{\alpha\in\Lambda_y} V_\alpha$ and $f^{-1}\ (O_y^*) \subseteq \bigcup_{\alpha\in\overset{*}{\Lambda}_y} W_\alpha$ then , $\tilde{O} = \{O_y : y \in Y\} \bigcup \{O_y^* : y \in Y\}$ is a $p-$open cover of Y. Hence , $\tilde{O}$ is $p-$open cover of Z . Since Z is $p-$Lindelof ,$\tilde{O}$ has a countable subcover$\left(\bigcup_{i=1}(O_{y_i})\right) \bigcup \left(\bigcup_{j=1}(\overset{*}{O}_{y\,j})\right)$ and $Z \subseteq \left(\bigcup_{i=1}(O_{y_i})\right) \bigcup \left(\bigcup_{j=1}(\overset{*}{O}_{y\,j})\right)$ .Thus, $f^{-1}\ (Z) \subseteq \left(\bigcup_{i=1} f^{-1}(O_{y_i})\right) \bigcup \left(\bigcup_{j=1} f^{-1}(\overset{*}{O}_{y\,j}\ )\right) \subseteq$ of a union of countable subset of $\tilde{U}$, i.e $f^{-1}\ (Z)$ is $p-$Lindelöf . □

Corollary 3.5.1. *A $p-$Lindelöf space is inverse invariant under $p-$Lindelö perfect functions.*

Corollary 3.5.2. *The composition of two $p-$Lindelö perfect functions is a $p-$Lindelö perfect function.*

Proposition 3.6. *If the composition $g \circ f$ of a $p-$continuous functions, $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ and $g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is a $p-$closed then the function $g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is $p-$closed.*

Proof. Let A be a σ_1-closed subset of Y,then $f^{-1}(A)$ is τ_1-closed subset of X. Since $g \circ f$ is $p-$closed,then $g(f\ (f^{-1}(A))) = g(A)$ is ρ_1-closed subset of Z ,Similarly, we can show that if B be a σ_2-closed in Y, then $g(B)$ is ρ_2-closed in Z .Thus g is a $p-$closed. □

Theorem 3.7. *If the composition function $g \circ f$ of a $p-$continuous funcions $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$,$g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is a $p-$Lindelö perfect, then the function $g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is $p-$Lindelö perfect.*

Proof. For every $z \in Z$, $g^{-1}(z) = f\ ((g \circ f\ )^{-1}(z))$ is a $p-$Lindelöf subset of Y, because $g \circ f$ is $p-$Lindelö perfect. □

Since g is $p-$closed by proposition 3.6, we get that g is p-Lindelo perfect .

Theorem 3.8. *If $f:(X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $p-$closed function ,then for any $B \subset Y$ the restriction $f_B:\ f^{-1}(B) \to B$ is $p-$closed .*

Proof. Let $B \subset Y$ .Consider the function $f:(X,\tau_1) \to (Y,\sigma_1)$. let A be a τ_1-closed subset of X. Then $f_B\ (A\bigcap f^{-1}(B)) = f(A)\bigcap B$ is σ_1-closed subset of B. Similarly we can show that if A a τ_2-closed subset of X. $f_B\ (A\bigcap f^{-1}(B)) = f(A)\bigcap B$ is σ_2-closed subset of B. Thus $f_B\ :\ f^{-1}(B) \to B$ is a $p-$closed. □

Theorem 3.9. *If $f:(X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $p-$Lindelö perfect function ,then for any $B \subset Y$ the restriction $f_B\ :\ f^{-1}(B) \to B$ is $p-$Lindelö perfect.*

Proof. The proof follows directly from theorem 3.8. □

The following two theorems can be found in [9]

Theorem 3.10. *Let $X=(X,\tau_1,\tau_2)$ be a $p-$Hausdörff space, then every τ_i-Lindelöf subset is τ_j-closed $(i \neq j, i, j = 1,2)$.*

Theorem 3.11. *A τ_i-closed proper subset of a $p-$Lindelöf space is τ_j $-$Lindelöf $(i \neq j, i, j = 1,2)$.*

