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STABLE LINEAR MULTISTEP METHODS WITH OFF-STEP POINTS FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: Of recent, stability has become an important concept and a qualitative property in any numerical integration scheme. In this work, we propose two stable linear multistep methods with off-step points for the numerical integration of ordinary differential equations whose development is collocation and interpolation based. The boundary locus techniques show that the proposed schemes are zero-stable, A-stable and $A(\alpha)$-stable for some step number k and are found suitable for stiff differential equations. Numerical results obtained compare favourably with some existing methods in literature.

Keywords: stiff differential equation; collocation; interpolation; multi-step; ordinary differential equation; numerical; stability.

Subject Classification codes: 65L05.

1. INTRODUCTION

Differential equations are equations resulting from modeling physical phenomena in sciences, social sciences, management, etc. In particular, ordinary differential equation (ODE) models have been playing a prominent role in physics, engineering, econometrics, biomedical sciences among

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other scientific fields. In fact, ODEs are the most widespread formalism to model dynamical systems in science and engineering. When the models appear in one or more derivatives, they are referred to as first or higher order differential equations, respectively. Systems of first order differential equation and can be expressed as:

$$y' = f(x, y), f: R\ xR^{m} \rightarrow \mathbb{R}^{m}, \quad x \in [x_0, x_N] \qquad (1)$$

It is generally known that the solutions of models are not generally written in closed form. In order to understand these solutions, it is often necessary to construct an approximation through computational methods which this work targets to achieve. This research work is concerned with the development and analysis of two new methods for solving first order initial value problems in ordinary differential equations with the aim of achieving high computational accuracy and whose solution can compete favourably with the exact solution in some selected problems without incurring high computational cost in implementation.

2. PRELIMINARIES

Lately, there are several numerical methods that have been developed by researchers for approximate solutions to models [7], [8], [9], etc. This is ranging from the one-step method such as the Euler method, Runge-Kutta methods, etc to the multistep methods such as the Adam Bashforth method (AB), Adam-Moulton (AM), backward different formula (BDF), trapezoidal rule, General linear methods (GLM), etc. Each of these methods has its computational advantages and disadvantages based on the type of ODEs to be solved. The process of using numerical methods to provide approximate solutions to ODEs models is known as “numerical integration” [2]. Differential equation (1) can be further classified into initial value problems and boundary value problems (BVPS). The equation (1) can be called an initial value problem if it has specified values assigned to it called the initial conditions of the unknown function at a given point in the domain of the solutions. This is written as:

$$y' = f(x, y), y\ (x_0) = y_0 \in \mathbb{R} \qquad (2)$$

A solution to (2) is the function $y(x)$ and satisfies the initial condition. The differential equation (1) is a boundary value problem, if the conditions can be specified in more than one point in the domain of the solution (Lambert, 1991). i.e.

$$y' = f(x, y), y\,(x_0) = y_0,\;\; y(x_1) = y_1,\quad \forall\, y_0,\; y_1 \in \mathbb{R} \tag{3}$$

Numerical methods for solving (2) have been known for a long time. Among the famous examples is the forward Euler method introduced as early as 1768 by Leonard Euler. Since then more good methods have been developed like the Runge-Kutta and the linear multistep methods. Numerical implementation solvers have also played vital roles in the solutions of ODEs through advancement of numerical codes and computer era.

Linear multistep methods as one of the promising methods for solving (1) and have been modified by researchers in recent time and have become useful methods for numerical integration of differential equations. Off grid collocation points have been introduced to the conventional linear multistep methods to improve stability and as well reduce errors during integration. This method is called “hybrid methods” which is the focal point of this research work. Two hybrid linear multistep methods that are $\boldsymbol{A}(\alpha)$- stable and A-stable and both having wide regions of absolute stability for the numerical integration of (1) is derived. Numerical methods with these properties are often used for special classes of ODEs especially for stiff differential equations [4], [5], [6]. The concept of stiffness shall be explained later in this research work. Most of the existing methods cannot approximate stiff differential equation due to small regions of absolute stability. The two methods are obtained by incorporating off-step points to the conventional second derivative linear multistep methods so as to overcome the constraints imposed by [1] on the stability of linear multistep methods. On the other hand, we shall examine their error constants, region of absolute stability and test their efficiency.

3. Statement of the Problem

Many methods have underperformed in some classes of problems in ordinary differential equations, especially stiff differential equations. This is due to the small region of absolute stability.

Numerical methods for the integration of stiff IVPs are often required to possess large region of absolute stability and smaller error constants for which small regions are constrained to this class of ODEs

The aim of this study is to develop, by means of interpolation and collocation, two high order hybrid methods for solving systems of first order stiff initial value problems in ordinary differential equations.

4. MAIN RESULTS

Derivation of the proposed hybrid methods: The first method considered in this work is expressed as

Method 1:

$$y_{n+k} = y_{n+k-1} + h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \eta_{vm} f_{n+vm}\right) + h^2(\lambda_{vm-1} f'_{n+vm-1} + \lambda_k f'_{n+k}) \qquad (4)$$

Order of the method: $p = k + 4$

Hybrid Predictors:

1. $$y_{n+vm-1} = \sum_{j=0}^{k} \alpha_j y_{n+j} + h^2 \lambda'_k \; f'_{n+k} \qquad (5)$$

of order $p^* = k+1$

2. $$y_{n+vm} = \sum_{j=0}^{k} \alpha'_j \; y_{n+j} + \beta_k h f_{n+k} + h^2 \lambda''_k \; f'_{n+k} \qquad (6)$$

of order $p^{**} = k+2$

where $\{\beta_j\}_{j=0}^{k}$, $\{\alpha_j\}_{j=0}^{k}$, $\{\alpha'_j\}_{j=0}^{k}$, $j = 0(1)k$, η_{vm}, λ_{vm-1}, λ'_k, and β_k, λ''_k are constant coefficients which depend on step-size are carefully and uniquely determined so that the methods achieved higher order of stability. The Equations (5) and (6) are hybrid predictors of the methods. The parameters of the off-step points are chosen according as:

$$vm = \frac{2k+1}{2}, \quad vm-1 = \frac{2k-1}{2}$$

The method (4) is an extended second derivative backward differentiation formula with off-step points. The parameters vm and $vm-1$ provide grid collocation points x_{n+vm}, x_{n+vm-1}, in the open interval (x_n, x_{n+k}), (Gear, 1965).

