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CERTAIN RESULTS IN b-METRIC SPACE USING SUBCOMPATIBLE, FAINTLY COMPATIBLE MAPPINGS

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Abstract: In this paper we prove a common fixed point theorem in b-metric space using faintly compatible, subcompatible, occasionally weakly compatible and reciprocally continuous mappings. Furthermore appropriate examples are provided to support our conclusions.

Keywords: b-metric space; subcompatible; occasionally weakly compatible; faintly compatible and reciprocally continuous mappings.

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1. INTRODUCTION

The fixed point theory plays very important role in analysis and it has got wide applications in many fields of mathematics. Many fixed point theorems have been developed on various areas in

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the current context. Introducing the concept of compatible mappings, Junck [1] generated many results in metric space. Later on many theorems were proved using weaker form compatible mappings namely weakly compatible mappings and occasionally weakly compatible mappings. In [2],[3],[4] and [5] many results can be witnessed using the above mentioned conditions. In the recent past b-metric space emerged as one of the generalizations of metric space. Czerwik [6] introduced the concept of b-metric space. Thereafter some more fixed point theorems were extracted in b-metric space, such as [7], [8] and [9] using a variety discovered a variety of constraints. J.R. Roshan et al. [10] used compatible and continuous mappings to prove a common unique fixed point theorem in b-metric space. We improve their result in this study by utilizing certain weaker conditions, such as faintly compatible mappings, occasionally weakly compatible, subcompatible and reciprocally continuous mappings.

2. PRELIMINARIES

2.1 Definition X is a nonempty set and $m \geq 1$then the mapping d from $X \times X$ to R^{+} is said to be a b-metric space if and only if for each $\phi, \varphi, \gamma \in X$

(i) $d(\phi, \varphi) = 0 \Leftrightarrow \phi = \varphi$

(ii) $d(\phi, \varphi) = d(\varphi, \phi)$

(iii) $d(\phi, \gamma) \leq m[d(\phi, \varphi) + d(\varphi, \gamma)]$

Definitions 2.2 A pair (T,S) of a b-metric space is said to be

(i) Compatible if $d(TS\eta_k, ST\eta_k) = 0$ whenever $\{\eta_k\}$sequence in X such that$T\eta_k = S\eta_k = \gamma$ for some $\gamma \in X$ as $k \to \infty$.

(ii) Weakly compatible mappings if$T\eta = S\eta$ for some $\eta \in X$ such that $TS\eta = ST\eta$ holds.

(iii) Occasionally weakly compatible if and only if there exists some η in X such that$T\eta = S\eta$ implies $TS\eta = TS\eta$.

(iv) Reciprocally continuous if and only if $\lim_{n\to\infty} TSu_n = T\eta$ and $\lim_{n\to\infty} STu_n = S\eta$ whenever (u_n) in X such that $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_n = \eta \in X$.

(v) Subcompatible if there exists a sequence $\{u_n\}$ in X such that $Tu_n = Su_n = \eta \in X$ as $n \to \infty$ and satisfy $d(TSu_n, STu_n) = 0$ as $k \to \infty$.

(vi) Conditionally compatible if and only if whenever the set of sequence $\{u_n\}$ satisfying $\lim_{n\to\infty} T(u_n) = \lim_{n\to\infty} S(u_n)$ is nonempty, there exists a sequence $\{v_n\}$ such that $\lim_{n\leftarrow\infty} T(v_n) = \lim_{n\to\infty} S(v_n) = w$ and $\lim_{n\to\infty} d(TSv_n, STv_n) = 0$.

(vii) Faintly compatible iff T and S are conditionally compatible and T and S commute on a nonempty subset of coincidence points whenever the set of coincidence is nonempty.

We now discuss some examples to find the relation among the above definitions.

Example 2.3: Suppose $X = [0,5]$ is a b-metric space with $d(u,v) = |u-v|^2$.

Define theself maps $T(u) = u^3,\ S(u) = \frac{u}{9} \quad \forall u \in [0,5]$.

Here $0, \frac{1}{3}$ are coincidence points for T,S

i.e $u = \frac{1}{3}, T\left(\frac{1}{3}\right) = S\left(\frac{1}{3}\right)$ but not $TS\left(\frac{1}{3}\right) \neq ST\left(\frac{1}{3}\right)$

and $u = 0, T(0) = S(0)$ but not $TS(0) = ST(0)$.

Therefore the maps T,S are occasionally weakly compatible but are not weakly compatible.

Example 2.4: Let $X = [0,9]$ is a b-metric space with $d(u,v) = |u-v|^2$.