Theorem 3.12. *If $f:(X,\tau_1,\tau_2) \overset{onto}{\longrightarrow} (Y,\sigma_1,\sigma_2)$ is a $p-$Lindelö perfect function ,where (X,τ_1,τ_2) is $p-$Lindelöf, and (Y,σ_1,σ_2) is $p-$Hausdörff ,then f is $p-$closed .*

Proof. If A is τ_1-closed subset of (X,τ_1,τ_2) ,then it is τ_2-Lindelöf because (X,τ_1,τ_2) is $p-$Lindelöf. Since f is $p-$continuous , $f(A)$ is a σ_2-Lindelöf subset of (Y,σ_1,σ_2). □

Since (Y,σ_1,σ_2) is $p-$Hausdörff, then $f(A)$ is a σ_1-closed subset of (Y,σ_1,σ_2) . Similarly we can show that if B is a τ_2-closed subset of (X,τ_1,τ_2) , then $f(B)$ is a σ_2-closed subset of (Y,σ_1,σ_2). Hence the result.

We can found the following definition in [21]

Definition 3.13. *If (X,τ_1,τ_2) is a bitopological space, then τ_1 is said to be locally compact with respect to τ_2 if each point of X has a τ_1 open neighborhood whose τ_2 closure is pairwise compact.*

A bitopological space (X,τ_1,τ_2) is said to be pairwise locally compact ($P-$locally compact) if it is $\tau_1$locally compact with respect to $\tau_2$and $\tau_2$locally compact with respect to τ_1.

Note that every $p-$compact space is $p-$locally compact.

Theorem 3.14. *Let $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ be $p-$continuous function from a $p-$Hausdörff space (X,τ_1,τ_2) in to a $p-$locally compact space (Y,σ_1,σ_2).*

Then the following are equivalent :

(i) f is a $p-$Lindelö perfect function,

(ii) For every $p-$Lindelöf subset $Z\subset Y$ the set $f^{-1}(Z)$ is a $p-$Lindelöf subset of X .

Proof. $(i)\Rightarrow(ii)$: the proof follows from theorem 3.8. □

$(ii)\Rightarrow(i)$: It is suffices to show that f is a $p-$closed function , i.e both functions $f_1:(X,\tau_1)\to(Y,\sigma_1)$ and $f_2:(X,\tau_2)\to(Y,\sigma_2)$ are closed functions. Let A be a τ_1-closed subset of X, and y be a cluster point of $f_1(A)$.Suppose $y\notin f_1(A)$.Since Y is $p-$locally compact, there is a σ_1-open set V containing y s.t $CL_{\sigma_2}(V)$ is $p-$compact, and so $p-$Lindelöf.

Now, $f_1^{-1}(CL_{\sigma_2}(V)\bigcap f(A)) = f_1^{-1}(CL_{\sigma_2}(V)) \bigcap A$ .By using (ii) $f_1^{-1}(CL_{\sigma_2}(V))$ is $p-$Lindelöf and A is a τ_2-closed , $p-$Lindelöf subset .

Also , $f_1(f_1^{-1}(CL_{\sigma_2}(V)) \bigcap A) = CL_{\sigma_2}(V)\bigcap f_1(A)$ is a $p-$Lindelöf subset which is σ_1-closed.

Now,$V-CL_{\sigma_2}(V)\bigcap f_1(A)=U$ is a σ_1-open set containing p and $U\bigcap f_1(A)=\phi$,which contradicts the fact that p is a cluster point. Hence $p\in f_1(A)$,i.e $f_1(A)$ is a σ_1-closed.

Thus $f_1:(X,\tau_1)\to(Y,\sigma_1)$ is a closed function.

By a similar method we can show that $f_2:(X,\tau_2)\to(Y,\sigma_2)$ is a closed function. Hence $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is $p-$closed function.