Derivation of proposed hybrid method 1: In order to obtain (4), we proceed by seeking the approximate solutions of the exact solution of (1) by assuming a continuous solution $y(x)$ of the form

$$y(x)=\sum_{j=0}^{k+4} b_j \varphi^j(x) \tag{7}$$

where $x\in[x_0, x_N]$, b_j, $j=1(1)k+4$ are unknown coefficients and $\varphi^j(x)$ are polynomial basis function of degree $k+4$. We take first and second derivatives of (7) and obtained

$$y'(x)=\sum_{j=1}^{k+4} jb_j\varphi^{j-1} \tag{8}$$

$$y''(x)=\sum_{j=2}^{k+4} j(j-1)b_j\varphi^{j-2} \tag{9}$$

Collocating (7) at x_{n+k-1} and interpolating (8) and (9) at $x_{n+j}, j=0(1)k$, x_{n+vm} and x_{n+vm-1} to obtain a system of equations through which the coefficients are obtained. The equations are obtained for each step number k. Now for step number $k=1$ is as follows

$$a_0 = y_n$$

$$a_1 = f_n$$

$$a_1 + 3ha_2 + \frac{27h^2a_3}{4} + \frac{27h^3a_4}{2} + \frac{405h^4a_5}{16} = f_{\frac{3}{2}+n}$$

$$a_1 + 2ha_2 + 3h^2a_3 + 4h^3a_4 + 5h^4a_5 = f_{1+n}$$

$$2a_2 + 3ha_3 + 3h^2a_4 + \frac{5h^3a_5}{2} = f'_{\frac{1}{2}+n}$$

$$2a_2 + 6ha_3 + 12h^2a_4 + 20h^3a_5 = f'_{1+n}$$

where the hybrid parameters are obtained as

$vm=\frac{2+1}{2}$ and $vm-1=\frac{2-1}{2}=\frac{1}{2}$

Solving with MATHEMATICA 10.0 software, we obtain the coefficients as

$a_0 = y_n$, $a_1 = f_n$, $a_2 = -\frac{6f_n - 6f_{1+n} + 4hf'_{\frac{1}{2}+n} + hf'_{1+n}}{2h}$,

$a_3 = -\frac{-101f_n + 117f_{1+n} - 16f_{\frac{3}{2}+n} - 96hf'_{\frac{1}{2}+n} + 3hf'_{1+n}}{27h^2}$, $a_4 = -\frac{19f_n - 27f_{1+n} + 8f_{\frac{3}{2}+n} + 21hf'_{\frac{1}{2}+n} - 6hf'_{1+n}}{9h^3}$,

$a_5 = \frac{4(5f_n - 9f_{1+n} + 4f_{\frac{3}{2}+n} + 6hf'_{\frac{1}{2}+n} - 3hf'_{1+n})}{45h^4}$

We now obtain the method for k=1

$$y_{n+1} = y_n + h\left(\frac{2f_n}{27} + \frac{13f_{n+1}}{15} + \frac{8f_{n+\frac{3}{2}}}{135}\right) + h^2\left(\frac{-11f'_{n+\frac{1}{2}}}{45} - \frac{19f'_{n+1}}{90}\right) \quad (10)$$

with the error constant as $c_6 = \frac{-13}{86400}$, and order $p = 5$

For k=2

We obtained the system of equations with the hybrid parameters as

$$vm = \frac{4+1}{2} = \frac{5}{2} \text{ and } vm - 1 = \frac{4-1}{2} = \frac{3}{2}$$

$$a_0 + ha_1 + h^2a_2 + h^3a_3 + h^4a_4 + h^5a_5 + h^6a_6 = y_{1+n}$$

$$a_1 = f_n,$$

$$a_1 + 2ha_2 + 3h^2a_3 + 4h^3a_4 + 5h^4a_5 + 6h^5a_6 = f_{1+n}$$

$$a_1 + 2ha_2 + 3h^2a_3 + 4h^3a_4 + 5h^4a_5 + 6h^5a_6 = f_{1+n}$$

$$a_1 + 4ha_2 + 12h^2a_3 + 32h^3a_4 + 80h^4a_5 + 192h^5a_6 = f_{2+n}$$

$$a_1 + 5ha_2 + \frac{75h^2a_3}{4} + \frac{125h^3a_4}{2} + \frac{3125h^4a_5}{16} + \frac{9375h^5a_6}{16} = f_{\frac{5}{2}+n}$$

$$2a_2 + 9ha_3 + 27h^2a_4 + \frac{135h^3a_5}{2} + \frac{1215h^4a_6}{8} = f'_{\frac{3}{2}+n}$$

$$2a_2 + 12ha_3 + 48h^2a_4 + 160h^3a_5 + 480h^4a_6 = f'_{2+n}$$

Solving in similar manner as in k=1 we obtain the coefficients as

$$a_0 = \frac{1}{25200}(-6249hf_n - 78940hf_{1+n} + 61845hf_{2+n} - 1856hf_{\frac{5}{2}+n} + 25200y_{1+n} - 45600h^2f'_{\frac{3}{2}+n} - 7110h^2f'_{2+n})$$

$$a_1 = f_n$$

$$a_2 = -\frac{114 f_n - 1100 f_{1+n} + 1050 f_{2+n} - 64 f_{\frac{5}{2}+n} - 800 h f'_{\frac{3}{2}+n} - 75 h f'_{2+n}}{70h}$$

$$a_3 = -\frac{-345 f_n + 5776 f_{1+n} - 5943 f_{2+n} + 512 f_{\frac{5}{2}+n} + 4608 h f'_{\frac{3}{2}+n} + 222 h f'_{2+n}}{252h^2}$$

$$a_4 = -\frac{1053 f_n - 23140 f_{1+n} + 25095 f_{2+n} - 3008 f_{\frac{5}{2}+n} - 19680 h f'_{\frac{3}{2}+n} + 150 h f'_{2+n}}{1680h^3}$$

$$a_5 = -\frac{2(-39 f_n + 1010 f_{1+n} - 1155 f_{2+n} + 184 f_{\frac{5}{2}+n} + 900 h f'_{\frac{3}{2}+n} - 60 h f'_{2+n})}{525h^4}$$

$$a_6 = -\frac{9 f_n - 260 f_{1+n} + 315 f_{2+n} - 64 f_{\frac{5}{2}+n} - 240 h f'_{\frac{3}{2}+n} + 30 h f'_{2+n}}{630h^5}$$

We obtain the method for k=2 as;

$$y_{n+2} = y_{n+1} + h\left(\frac{13 f_n}{8400} + \frac{37 f_{n+1}}{1260} + \frac{221 f_{n+2}}{240} + \frac{76 f_{n+\frac{5}{2}}}{1575}\right) + h^2\left(\frac{-2 f'_{n+\frac{3}{2}}}{7} - \frac{173 f'_{n+2}}{840}\right) \qquad (11)$$

With error constant

$$c_7 = \frac{-67}{1209600} \text{ and order } p = 6.$$

We therefore generalized the nth step number to a matrix of system of difference equation to give

$$\begin{pmatrix} 1 & x_n & x_n^2 & . & . & . & x_n^{k+4} \\ 0 & 1 & 2x_n & . & . & . & (k+4)x_n^{k+3} \\ 0 & 1 & 2x_{n+vm} & . & . & . & (k+4)x_{n+vm}^{k+2} \\ . & 1 & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 1 & 2 & . & . & . & (k+3)(k+4)_{n+1}^{k+2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_{k+4} \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+vm} \\ . \\ . \\ . \\ f'_{n+k} \end{pmatrix} \qquad (12)$$

Solving equation (12) with Mathematica 10.0 we obtain other members of the family of methods in (4).