Define $T(u) = \begin{cases} 2 - \frac{u}{4} & if\ 0 \le u < 2 \\ u & if\ 2 \le u \le 9 \end{cases}$ and $S(u) = \begin{cases} 2 + 4u & if\ 0 \le u \le 2 \\ 2 & if\ 2 < u \le 9 \end{cases}$

Let $u_n = \frac{1}{9n}$ for $n \ge 1$.

Then $Tu_n = 2 - \frac{1}{36n} \to 2,\ Su_n = 2 + \frac{4}{9n} \to 2\ \ as\ \ n \to \infty.$

Also $TSu_n = TS\left(\frac{1}{9n}\right) = T\left(2 + \frac{4}{9n}\right) = 2 + \frac{4}{9n} \to 2\ \ as\ n \to \infty.$

Now $STu_n = ST\left(\frac{1}{9n}\right) = S\left(2 - \frac{1}{36n}\right) = 2 - \frac{1}{4} + \frac{1}{144n} \to \frac{7}{4}\ \ as\ n \to \infty$

so that the pair (T,S) is not compatible.

If $u_n = 2 + \frac{1}{5n}$ for $n \geq 1$

then $Tu_n = T\left(2 + \frac{1}{5n}\right) = 2 + \frac{1}{5n} \to 2\ \ and\ \ Su_n = S\left(2 + \frac{1}{5n}\right) = 2\ \ as\ \ n \to \infty.$

Now $TSu_n = TS\left(2 + \frac{1}{5n}\right) = T(2) = 2\ \ and\ \ STu_n = ST\left(2 + \frac{1}{5n}\right) = S\left(2 + \frac{1}{5n}\right) = 2\ as\ n \to \infty.$

Therefore the pair (T,S) is sub compatible.

The following theorem was proved in [10].

2.5 Theorem Let the self maps f,g,S and T be defined on a b-metric space (X,d) which is complete with the given conditions:

(b1) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$

(b2) $d(fu, gv) \leq \frac{q}{k^4} \max\left\{d(Su,Tv), d(fu,Su), d(g\beta,T\beta), \frac{1}{2}\left(d(Su,gv) + d(fu,Tv)\right)\right\}$

holds for every $u,v\ in\ X$ with $0 < q < 1$.

(b3) The self mappings T and S are both continuous

(b4) two pairs (f,S) and (g,T) are compatible.

Then the above four maps having a single fixed point which is common.

Now we improve Theorem (2.5) under some modified conditions.

To do so, we need to recall the following lemma.

2.6 Lemma[10]: Let (X,d) be a $b-$metric space with $k \geq 1$ and two sequences $\{\alpha_n\}$ and

$\{\beta_n\}$ are b-convergent to α and β respectively. Then we have

$$\frac{1}{k^2} d(\alpha,\beta) \le \lim_{n\to\infty} \inf\, d(\alpha_n,\beta_n) \le \lim_{n\to\infty} \sup d(\alpha_n,\beta_n) \le k^2 d(\alpha,\beta).$$

3. MAIN RESULTS

3.1. Theorem:

Let (X,d) be b-metric space which is complete and four self mappings f,g,S, and T are satisfying

(b1) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$

(b2) $$d(fu,gv) \le \frac{q}{k^4} \max\left\{ d(Su,Tv), d(fu,Su), d(gv,Tv), \frac{1}{2}\big(d(Su,gv)+d(fu,Tv)\big)\right\},$$

holds for every u,v in X with $0<q<1$.

(b3) the pair (g,T) is occasionally weakly compatible

(b4) the pair (f,S) is sub compatible and reciprocally continuous.

Then the four maps f, g, S, and T, share a unique common fixed point.

Proof:

From (b1) we can construct a sequence $\{v_j\}$ for $n \ge 1$

$v_{2j} = fu_{2j} = Tu_{2j+1}$ and $v_{2j+1} = gu_{2j+1} = Su_{2j+2}$ for all $j \ge 0$.

Now we show that $\{v_j\}$ is a cauchy sequence.

Consider $d(v_{2j}, v_{2j+1}) = d(fu_{2j}, gu_{2j+1})$

$$\le \frac{q}{k^4} \max\{d(Su_{2j},Tu_{2j+1}), d(fu_{2j},Su_{2j}), d(gu_{2j+1},Tu_{2j+1}), \frac{1}{2}(d(Su_{2j},gu_{2j+1}) + d(fu_{2j},Tu_{2j+1}))\}$$

$$= \frac{q}{k^4} \max\{d(v_{2j-1},v_{2j}), d(v_{2j},v_{2j-1}), d(v_{2j+1},v_{2j}), \frac{1}{2}(d(v_{2j-1},v_{2j+1}) + d(v_{2j},v_{2j}))\}.$$

$$= \frac{q}{k^4} \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1}), \frac{d(v_{2j-1}, v_{2j+1})}{2}\}$$

$$\leq \frac{q}{k^4} \max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1}), \frac{k}{2}(d(v_{2j-1}, v_{2j}) + d(v_{2j}, v_{2j+1}))\}.$$

If $d(v_{2j}, v_{2j+1}) > d(v_{2j-1}, v_{2j})$ for some j, then the above inequality gives

$$d(v_{2j}, v_{2j+1}) \leq \frac{q}{k^3} d(v_{2j}, v_{2j+1})$$

a contradiction.