Theorem 3.15. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $p-$continuous bijection function. If (Y,σ_1,σ_2) is $p-$Hausdörff space, and (X,τ_1,τ_2) is $p-$Lindelöf, then f is $p-$homeomorphic function.*

Proof. It's enough to show that f is $p-$closed.Let F be a τ_i-closed proper subset of X. and hence F is proper τ_j-p-Lindelöf, for $i\neq j,i,j=1,2$, by using theorem 3.10 □

Hence , $f(F)$ is a $\sigma_j - p-$Lindelof, but (Y,σ_1,σ_2) is $p-$Hausdörff space, by theorem 3.11, $f(F)$ is σ_i-closed, i.e. f is p-homeomorphic function.

Definition 3.16. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $p-$strongly function, if for every $p-$open cover $\tilde{U}=\{U_\alpha:\alpha\in\Lambda\}$ of X there exists a $p-$open cover*

$\tilde{V}=\left\{V_\gamma:\gamma\in\Gamma\right\}$of Y, s.t $f^{-1}(V\ )\subseteq\bigcup\{U_\alpha:\alpha\in\Lambda_1\ :\ \Lambda_1$ is a countable subset of $\Lambda\}\,\forall\, V\in\tilde{V}$.

Theorem 3.17. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $p-$strongly onto function, then (X,τ_1,τ_2) is $p-$Lindelöf, if (Y,σ_1,σ_2) is so.*

Proof. Let $\tilde{U}\ =\{U_\alpha:\alpha\in\Lambda\}$ be a $p-$open cover (X,τ_1,τ_2).Since f is a $p-$strongly function, there exists $p-$open cover $\tilde{V}=\left\{V_\gamma:\gamma\in\Gamma\right\}$ of (Y,σ_1,σ_2), □

such that $f^{-1}(V\ )\subseteq\bigcup\{U_\alpha:\alpha\in\Lambda_1\ :\ \Lambda_1$ is a countable subset of $\Lambda\}\,\forall\, V\in\tilde{V}$.

but (Y,σ_1,σ_2) is $p-$Lindelöf , so there existsis a countable subset Γ_1 of Γ s.t $Y=\bigcup\limits_{\gamma\in\Gamma_1}V_\gamma$. Hence $X=\bigcup\limits_{\gamma\in\Gamma_1}f^{-1}(V_\gamma)$.So each $f^{-1}(V_\gamma)$ contains in a countable number of members of $\tilde{U}$ Thus X is $p-$Lindelöf.

Theorem 3.18. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $p-$Lindelö perfect function such that $\forall y\in Y$, $f^{-1}(y)$ is $p-$countably compact,. If (Y,σ_1,σ_2) is a $p-$countably compact , then (X,τ_1,τ_2) is so.*

Proof. Let $\tilde{U}=\{U_\alpha{:}\alpha\in\Lambda\}$ be a $p-$open cover of (X,τ_1,τ_2). Since f is a $p-$Lindelö perfect function, then $\forall y\in Y$, $f^{-1}(y)$ is $p-$Lindelöf, $\exists$ a countable subsets Λ_y , $\overset{*}{\Lambda}_y$ of Λ , □

s.t $f^{-1}(y)\subseteq\bigcup\limits_{\alpha\in\Lambda_y}\{V_\alpha:\alpha\in\Lambda_y\}\bigcup\bigcup\limits_{\alpha\in\Lambda_y^*}\{W_\alpha:\alpha\in\overset{*}{\Lambda}_y\}$,where $\{V_\alpha:\alpha\in\Lambda_y\}$ is τ_1-open subsets of X and $\{W_\alpha:\alpha\in\overset{*}{\Lambda}_y\}$ is τ_2-open subsets of X. Now , $Oy(\alpha,y)=Y-f(X-\bigcup\limits_{\alpha\in\Lambda_y}V_\alpha)$ is a σ_1-open set containing y and $O_y^*(\alpha,y)\ =Y-f\ (X-\bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y}W_\alpha:\alpha\in\Lambda)$ is a σ_2-open set containing y .Also , $f^{-1}(O_y(\alpha,y)\ \subseteq\bigcup\limits_{\alpha\in\Lambda_y}V_\alpha$ and $f^{-1}\ (O_y^*\ (\alpha,y)\subseteq\bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y}W_\alpha$.