4.1.2 Derivation of hybrid predictor 1 for method 1: If the solution of the formula (4) at the point x_{n+vm} is given as the polynomial interpolant

$$y(x_{n+vm}) = \sum_{j=0}^{k+1} c_j x^j \tag{13}$$

where $(c_j)_{j=0}^{k+1}$, $j = 0(1)\,\mathrm{k+1}$the unknown coefficients to be determined, x^j is the polynomial basis function. The second derivative of (12) gives

$$y''(x_{n+vm}) = \sum_{j=2}^{k+1} j(j-1)c_j x^{j-2} \tag{14}$$

Collocating at point x_{n+vm} and points x_{n+j}, $j = 0(1)\,\mathrm{k}$ and interpolating at points x_{n+vm} and x_{n+k} to obtain a system of equation for each value of k

Now, let us consider for $k = 1$, we have the system of equations

$$a_0 = y_n$$

$$a_0 + ha_1 + h^2a_2 + h^3a_3 = y_{1+n}$$

$$a_1 + 2ha_2 + 3h^2a_3 = f_{1+n}$$

$$2a_2 + 6ha_3 = f'_{1+n}$$

Solving with the Mathematica 10.0 to obtain the values as

$$a_0 = y_n$$

$$a_1 = -\frac{4hf_{1+n} + 6y_n - 6y_{1+n} - h^2f'_{1+n}}{2h}$$

$$a_2 = -\frac{-3hf_{1+n} - 3y_n + 3y_{1+n} + h^2f'_{1+n}}{h^2}$$

$$a_3 = -\frac{2hf_{1+n} + 2y_n - 2y_{1+n} - h^2f'_{1+n}}{2h^3}$$

With this we obtain the hybrid formula as

$$y_{n+\frac{3}{2}} = \frac{3hf_{n+1}}{8} - \frac{y_n}{8} + \frac{9y_{n+1}}{8} + \frac{3h^2 f'_{n+1}}{16} \tag{15}$$

of order $p^* = 3$ and error constant

$$c_4 = \frac{1}{128}$$

For $k = 2$, $vm = \frac{5}{2}$

We have the system of equations as

$$a_0 = y_n$$

$$a_0 + ha_1 + h^2 a_2 + h^3 a_3 = y_{1+n}$$

$$a_0 + 2ha_1 + 4h^2 a_2 + 8h^3 a_3 = y_{2+n}$$

$$2a_2 + 12ha_3 = f'_{2+n}$$

Solving to obtain

$$a_0 = y_n$$

$$a_1 = -\frac{11y_n - 16y_{1+n} + 5y_{2+n} - 2h^2 f'_{2+n}}{6h}$$

$$a_2 = -\frac{-2y_n + 4y_{1+n} - 2y_{2+n} + h^2 f'_{2+n}}{2h^2}$$

$$a_3 = -\frac{y_n - 2y_{1+n} + y_{2+n} - h^2 f'_{2+n}}{6h^3}$$

Through these parameters, we now obtain the method of order $p^* = 4$ and error constant $c_5 = 1/256$ as

$$y_{n+\frac{5}{2}} = \frac{15hf_{n+2}}{64} + \frac{3y_n}{128} - \frac{5y_{n+1}}{16} + \frac{165y_{n+2}}{128} + \frac{15h^2 f'_{n+2}}{64} \qquad (16)$$

Continuing in this form other values for k are obtained.

Derivation of hybrid predictor 2 for method 1: The hybrid predictor methods in (6) are obtained following similar approach as in (5). we obtain the methods and error constants as follows

For $k = 1$, $vm - 1 = \frac{1}{2}$

$$y_{n+\frac{1}{2}} = \frac{y_n}{2} + \frac{y_{n+1}}{2} - \frac{h^2 f'_{n+1}}{8}, \; c_3 = \frac{1}{6} \qquad (17)$$

For k=2, $vm-1=\frac{3}{2}$

$$y_{n+\frac{3}{2}}=-\frac{y_n}{16}+\frac{5y_{1+n}}{8}+\frac{7y_{2+n}}{16}-\frac{1}{16}h^2f'_{2+n}$$

$$c_4=\frac{7}{384} \tag{18}$$

For $k=3$

We have the method and the predictors as

$$y_{n+3}=y_{n+2}+h\left(-\frac{67f_n}{219240}+\frac{26f_{n+1}}{5075}-\frac{1807f_{n+2}}{73080}+\frac{3827f_{n+3}}{3915}+\frac{1936f_{n+\frac{7}{2}}}{45675}\right)+h^2\left(-\frac{1004f'_{n+\frac{5}{2}}}{3045}-\frac{7550f'_{n+3}}{36540}\right)$$

of order $p=7$ and error constant $c_8=-\frac{881}{35078400}$

and the hybrid formulas as

$$y_{n+\frac{5}{2}}=\frac{7y_n}{352}-\frac{25y_{n+1}}{176}+\frac{255y_{n+2}}{352}+\frac{35y_{n+3}}{88}-\frac{15}{352}h^2f'_{n+3}$$

$p*=4$ and error constant $c_5=\frac{23}{2816}$

$$y_{n+\frac{7}{2}}=\frac{35}{384}hf_{n+3}-\frac{5y_n}{576}+\frac{21y_{n+1}}{256}-\frac{35y_{n+2}}{64}+\frac{3395y_{n+3}}{2304}+\frac{35}{128}h^2f'_{n+3}$$

$p**=5$ and error constant $c_6=\frac{7}{3072}$

For $k=4$

$$y_{n+4}=y_{n+3}+h\left(\frac{881f_n}{8968320}-\frac{853f_{n+1}}{653940}+\frac{2141f_{n+3}}{653940}+\frac{3103457f_{n+4}}{2989440}+\frac{19136f_{n+\frac{9}{2}}}{490455}\right)+$$
$$h^2\left(-\frac{456f'_{n+\frac{7}{2}}}{1211}-\frac{73723f'_{n+4}}{348768}\right)$$

of order $p=7$ and error constant $c_8=\frac{-116411}{8788953600}$

With hybrid formulas

$$y_{n+\frac{7}{2}}=-\frac{23y_n}{2560}+\frac{21y_{1+n}}{320}-\frac{301y_{2+n}}{1280}+\frac{259y_{3+n}}{320}+\frac{189y_{4+n}}{512}-\frac{21}{640}h^2f'_{4+n}$$

of order $p^*=5$ and error constant $c_6=\frac{343}{76800}$

$$y_{n+\frac{9}{2}}=-\frac{105hf_{4+n}}{2048}+\frac{35y_n}{8192}-\frac{5y_{1+n}}{128}+\frac{189y_{2+n}}{1024}-\frac{105y_{3+n}}{128}+\frac{13685y_{4+n}}{8192}+\frac{315h^2f'_{4+n}}{1024}$$

of order $p^{**}=6$ and error constant $c_7=\frac{3}{2048}$

Method for $k=5$

$$\boldsymbol{y_{n+5}}=h\left(-\frac{116411f_n}{2800413000}+\frac{36467f_{1+n}}{69828480}-\frac{61937f_{2+n}}{17820810}+\frac{4024217f_{3+n}}{203666400}-\frac{1612007f_{4+n}}{10183320}+\frac{4498911341f_{5+n}}{4073328000}+\frac{54477824f_{\frac{11}{2}+n}}{1470216825}\right)+y_{4+n}+h^2\left(-\frac{181304f'_{\frac{9}{2}+n}}{424305}-\frac{14837593f'_{5+n}}{67888800}\right)$$

of order $p=9$

With hybrid formulas as

$$y_{n+\frac{9}{2}}=\frac{343y_n}{70144}-\frac{5445y_{1+n}}{140288}+\frac{4977y_{2+n}}{35072}-\frac{23835y_{3+n}}{70144}+\frac{62055y_{4+n}}{70144}+\frac{48699y_{5+n}}{140288}-\frac{945h^2f'_{5+n}}{35072}$$

of order $p^*=6$ and error constant $c_7=\frac{771}{280576}$

$$y_{n+\frac{11}{2}}=-\frac{3927hf_{5+n}}{20480}-\frac{63y_n}{25600}+\frac{385y_{1+n}}{16384}-\frac{55y_{2+n}}{512}+\frac{693y_{3+n}}{2048}-\frac{1155y_{4+n}}{1024}+\frac{768383y_{5+n}}{409600}+\frac{693h^2f'_{5+n}}{2048}$$

of order $p^{**}=7$ and error constant $c_8=\frac{33}{32768}$

TABLE 3: Discrete Coefficients of the method (4)