Hence $d(v_{2j}, v_{2j+1}) \leq d(v_{2j-1}, v_{2j}) \quad \forall \ j \in N$.

Therefore the above gives

$$d(v_{2j}, v_{2j+1}) \leq \frac{q}{k^3} d(v_{2j-1}, v_{2j}).\text{-------------------}(1)$$

Similarly

$$d(v_{2j-1}, v_{2j}) \leq \frac{q}{k^3} d(v_{2j-2}, v_{2j-1}).\text{------------}(2)$$

Using (1) and (2) we get

$d(v_j, v_{j-1}) \leq \omega d(v_{j-1}, v_{j-2})$, where $\omega = \frac{q}{k^3} < 1$ and $j \geq 2$.

we obtain $\forall$ for all $j \geq 2$,

$$d(v_j, v_{j-1}) \leq \omega d(v_{j-1}, v_{j-2}) \leq \dots\dots\dots \leq \omega^{j-1} d(v_1, v_0).\text{----------}(3)$$

So for all $j > l$, we have

$$d(v_j, v_l) \leq kd(v_l, v_{l+1}) + k^2 d(v_{l+1}, v_{l+2}) + \dots\dots\dots + k^{j-l-1} d(v_{j-1}, v_j).$$

From equation (3),we get

$$d(v_j, v_l) \leq (k\omega^l + k^2\omega^{l+1} + \dots\dots\dots\dots + k^{j-l-1}\omega^{j-1}) d(v_1, v_0)$$

$$\leq k\omega^l (1 + k\omega + k^2\omega^2 + \dots\dots\dots\dots) d(v_1, v_0)$$

$$\leq \frac{k\omega^l}{1 - k\omega} d(v_1, v_0).$$

Take $l, j \to \infty$,we get $d(v_j, v_l) \to 0$ as $k\omega < 1$.

Hence the sequence $\{v_j\}$ is a cauchy sequence in X and since X is complete, $\{v_j\}$ converges to λ in X such that $\lim_{j\to\infty} fu_{2j} = \lim_{j\to\infty} Tu_{2j+1} = \lim_{j\to\infty} gu_{2j+1} = \lim_{j\to\infty} Su_{2j+2} = \lambda$.

Again since the pair (f,S) sub compatible therefore there exists a sequence $\{u_j\}$ in X such that $\lim_{j\to\infty} Ku_j = \lambda = \lim_{j\to\infty} Mu_j$ for u in X satisfying $\lim_{j\to\infty} d(fSu_j, Sfu_j) = 0$.

By (C4) we have the pair (f,S) reciprocally continuous then $\lim_{j\to\infty} d(fSu_j, f\lambda) = 0$ and $\lim_{j\to\infty} d(Sfu_j, S\lambda) = 0$.

This implies $d(f\lambda, S\lambda) = 0.$

This gives $f\lambda = S\lambda.$

We claim $f\lambda = \lambda.$

Putting $u = \lambda$ and $v = u_{2j+1}$ in (b2), we get

$$d(f\lambda, gu_{2j+1}) \le \frac{q}{k^4} \max\left\{ d(S\lambda, Tu_{2j+1}), d(f\lambda, S\lambda), d(gu_{2j+1}, Tu_{2j+1}), \frac{1}{2}\left(d(S\eta, gu_{2j+1}) + d(f\lambda, Tu_{2j+1})\right) \right\}$$

take the sup limit as $j \to \infty$ on both the sides and by Lemma(2.6),which gives

$$d(f\lambda, \lambda) \le \lim_{j\to\infty} Sup\, d(f\lambda, gu_{2j+1})$$

$$\frac{q}{k^4} \max\left\{ \begin{array}{l} \lim_{j\to\infty} Sup\, d(S\lambda, Tu_{2j+1}), \lim_{j\to\infty} Sup\, d(f\lambda, S\lambda), \lim_{j\to\infty} Sup\, d(gu_{2j+1}, Tu_{2j+1}), \\ \frac{1}{2}\left(\lim_{j\to\infty} Sup\, d(S\lambda, gu_{2j+1}) + \lim_{j\to\infty} Sup\, d(f\lambda, Tu_{2j+1}) \right) \end{array} \right\}$$