Let $\tilde{O} = \{O_y(\alpha,y)\ :\ y \in\ Y\} \bigcup \{O_y^*(\alpha,y) : y \in Y\ \}$ then $\tilde{O}$ a $p-$countable cover of Y . Since (Y,σ_1,σ_2) is p -countably compact , $\tilde{O}$ has a $p-$countable subcover say , $\tilde{O}^* = \{O_y(\alpha_i,y)\ \ : i \in N\ ,\ y \in\ \ Y\} \bigcup \{O_y^*(\alpha_i,y) : i \in N\ ,\ y \in Y\ \}$. So $(\ X,\tau_1,\tau_2) = \bigcup\limits_{i\in N} f^{-1}(O_y(\alpha_i,y)) \bigcup \bigcup\limits_{i\in N} f^{-1}(O_y^*(\alpha_i,y))$.Hence (X,τ_1,τ_2) is a $p-$countably compact.

The following theorem shows that a $p-$paracompactness is an inverse invariant under $p-$Lindelö perfect function.

Theorem 3.19. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a $p-$Lindelö perfect function . If (Y,σ_1,σ_2) is a $p-$regular $p-$paracompact space then (X,τ_1,τ_2) is so.*

Proof. Let $\tilde{U} = \{U_\alpha{:}\alpha \in \Lambda\}$ be a p-open cover of (X,τ_1,τ_2) , since f is a $p-$Lindelö perfect function, then $\forall y \in Y,\ f^{-1}(y)$ is $p-$Lindelöf, $\exists$ a countable subsets Λ_y, $\overset{*}{\Lambda}_y$ of Λ , □

s.t $f^{-1}(y) \subseteq \bigcup\limits_{\alpha\in\Lambda_y} \{V_\alpha : \alpha \in \Lambda_y\} \bigcup \bigcup\limits_{\alpha\in\ \Lambda_y^*} \{W_\alpha : \alpha \in \overset{*}{\Lambda}_y\}$, where $\{V_\alpha : \alpha \in \Lambda_y\}$ is τ_1-open , $\{W_\alpha : \alpha \in \overset{*}{\Lambda}_y\}$ is τ_2-open. Let

$O_y = Y - f\ \ (X - \bigcup\limits_{\alpha\in\Lambda_y} V_\alpha)$ is a σ_1-open set containing y, and $O_y^* =\ Y - f\ \ (X - \bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y} W_\alpha)$ is a σ_2-open set containing y , where

$f^{-1}(O_y) \subseteq \bigcup\limits_{\alpha\in\Lambda_y} V_\alpha,\ \ f^{-1}\ (O_y^*) \subseteq \bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y} W_\alpha$. Now , $\tilde{O} = \{O_y : y \in\ \ Y\ \} \bigcup \{O_y^* : y \in\ \ Y\ \}$ is a $p-$open cover of Y .Since (Y,σ_1,σ_2) is p $-$paracompact $\tilde{O}$ has a $p-$open locally finite

parallel refinement say $\tilde{H} = \{H_B : B \in \Gamma_1\ \} \bigcup \{H_B^* : B \in \Gamma_2\ \}$, where $\{H_B : B \in \Gamma_1\ \}$ is σ_1-locally finite parallel refinement of $\{O_y : y \in\ \ Y\ \}$ and $\{H_B^* : B \in \Gamma_2\ \}$ is σ_2-locally finite parallel

refinement of $\{O_y^* : y \in\ \ Y\ \}$. Let $S_1 = \{f^{-1}(H_B) \bigcap V_{\alpha_i}\ : B \in \Gamma_1, \alpha_i \in \Lambda_y\}$ then S_1 is a τ_{1-} open locally finite parallel refinement of $\{V_\alpha : \alpha \in \Lambda\}$. Also $S_2 = \{f^{-1}(H_B^*) \bigcap W_{\alpha_i}\ : B \in \Gamma_2\ , \alpha_i \in \Lambda_y^*\}$ is a τ_2- open locally finite parallel refinement of $\{W_\alpha : \alpha \in \Lambda\}$. Let $\tilde{S} = \{S_1 \bigcup S_2\}$,then $\tilde{S}$ is a $p-$open locally finite parallel refinement of $\tilde{U}$, so (X,τ_1,τ_2) is $p-$paracompact $p-$regular space.