K	1	2	3	4	5
β_0	$\frac{2}{27}$	$\frac{13}{1260}$	$\frac{-67}{219240}$	$\frac{881}{8968320}$	$\frac{116411}{2800413000}$
β_1	$\frac{13}{15}$	$\frac{37}{1260}$	$\frac{26}{5075}$	$\frac{853}{653940}$	$\frac{3647}{69828480}$
β_2	0	$\frac{221}{240}$	$\frac{180}{73080}$	$\frac{2141}{653940}$	$\frac{61937}{17820810}$
β_3	0	0	$\frac{382}{3915}$	$\frac{19136}{490455}$	$\frac{4024217}{203666400}$
β_4	0	0	0	$\frac{1234}{480440}$	$-\frac{1612007}{10183320}$
β_5	0	0	0	0	$\frac{4498911341}{4073328000}$
λ_{vm}	$\frac{8}{135}$	$\frac{76}{1575}$	$\frac{1936}{45675}$	$\frac{19136}{490455}$	$\frac{54477824}{1470216825}$
λ_{vm-1}	$\frac{-11}{45}$	$\frac{-2}{7}$	$\frac{1004}{3045}$	$-\frac{456}{1211}$	$\frac{181304}{424305}$
λ_k	$\frac{19}{90}$	$\frac{173}{840}$	$\frac{7850}{36540}$	$\frac{-73723}{345768}$	$\frac{14837593}{67888800}$

TABLE 4: Discrete Coefficients of the Predictor (5)

k	1	2	3	4	5
α_0	$\frac{1}{2}$	$-\frac{1}{16}$	$\frac{7}{352}$	$\frac{23}{2560}$	$\frac{343}{70144}$
α_1	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{25}{176}$	$\frac{21}{320}$	$\frac{5445}{140288}$
α_2	0	$\frac{7}{16}$	$\frac{255}{352}$	$\frac{301}{1280}$	$\frac{4977}{35072}$
α_3	0	0	$\frac{35}{88}$	$\frac{259}{320}$	$-\frac{23835}{70144}$
α_4	0	0	0	$\frac{189}{512}$	$\frac{62055}{70144}$
α_5	0	0	0	0	$\frac{48699}{140288}$
λ'_k	$\frac{-1}{8}$	$-\frac{1}{16}$	$-\frac{15}{352}$	$-\frac{21}{640}$	$\frac{945}{35072}$

Derivation of proposed hybrid Method 2

$$y_{n+k} = \alpha_{k-1} y_{n+k-1} + h \sum_{j=0}^{k} \varsigma_j f_{n+j} + h\lambda f_{n+\frac{1}{2}} + h^2 \sum_{j=0}^{k} \sigma_j f'_{n+k} \quad (19)$$

of order $p_1{*} = 2k+3$

With predictor

$$y_{n+\frac{1}{2}} = \sum_{j=0}^{k} \eta_j y_{n+j} + h\lambda_k f_{n+k} + h^2 \zeta_k f_{n+k} \quad (20)$$

of order $p{**}_1 = k+2$

The coefficients $\left[\varsigma_j\right]_{j=0}^{k}, \lambda, \left[\sigma_j\right]_{j=0}^{k}, \alpha_{k-1}$ are to be determined. We normalized $\alpha_{k-1} = 1$. The methods (18) have only one off-step point with a fixed parameter at $x_{n+\frac{1}{2}}$ for stability and for each value of k. This method differs from the methods (4) since it has only one fixed hybrid predictor unlike the latter has two off-step points with variable hybrid parameters. Interpolation and collocation approach is adopted in its derivation as in methods (4) above. We obtain the constant parameters of the methods for k=1 below:

$$a_0 = y_n$$

$$a_1 = f_n$$

$$a_2 = \frac{f'_n}{2}$$

$$a_3 = -\frac{11f_n - 16f_{\frac{1}{2}+n} + 5f_{1+n} + 4hf'_n - hf'_{1+n}}{3h^2}$$

$$a_4 = -\frac{-18f_n + 32f_{\frac{1}{2}+n} - 14f_{1+n} - 5hf'_n + 3hf'_{1+n}}{4h^3}$$

$$a_5 = -\frac{2(4f_n - 8f_{\frac{1}{2}+n} + 4f_{1+n} + hf'_n - hf'_{1+n})}{5h^4}$$

Member of family of methods in (56) and predictor for $k=1$ are as follows;

$$y_{n+1} = h\left(\frac{7f_n}{30} + \frac{8}{15} f_{\frac{1}{2}+n} + \frac{7f_{1+n}}{30}\right) + y_n + h^2\left(\frac{f'_n}{60} - \frac{f'_{1+n}}{60}\right)$$

of order $p*_1 = 5$ and error constant $c_6 = \frac{1}{604800}$

With hybrid point

$$\frac{h}{2} + y_n = \frac{3hf_n}{16} + \frac{11y_n}{16} + \frac{5y_{1+n}}{16} - \frac{1}{32}h^2 f'_{1+n}$$

of order $p** = 3$

For k=2 we have the method and its predictor as

$$y_{n+2} = y_{n+1} + h\left(-\frac{81f_n}{560} + \frac{512}{945} f_{n+\frac{1}{2}} + \frac{8f_{n+1}}{35} + \frac{5659f_{n+2}}{15120}\right) + h^2\left(-\frac{43f'_n}{1680} + \frac{67f'_{n+1}}{210} - \frac{209f'_{n+2}}{5040}\right)$$ of

order 7 and error constant $c_8 = \frac{67}{4233600}$

With the predictor

$$\frac{h}{2} + y_n = \frac{39}{64}hf_{2+n} + \frac{27y_n}{128} + \frac{27y_{1+n}}{16} - \frac{115y_{2+n}}{128} - \frac{9}{64}h^2 f'_{2+n}$$

of order 4 and error constant $c_5 = \frac{9}{1280}$

Method for $k = 3$

$$y_{n+3} = y_{n+2} + h\left(-\frac{36613f_n}{272160} + \frac{4006f_{n+\frac{1}{2}}}{70875} - \frac{167f_{n+1}}{1440} + \frac{5947f_{n+2}}{18144} + \frac{2431231f_{n+3}}{6804000}\right) +$$
$$h^3\left(-\frac{403f'_n}{18144} + \frac{473f'_{n+1}}{1440} + \frac{10163f'_{n+2}}{30240} - \frac{16741f'_{n+3}}{453600}\right)$$

of order 9 and error constant

$$c_{10} = \frac{649}{228614400}$$

With hybrid predictor

$$\frac{h}{2} + y_n = -\frac{335}{384}hf_{3+n} + \frac{125y_n}{576} + \frac{375y_{1+n}}{256} - \frac{125y_{2+n}}{64} + \frac{2929y_{3+n}}{2304} + \frac{25}{128}h^2 f'_{3+n}$$

of order 5 and error constant $c_6 = -\frac{25}{3072}$

Method for $k = 4$

$$y_{n+4}=y_{n+3}+h\left(-\frac{1955783 f_n}{15966720}+\frac{4513792 f_{n+\frac{1}{2}}}{7640325}+\frac{1097 f_{n+2}}{47520}-\frac{909679 f_{n+3}}{4989600}+\frac{8048951 f_{n+4}}{23708160}\right)+$$
$$h^2\left(-\frac{1873 f'_n}{98560}+\frac{28067 f'_{n+1}}{66528}+\frac{344719 f'_{n+2}}{665280}+\frac{3941 f'_{n+3}}{9504}-\frac{609869 f'_{n+4}}{18627840}\right)$$

of order 11 and error constant $c_{12}=\frac{36343}{73766246400}$

With hybrid predictor

$$\frac{h}{2}+y_n=\frac{6965hf_{4+n}}{6144}+\frac{1715y_n}{8192}+\frac{1715y_{1+n}}{1152}-\frac{1715y_{2+n}}{1024}+\frac{343y_{3+n}}{128}-\frac{125555y_{4+n}}{73728}-\frac{245h^2f'_{4+n}}{1024}$$

of order 6 and error constant $c_7=\frac{49}{6144}$

Method for $k=5$

$$y_{n+5}=h(-\frac{8328829f_n}{67567500}+\frac{9048948736f_{\frac{1}{2}+n}}{13408770375}+\frac{172675f_{1+n}}{648648}+\frac{2036f_{2+n}}{4455}+\frac{71156f_{3+n}}{3378375}$$
$$+\frac{29962403f_{4+n}}{79459380}+\frac{3579881423f_{5+n}}{10945935000})+y_{4+n}$$
$$+h^2\left(-\frac{122671f'_n}{6756750}+\frac{1003f'_{1+n}}{1716}+\frac{445646f'_{2+n}}{405405}+\frac{804472f'_{3+n}}{675675}\right.$$
$$\left.+\frac{850151f'_{4+n}}{1891890}-\frac{3649783f'_{5+n}}{121621500}\right)$$

of order 13 and error constant $c_{14}=\frac{17317}{18261482250}$

With hybrid

$$\frac{h}{2}+y_n=-\frac{28413hf_{5+n}}{20480}+\frac{5103y_n}{25600}+\frac{25515y_{1+n}}{16384}-\frac{945y_{2+n}}{512}+\frac{5103y_{3+n}}{2048}-\frac{3645y_{4+n}}{1024}$$
$$+\frac{883477y_{5+n}}{409600}+\frac{567h^2f'_{5+n}}{2048}$$

of order 7 and error constant $c_8=\frac{-243}{32768}$

Zero-stability of the proposed methods: Given the first hybrid method as in equation (4)

$$y_{n+k} = y_{n+k-1} + h\left(\sum_{j=0}^{k} \beta_j f_{n+j} + \eta_{vm} f_{n+vm}\right) + h^2(\lambda_{vm-1} f'_{n+vm-1} + \lambda_k f'_{n+k})$$

With members for different k values are as follows:

For $\boldsymbol{k} = 2$

$$y_{n+2} = y_{n+1} + h\left(\frac{13 f_n}{8400} + \frac{37 f_{n+1}}{1260} + \frac{221 f_{n+2}}{240} + \frac{76 f_{n+\frac{5}{2}}}{1575}\right) + h^2\left(\frac{-2 f'_{n+\frac{3}{2}}}{7} - \frac{173 f'_{n+2}}{840}\right)$$

The first characteristic polynomial can be obtain by applying the shift operator to obtain

$$\boldsymbol{P}(\boldsymbol{r}) = \boldsymbol{r}^2 - \boldsymbol{r}$$

Solving to obtain

$$0 = \boldsymbol{r}^2 - \boldsymbol{r}$$
$$\Rightarrow \boldsymbol{r}(\boldsymbol{r} - 1) = 0$$
$$\boldsymbol{r} = 0 \; \boldsymbol{or} \; \boldsymbol{r} = 1$$

Hence, the method is Zero-stable since a root lie inside the unit disc and a unit root on the disc.

Given the hybrid method 2 for $\boldsymbol{k} = 2$

$$y_{n+2} = y_{n+1} + h\left(-\frac{81 f_n}{560} + \frac{512}{945} f_{n+\frac{1}{2}} + \frac{8 f_{n+1}}{35} + \frac{5659 f_{n+2}}{15120}\right) + h^2\left(-\frac{43 f'_n}{1680} + \frac{67 f'_{n+1}}{210} - \frac{209 f'_{n+2}}{5040}\right)$$

Taking the first characteristics polynomial

$$\boldsymbol{y}_{\boldsymbol{n}+2} - \boldsymbol{y}_{\boldsymbol{n}+1} = 0$$

$$\boldsymbol{r}^2 \boldsymbol{y}_{\boldsymbol{n}} - \boldsymbol{r}\boldsymbol{y}_{\boldsymbol{n}} = 0$$
$$\boldsymbol{r}^2 - \boldsymbol{r} = 0$$
$$\boldsymbol{r} = 0 \; , \boldsymbol{r} = 1$$

The hybrid method 2 is also zero-stable.

Stability structure and error constant of the proposed hybrid method: In this section, we shall investigate the stability properties of the hybrid linear multistep methods 1 and 2 for fixed value k. The resulting schemes are applied on the scalar test problem $\boldsymbol{y}' = \lambda \boldsymbol{y}$ to obtain the stability polynomials. From (4) and (7), we can deduce the stability of the proposed hybrid methods as follows:

$$\boldsymbol{R}(\boldsymbol{z}) = \boldsymbol{r}^{\boldsymbol{k}} - \boldsymbol{r}^{\boldsymbol{k}-1} - \boldsymbol{z}\left(\sum_{\boldsymbol{j}=0}^{\boldsymbol{k}} \beta_{\boldsymbol{j}} \boldsymbol{r}^{\boldsymbol{j}} + \boldsymbol{z}\lambda_{\boldsymbol{k}} \boldsymbol{r}^{\boldsymbol{k}}\right) - \boldsymbol{z}\eta_{\boldsymbol{vm}} \boldsymbol{R}(\boldsymbol{zvm}) - \boldsymbol{z}^2 \lambda_{\boldsymbol{vm}-1} \boldsymbol{R}(\boldsymbol{zvn}-1) \tag{21}$$

$$R(zvn-1) = \left(\sum_{j=0}^{k} \alpha_j r^j + z^2 \lambda'_k r^k\right)$$

$$\boldsymbol{R}(\boldsymbol{zvm}) = \sum_{\boldsymbol{j}=0}^{\boldsymbol{k}} \alpha_{\boldsymbol{j}} \boldsymbol{r}^{\boldsymbol{j}} + \boldsymbol{z}\beta_{\boldsymbol{k}} \boldsymbol{r}^{\boldsymbol{k}} + \boldsymbol{z}^2 \boldsymbol{r}^{\boldsymbol{k}} \lambda''_{\boldsymbol{k}}$$

From the stability polynomials we now investigate the $\boldsymbol{A}-$stability and $\boldsymbol{A}(\alpha)-$stability of the proposed methods.