4. MAIN RESULTS IN S-LINDELÖ PERFECT FUNCTIONS

Definition 4.1. *A function $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is called $S-$Lindelö perfect,if f is $p-$continuous, $p-$closed, and for each $y\in Y$, $f^{-1}(y)$ is $S-$Lindelöf.*

Corollary 4.1.1. *In above definition if $f^{-1}(y)$ is countable then f is $S-$Lindelö perfect function.*

Example 4.2. *Let $f:(R,\tau_{cof},\tau_{dis})\longrightarrow f:(R,\tau_{cof},\tau_{dis})$ be the identity function, where τ_f and τ_d are denoted the cofinite topology on R and discrete topologies,respectively.Then f is $S-$Lindelö perfect function. Since f is $P-$continuous , $P-$closed and for each $y\in Y$ any $\tau_1\tau_2-$open cover $\tilde{U}$ of $f^{-1}(y)$ has a countable subcover. Hence $f^{-1}(y)$ is $S-$Lindelöf.*

Remark 4.3. *Since every $P-$open cover of the space (X,τ_1,τ_2) is a $\tau_1\tau_2-$open cover ,it is clear that every $S-$Lindelöf space is $P-$Lindelöf. Hence every $S-$Lindelö perfect function is $P-$Lindelö perfect function.*

Theorem 4.4. *If $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is a $S-$Lindelö perfect function, then for every $S-$Lindelöf subset $Z\subseteq(Y,\sigma_1,\sigma_2)$, the inverse image $f^{-1}(Z)$ is $S-$Lindelöf.*

Proof. Let $\tilde{U}=\{U_\alpha{:}\alpha\in\Lambda\}$ be a $\tau_1\tau_2-$open cover of (X,τ_1,τ_2) , since f is a $S-$Lindelö perfect function, then $\forall y\in Y$, $f^{-1}(y)$ is $S-$Lindelöf subset of X. So there exists a countable subsets Λ_y , Λ_y^* of Λ ,s.t $f^{-1}(y)\subseteq\left(\bigcup\limits_{\alpha\in\Lambda_y}\{V_\alpha:\alpha\in\Lambda_y\}\right)\bigcup\left(\bigcup\limits_{\alpha\in\Lambda_y^*}\{W_\alpha:\alpha\in\overset{*}{\Lambda}_y\}\right)$, where $\{V_\alpha:\alpha\in\Lambda_y\}$ is τ_1-open subsets of X and $\{W_\alpha:\alpha\in\overset{*}{\Lambda}_y\}$ is τ_2-open subsets of X. Now , let $O_y=Y-f\,(X-\bigcup\limits_{\alpha\in\Lambda_y}V_\alpha)$ is a σ_1-open subset of Y and $O_y^*=\;Y-f\,(X-\bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y}W_\alpha)$ is also a σ_2-open subset of Y .Then $y\in O_y\cup O_y^*$ .Since $f^{-1}(O_y)\subseteq\bigcup\limits_{\alpha\in\Lambda_y}V_\alpha$ or $f^{-1}\,(O_y^*)\subseteq\bigcup\limits_{\alpha\in\overset{*}{\Lambda}_y}W_\alpha$ then , $\tilde{O}=\{O_y:y\in\;Y\}\bigcup\{O_y^*:y\in\;Y\}$ is a $\tau_1\tau_2-$open cover of Y. Hence , $\tilde{O}$ is $\tau_1\tau_2-$open cover of Z . Since Z is$S-$Lindelof ,$\tilde{O}$ has a countable subcover$\left(\bigcup\limits_{i=1}(O_{y_i})\right)\bigcup\left(\bigcup\limits_{j=1}(\overset{*}{O}_{y\,j})\right)$ and