Stability structure of the proposed hybrid method 1

Adopting the boundary locus techniques, the stability plots of method 1 are shown below.

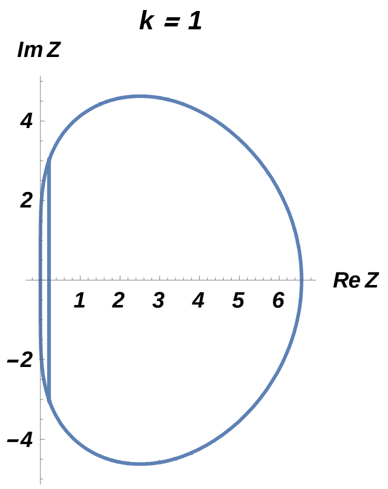

FIG. 1: Parametric plot for $k=1$ **(method 1)**

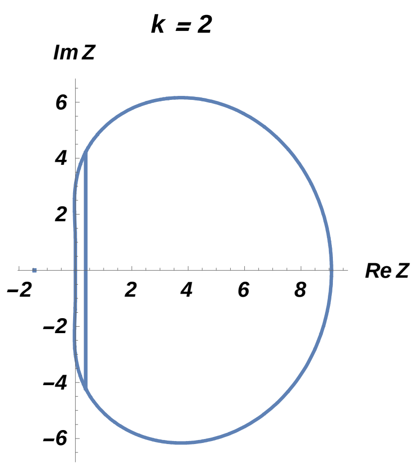

FIG. 2: Parametric plot for $k=2$ **(method 1)**

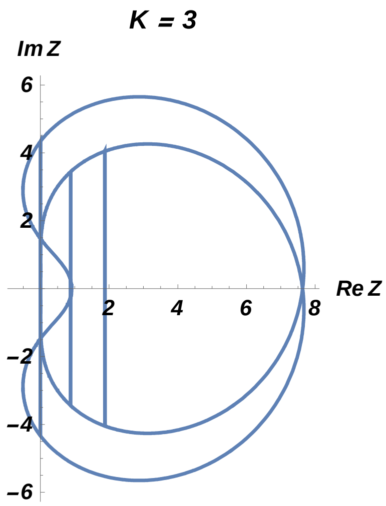

FIG. 3: Parametric plot for $k = 3$ (method 1)

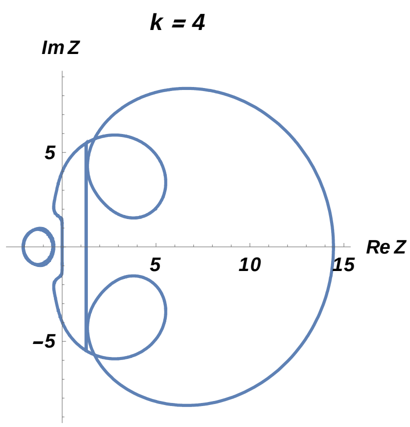

FIG. 4: Parametric plot for $k = 4$ (method 1)

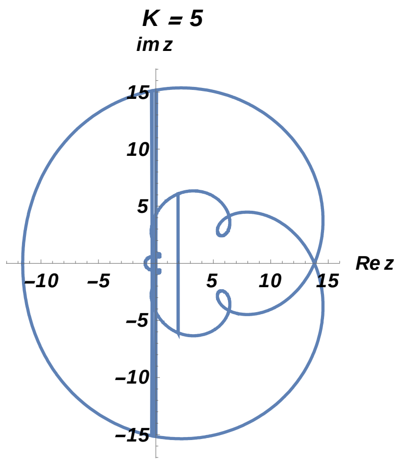

FIG. 5: Parametric plot for $k = 5$ (method 1)

From the plots above, it is seen that the method1 is A-stable for step number k=1 to 2 and $A(\alpha)-$ stable for step number 3 and 4. The method becomes unstable for step-number 5 and above.

Stability structure of the proposed hybrid method 2: From equation (56) we derived the general stability polynomial as

$$R_2(z) = r^k - r^{k-1} - z\sum_{j=0}^{k} c_j r^j - z_{n+\frac{1}{2}} R\left(\tfrac{I}{2}z\right) + z^2 \sum_{j=0}^{k} \sigma_j r^j \tag{22}$$

where

$$R\left(\tfrac{I}{2}z\right) = \sum_{j=0}^{k} \eta_j r^j + z\lambda_k r^k + z^2 \gamma_k r^k$$

From the stability polynomials we now investigate the $A-$stability and $A(\alpha)-$stability of the proposed hybrid method 2.

Adopting the boundary locus techniques, the stability plots of method 2 are shown below.