$Z \subseteq \left(\bigcup_{i=1}(O_{y_i})\right) \bigcup \left(\bigcup_{j=1}(\overset{*}{O}_{y_j})\right)$ .Thus, $f^{-1}\ (Z) \subseteq \left(\bigcup_{i=1} f^{-1}(O_{y_i})\right) \bigcup \left(\bigcup_{j=1} f^{-1}(\overset{*}{O}_{y_j}\ )\right) \subseteq$ of a union of countable subset of $\tilde{U}$, i.e $f^{-1}\ (Z)$ is $S-$Lindelöf . □

Corollary 4.4.1. *A $S-$Lindelöf space is inverse invariant under $S-$Lindelö perfect functions.*

Corollary 4.4.2. *The composition of two $S-$Lindelö perfect functions is a $S-$Lindelö perfect function.*

Theorem 4.5. *If the composition $g \circ f$ of the $p-$continuous funcions, $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$,$g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is a $S-$Lindelö perfect, then the function $g : (Y,\sigma_1,\sigma_2) \overset{onto}{\to} (Z,\rho_1,\rho_2)$ is $S-$Lindelö perfect.*

Proof. For every $z \in Z$, $g^{-1}(z) = f\ ((g \circ f\ )^{-1}(z)) =$ S-Lindelöf , because $g \circ f$ is $S-$Lindelö perfect. Since g is $p-$closed by proposition (3.6), we get that g is $S-$Lindelö perfect. □

Theorem 4.6. *If $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $S-$Lindelö perfect function , then for any $B \subset Y$ the restriction $f_B\ :\ f^{\ -1}(B)\ \to B$ is $S-$Lindelö perfect .*

Proof. The proof is similar to the proof of theorem 3.5.

The following theorem is easy to prove similarly to the theorem 3.8 □

Theorem 4.7. *If $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $S-$Lindelö perfect function ,then for any $B \subset Y$ the restriction $f_B\ :\ f^{\ -1}(B)\ \to B$ is $S-$Lindelö perfect.*

Theorem 4.8. *If $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $S-$perfect ,where (X,τ_1,τ_2) is $P-$Lindelöf, and (Y,σ_1,σ_2) is $p-$Hausdorff , then f is $p-$closed .*

Proof. The proof follows from remark 4.3 and theorem 3.12. □

since every $P-$compact space (X,τ_1,τ_2) is $P-$Lindelöf we give the following important result:

Corollary 4.8.1. *If $f : (X,\tau_1,\tau_2) \overset{onto}{\to} (Y,\sigma_1,\sigma_2)$ is $S-$Lindelö perfect ,where (X,τ_1,τ_2) is $P-$compact, and (Y,σ_1,σ_2) is $p-$Hausdorff , then f is $p-$closed .*

Definition 4.9. *If (X,τ_1,τ_2) is a bitopological space, then τ_1 is said to be $S-$locally compact with respect to τ_2 if each point of X has a $\tau_1$open neighborhood whose τ_2 closure is $S-$compact.*

A bitopological space (X,τ_1,τ_2) is said to be $S-$ locally compact if τ_1 is $S-$locally compact with respect to $\tau_2$and $\tau_2$is $S-$locally compact with respect to τ_1.

Note that every $S-$compact space is $S-$locally compact.

Theorem 4.10. *The $P-$Hausdroff space is invariant under $S-$Lindelö perfect functions.*

Proof. Let (X,τ_1,τ_2) be a $P-$Hausdroff space, $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $S-$Lindelö perfect function, and $y_1\neq y_2$ in (Y,σ_1,σ_2),then $f^{-1}(y_1)$,$f^{-1}(y_2)$ are disjoint and $S-$Lindelöfness subset of (X,τ_1,τ_2) . Since (X,τ_1,τ_2) be a $P-$Hausdroff space ,there exists a τ_{1-}neighborhood U in X , and τ_{2-}neighborhood$V$ in X s.t $f^{-1}(y_1)\subseteq U$,$f^{-1}(y_2)\subseteq V$ and $U\bigcap V=\phi$.Now , the sets $Y-f(X-U)$ is a σ_1-open subset in (Y,σ_1,σ_2) containing y_1 and $Y-f(X-V)$ is a σ_2-open subset in (Y,σ_1,σ_2) containing y_2 , s.t $[Y-f(X-U)\bigcap Y-f(X-V)]=Y-[f(X-U)\bigcup f(X-V)]$ □

$=Y-f(X-U\bigcap V)=Y-f(X)=\phi$.Hence (Y,σ_1,σ_2) is $P-$Hausdroff space.

The following theorem is easy to prove :

Theorem 4.11. *The $P-$Hausdroff space is inverse invariant under $S-$Lindelö perfect functions.*

Definition 4.12. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called $S-$strongly function, if for every $\tau_1\tau_2-$open cover $\tilde{U}=\{U_\alpha:\alpha\in\Lambda\}$ of X there exists a $\tau_1\tau_2$-open cover*

$\tilde{V}=\{V_\gamma:\gamma\in\Gamma\}$of Y, s.t $f^{-1}(V)\subseteq\bigcup\{U_\alpha:\alpha\in\Lambda_1$: Λ_1 is a countable subset of $\Lambda\}\,\forall\,V\in\tilde{V}$.

Theorem 4.13. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a $p-$closed onto function, and $f^{-1}(y)$ is $S-$Lindelöf for all $y\in Y$, then f is $S-$strongly function.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a $\tau_1\tau_2$-open cover of (X, τ_1, τ_2). Since $f^{-1}(y)$ is $S-$Lindelöf for all $y \in Y$, there exists a countable subset $\Lambda_1 \subset \Lambda$ such that $f^{-1}(y) \subseteq \bigcup_{\alpha\in\Lambda_1} U_\alpha$. Let $O_y = Y - f(X - \bigcup_{\alpha\in\Lambda_1} U_\alpha)$, then O_y is $\sigma_1\sigma_2-$open subset of Y. Now , define $\tilde{O} = \{O_y : y \in Y\}$, then $\tilde{O}$ is $\sigma_1\sigma_2-$open cover of Y. Hence $f^{-1}(O_y)$ is contained in a countable number of members of $\tilde{U}$ thus f is $S-$strongly function. □

Theorem 4.14. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $p-$continuous $S-$strong function and $K \in \sigma_1 \cup \sigma_2$ be a $S-$Lindelöf subset of (Y, σ_1, σ_2).Then $f^{-1}(K)$ is $S-$Lindelöf subset of (X, τ_1, τ_2).*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a $\tau_1\tau_2$-open cover $f^{-1}(K)$. Let $\tilde{W} = \tilde{U} \bigcup \{X - f^{-1}(K)\}$, then $\tilde{W}$ is a $\tau_1\tau_2$-open cover of (X, τ_1, τ_2). Since f is $S-$strongly function, there exists a $\tau_1\tau_2$-open cover$\tilde{V} = \{V_\gamma : \gamma \in \Gamma\}$of (Y, σ_1, σ_2) such that $f^{-1}(K)$ contained in the union of countable members of $\tilde{U}$, But K is $S-$Lindelöf subset of (Y, σ_1, σ_2), so K is contained in the countable members of $\tilde{V}$. Hence $f^{-1}(K)$ is $S-$Lindelöf subset of (X, τ_1, τ_2). □

Theorem 4.15. *Let $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$, be any two bitopological spaces .If (X, τ_1, τ_2) is $S-$Lindelöf ,then the projection function, $\pi : (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2) \to (Y, \sigma_1, \sigma_2)$ is closed.*

Proof. If (X, τ_1, τ_2) is $S-$Lindelöf, then both topological spaces (X, τ_1) and (X, τ_2) are $S-$Lindelöf. Thus the projection functions:$\pi_1 : (X \times Y, \tau_1 \times \sigma_1) \to (Y, \sigma_1)$ and $\pi_2 : (X \times Y, \tau_2 \times \sigma_2) \to (Y, \sigma_2)$ are closed functions. Hence π is closed function. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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