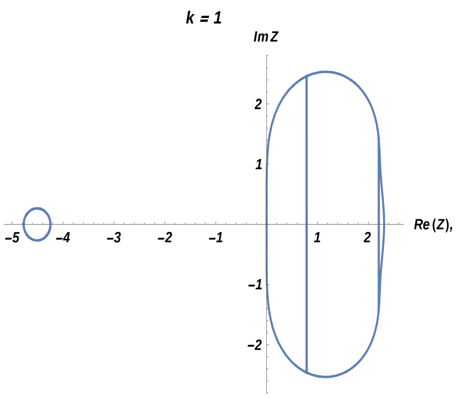

FIG. 6: Parametric plot for $k=1$ **(method 2)**

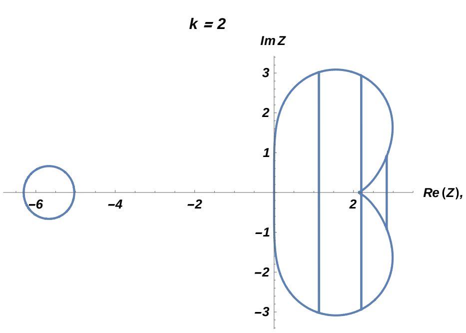

FIG. 7: Parametric plot for $k=2$ **(method 2)**

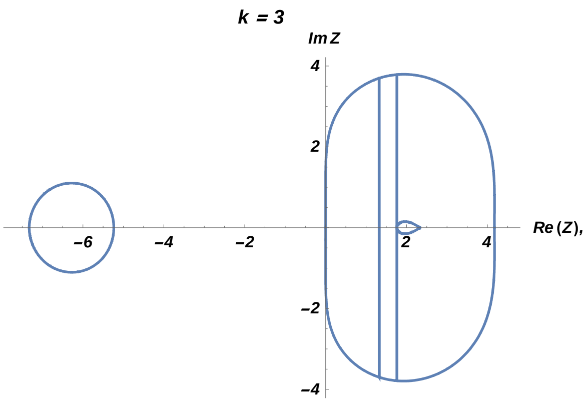

FIG. 8: Parametric plot for $k = 3$ **(method 2)**

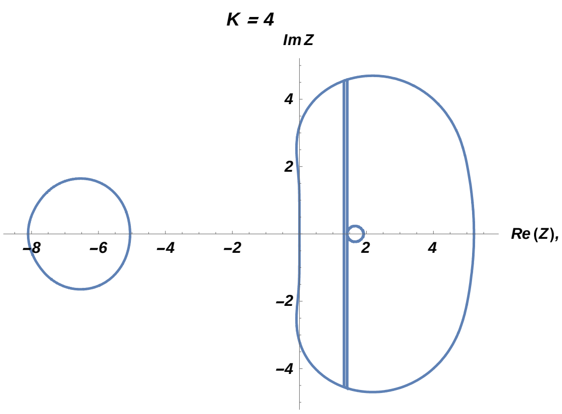

FIG. 9: Parametric plot for $k = 4$ **(method 2)**

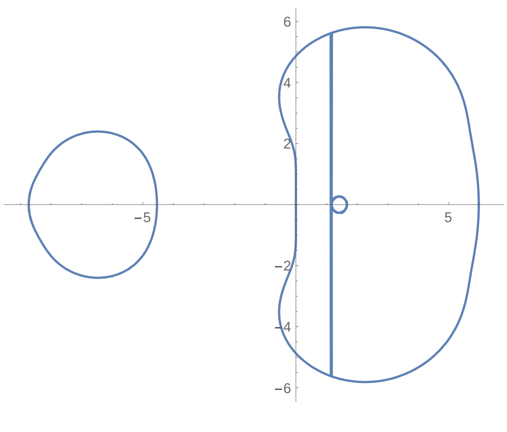

FIG. 10: Parametric plot for $k = 5$ **(method 2)**

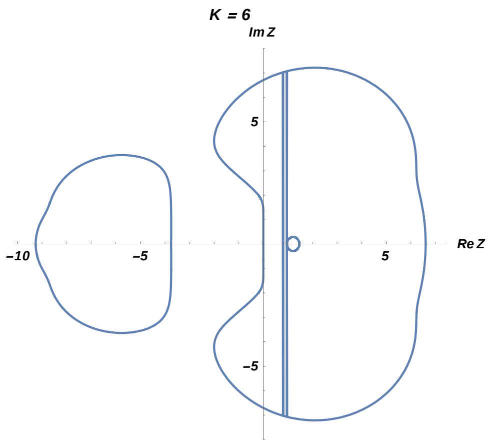

FIG. 11: Parametric plot for $k = 6$ (method 2)

From the plots above, it is seen that the method is A-stable for step number k=1 to 3 and $A(\alpha)-$ stable for step number 4 to 6. The method becomes unstable for step-number 7 and above. These methods are strong enough to be used for stiff differential equations.

4.3.3 Error constants of the proposed hybrid methods

Consider the operator,

$$L\left(y\left(x_n,h\right)\right)=y\left(x_n+kh\right)-y\left(x_n+\left(k-1\right)h\right)-$$
$$h\sum_{j=0}^{k}\varsigma_j y'\left(x_n+jh\right)-h\lambda_{n+\frac{1}{2}}y'\left(x_n+\frac{1}{2}h\right)-h^2\sum_{j=0}^{k}\sigma_j y''\left(x_n+kh\right)$$

We expand term by term in Taylor series about x_n and obtain

$$y\left(x_n\right)+khy'\left(x_n\right)+\left(kh\right)^2\frac{y''\left(x_n\right)}{2!}+\ldots+\frac{\left(kh\right)^p y^p\left(x_n\right)}{p!}+\ldots$$

$$-\left(y\left(x_n\right)+\left(k-1\right)hy'\left(x_n\right)+\frac{\left(k-1\right)^2h^2y''\left(x_n\right)}{2!}+\ldots+\frac{\left(k-1\right)^p h^p y^{(p)}\left(\left(x_n\right)\right)}{p!}\ldots\right)$$

$$-\left(h\sum_{j=0}^{k}\varsigma_j y'\left(x_n\right)+jhy''\left(x_n\right)+\frac{\left(jh\right)^2y'''\left(x_n\right)}{2!}+\ldots+\frac{\left(jh\right)^p y^{p+1}\left(x_n\right)}{p!}+\ldots\right)$$

$$-h\lambda_{n+\frac{1}{2}}\left(y'\left(x_n\right)+\tfrac{1}{2}hy''\left(x_n\right)+\frac{\frac{1}{4}h^2y'''\left(x_n\right)}{2!}+\ldots+\frac{\left(\frac{1}{2}h\right)^p y^{(p+1)}\left(x_n\right)}{p!}\right)+\ldots$$

$$-h^2 \sum_{j=0}^{k} \sigma_j \left(y''(x_n) + khy'''(x_n) + (kh)^2 \frac{y^{iv}(x_n)}{2!} + \ldots + \frac{(kh)^p \, y^{(p+2)}(x_n)}{p!} + \ldots \right)$$

Recall that

$$L\{y(x_n);h\} = C_0 y(x_n) + hC_1 y'(x_n) + h^2 C_2 y''(x_n) + \ldots + h^p C_p y^p(x_n) + O\left(h^{p+1}\right)$$

Collecting terms in power of h we obtain

$$C_0 = 0$$

$$C_1 = 1 - \sum_{j=0}^{k} \varsigma_j - \lambda_{n+\frac{1}{2}}$$

$$C_2 = \frac{K^2}{2!} - \frac{(K-1)^2}{2!} - \left(\sum_{J=0}^{K} \varsigma_j j + \sigma_j k \right) - \frac{1}{2}$$

.

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.

$$C_p = \frac{k^p}{p!} - \frac{(k-1)^p}{p!} - \frac{\left(\sum_{j=0}^{k} \varsigma_j j^{p-1} + \sigma_j k^{p-1} \right)}{(p-1)!}$$

$$C_{p+1} = \frac{k^{p+1}}{p+1!} - \frac{(k-1)^{p+1}}{p+1!} - \frac{\left(\sum_{j=0}^{k} \varsigma_j j^p + \sigma_j k^p \right)}{(p)!}$$

In similar manner, the error constant of the hybrid method 2 is obtain as

$$c_{p+1} = \frac{\left(\frac{1}{2}\right)^{p+1}}{p+1!} - \frac{\sum_{j=0}^{k} \eta_j j^{p+1}}{p+1!} - \frac{\lambda_k k^p}{p!} - \frac{k^{p-1}}{p-1!}$$

3. IMPLEMENTATION OF NUMERICAL SCHEME

The numerical implementation is carried out in fixed and variable step-sizes. The Newton-Ralphson iterative method is adopted to resolve implicitness of the proposed method during the implementation.

We will consider three test problems for implementation with the proposed hybrid methods in this section.

Problem 1

$$y'_1(x) = -\left(2+e^{-1}\right)y_1(x) + e^{-1}y_2^2(x), \quad y_1(0)=1$$
$$y'_2(x) = y_1(x) - y_2(x) - y_2^2(x), \qquad y_2(0)=1$$

$$x \in [0,1] \text{ , } h = 0.0001$$

The exact solution is given as $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-x}$. This is a singular perturbed problem suggested in Rosenbrock (1981). These problems become stiff as $e \to 0$. Applying the proposed hybrid method in (22) to the IVP above using Newton iterative scheme:

$$Y_{n+k}^{(\mu+1)} = y_{n+k}^{(\mu)} - J\left(y_{n+k}^{(\mu)}\right)^{-1} F\left(y_{n+k}^{(\mu)}\right)$$

In order to resolve the implicitness, where

$$F\left(y_{n+k}^{(\mu)}\right) = y_{n+k}^{(\mu)} - \alpha_{k-1}y_{n+k-1} - h\sum_{j=0}^{k}\beta_j f\left(x_{n+j}, y_{n+j}\right) - h^2\lambda_{vm-1} f\left(x_{n+vm-1}, y_{n+vm-1}^{(\mu)}\right) - h^2\lambda_k f'\left(x_{n+k}, y_{n+k}^{(\mu)}\right)$$ with

hybrid predictors

$$y_{n+vm-1}^{(\mu)} = \sum_{j=0}^{k}\alpha_j y_{n+j} + h^2\lambda'_k\ f'\left(x_{n+k}, y_{n+k}^{(\mu)}\right) \quad \textit{and}$$
$$y_{n+vm}^{(\mu)} = \sum_{j=0}^{k}\alpha'_j\ y_{n+j} + \beta_k hf\left(x_{n+k}.\, y_{n+k}^{(\mu)}\right) + h^2\lambda''_k\ f'\left(x_{n+k}, y_{n+k}^{(\mu)}\right).$$

The Jacobian matrix is given as

$$J\left(y_{n+k}^{(\mu)}\right) = \left(\delta F\left(y_{n+k}^{(\mu)}\right)\right) / \left(\delta y_{n+k}^{(\mu)}\right).$$

We used the explicit Trapezoidal rule to generate the starting method. Result is tabulated below;

TABLE 5: Result of problem 1

	Proposed method 1	**Ode15s**
e	$\|e\|_2$	$\|e\|_2$
10^{-1}	4.3×10^{-5}	1.1×10^{-1}
10^{-2}	4.3×10^{-5}	1.0×10^{-2}
10^{-3}	4.4×10^{-5}	2.2×10^{-3}
10^{-4}	4.4×10^{-5}	2.5×10^{-3}

Problem 2

A stiff system of equation

$$y'_1(x) = -8y_1(x) + 7y_2(x)$$
$$y'_2(x) = 42y_1(x) - 43y_2(x) \quad y(0) = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad x \in [0,10]$$

$$y_1(x) = 2e^{-x} - e^{-50x}$$
$$y_2(x) = 2e^{-x} + 6e^{-50x}$$

Implementing the problem with our proposed method for k=1in variable step-size techniques, we estimate the local error $\left\| y_{n+1}^{(e)} - y_{n+1} \right\|$ at each step and we controlled it by taking a new step-size as

$$h_{new} = \left(\frac{TOL}{\left\| y_{n+1}^{(e)} - y_{n+1} \right\|} \right)^{\frac{1}{p+1}} \times h_{old} \tag{21}$$

where

$$y_{n+1}^{(e)} = y_n + h\left(\frac{2f_n}{27} + \frac{13f_{n+1}}{15} + \frac{8f_{n+\frac{3}{2}}}{135} \right) + h^2 \left(\frac{-11f'_{n+\frac{1}{2}}}{45} - \frac{19f'_{n+1}}{90} \right) \quad , p = 5$$

With predictors

$$y_{n+\frac{3}{2}} = \frac{3hf_{n+1}}{8} - \frac{y_n}{8} + \frac{9y_{n+1}}{8} + \frac{3h^2 f'_{n+1}}{16}, \qquad p = 3$$

$$y_{n+\frac{1}{2}} = \frac{y_n}{2} + \frac{y_{n+1}}{2} - \frac{h^2 f'_{n+1}}{8}, \qquad p = 2$$

If the local estimate is less than the tolerance i.e $\left\| y_{n+1}^{(e)} - y_{n+1} \right\| \le TOL$, we accept the current step. We can now use (**) to predict the next step-size. But if $\left\| y_{n+1}^{(e)} - y_{n+1} \right\| > TOL$, reject the step. The result of the above experiment is tabulated below:

TABLE 6: Result of problem 2

Method	TOL	FC	FS	TS
SDMM	10^{-2}	96	12	73
Proposed method 1	10^{-2}	73	4	38
SDMM	10^{-4}	284	14	132
Proposed method 1	10^{-4}	200	4	58
SDMM	10^{-6}	357	6	233
Proposed method 1	10^{-6}	270	4	140

Problem 3

Consider the stiff system:

$$y'_1 = -y_1 - 30y_2 + 30e^{-x},$$
$$y'_2 = 30y_1 - y_2 - 30e^{-x},$$

With initial value $y(0) = (1,1)^T$ and exact solution $y_1(x) = y_2(x) = e^{-x}$. Using step-size $h = 0.002$, the numerical result is compared with the Hybrid Extended Backward Differentiation Formulas (HEBDF) for Stiff Systems proposed by Ezzzeddine & Hojjatti (2011).the numerical solution is in the table below.

TABLE 7: Result of problem 3

x	y_i	Error in HEBDF	Error in method 2
0.04	y_1	4E-20	4E-21
	y_2	1.81E-18	1.33E-20
0.2	y_1	2.5E-19	2.33E-21
	y_2	6.2E-19	7.1E-22
2.0	y_1	2E-20	2.2E-21
	y_2	3.8E-19	3.5E-20

Problem 4.

Consider the nonlinear system

$$y'_1 = -1002y_1 - 1000y_2^2$$
$$y'_2 = y_1 - y_2(1 + y_2)$$

With initial value $y(0) = (1,1)^T$, the theoretical solution is $y_1(x) = e^{-2x}, y_2(x) = e^{-x}$ in the interval $[0,5]$. We integrate the system by our proposed methods with step-size $h = 0.005$ and obtain the following results:

TABLE 8: Result of problem 4

x	y_i	**Error in HEBDF**	**Error in our proposed method 2**
0.4	y_1	5.46E-17	5.22E-17
	y_2	4.05E-17	3.99E-17
5.0	y_1	7.08E-20	6.80E-20
	y_2	5.25E-18	4.88E-19

TABLE 9: Result of problem 4 in the interval [0, 30] with a step-size h=0.01

x	y_i	**Exact Solution**	**Error in HEBDF**	**Error in method 2**
10.0	y_1	2.0611536224385578280E-9	3.88E-25	3.66E-25
	y_2	4.5399929724848515 36E-5	4.28E-21	4.05E-22
20.0	y_1	4.2483542552915889953E-18	1.50E-33	2.00E-34
	y_2	2.0611536224385578280E-27	3.65E-25	3.55E-25
30.0	y_1	8.7565107626965203385E-27	6.78E-32	6.00E-32
	y_2	9.3576229688401746049E-14	2.46E-29	2.33E-29

4. CONCLUSION

This study is an extension and modification of linear multistep methods. The modification is done by introducing hybrid points and adding second derivative points to the conventional linear multistep methods. Two methods are derived each with hybrid collocation points. We employed the interpolation and collocation approach in the derivation using Mathematica software. The implementation is carried out in fixed and variable step-size. Mathematica and MATLAB programming software are used in the derivations and implementation of the two new classes of hybrid linear multistep methods with predictors. These methods have small error constants and are zero-stable and *A*-stable at higher orders which are important properties for numerical integrations.

The experimental results in table 5-9 show that the methods are competitive with other existing methods in literature.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest

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