$$\le \frac{q}{k^4} \max\left\{ k^2 d(f\lambda, \lambda), 0, 0, \frac{k^2}{2}\left(d(f\lambda, \lambda) + d(f\lambda, \lambda)\right) \right\}$$

$$= \frac{q}{k^4} k^2 d(f\lambda, \lambda)$$

$$= \frac{q}{k^2} d(f\lambda, \lambda)$$

$$\frac{d(f\lambda, \lambda)}{k^2} \le \frac{q}{k^2} d(f\lambda, \lambda)$$

as $0 < q < 1$,so $f\lambda = \lambda.$

Therefore $f\lambda = S\lambda = \lambda.$---------(4)

Now $v = f\lambda \in f(X) \subseteq T(X).$

By (b1) $f\lambda = Tw$ for some $w \in X$.---------------(5)

Now we claim $\lambda = gw.$

Putting $u = \lambda$, $v = w$ in (b2) we obtain

$$d(f\lambda, gw) \leq \frac{q}{k^4} \max\left\{d(S\lambda, Tw)\beta, d(f\lambda, Sv\lambda), d(gu, Tw), \frac{1}{2}(d(S\lambda, gw) + d(f\lambda, Tw))\right\}$$

By using $\lambda = f\lambda = S\lambda = Tw$

$$d(\lambda, gw) \leq \frac{q}{k^4} \max\left\{d(\lambda, \lambda)\beta, d(\lambda, \lambda), d(gw, \lambda), \frac{1}{2}(d(\lambda, gw) + d(\lambda, \lambda))\right\}$$

take the sup limit as $j \to \infty$ on both the sides and by Lemma (2.6), which gives

$$\frac{d(\lambda, gw)}{k^2} \leq \lim_{j\to\infty} Sup\, d(\lambda, gw) \leq \frac{q}{k^4} \max\left\{0,0, d(gw, \lambda), \frac{k^2}{2}(d(\lambda, gw) + 0)\right\}$$

$$\frac{d(\lambda, gw)}{k^2} \leq \frac{q}{k^4} \max\{0,0, k^2 d(gw, \lambda), k^2 d(\lambda, gw)\}$$

$$\frac{d(\lambda, gw)}{k^2} \leq \frac{q}{k^4} k^2 d(gw, \lambda)$$

$$d(\lambda, gw) \leq q d(gw, \lambda)$$

since $0 < q < 1$,so $gw = \lambda.$

This gives $gw = Tw = \lambda.$--------------(6)

By (b3) we have the pair (g, T) is occasionally weakly compatible then there is $w \in X$ with $gw = Tw$ implies $gTw = Tgw$ this gives $g\lambda = T\lambda$.

Now again show that $g\lambda = \lambda$.

Putting $u = \lambda$, $v = \lambda$ in (b2) we get

$$d(f\lambda, g\lambda) \le \frac{q}{k^4} \max\left\{ d(S\lambda, T\lambda)\beta, d(f\lambda, S\lambda), d(g\lambda, T\lambda), \frac{1}{2}(d(S\lambda g\lambda) + d(f\lambda, T\lambda)) \right\}$$

take the sup limit as $j \to \infty$ on both the sides and by Lemma(2.6),which gives

$$\frac{d(\lambda, g\lambda)}{k^2} \le \lim_{j\to\infty} Sup d(\lambda, g\lambda) \le \frac{q}{k^4} \max\left\{ k^2 d(\lambda, g\lambda), 0, 0, \frac{k^2}{2}(d(\lambda, g\lambda) + d(\mu, g\lambda)) \right\}$$

$$\frac{d(\lambda, g\lambda)}{k^2} \le \frac{q}{k^4} \max\{ k^2 d(\lambda, g\lambda), k^2 d(\lambda, g\lambda) \}$$

$$\frac{d(\lambda, g\lambda)}{k^2} \le \frac{q}{k^2} d(\lambda, g\lambda)$$

As $0 < q < 1$,so $g\lambda = \lambda$.

$\Rightarrow g\lambda = T\lambda = \lambda$.--------------(7)

We get from above (4) and (7).

$f\lambda = S\lambda = g\lambda = T\lambda = \lambda.$

Therefore λ is the required common fixed point.

Uniqueness can be easily obtained.

Now we discuss a suitable example to support our Theorem.

3.2. Example:

Assume that $X = [0, 4]$ is a $b-$metric space with $d(u,v) = |u - v|^2$ where $u, v \in X$.

Introduce the self maps as

$$T(\eta) = S(\eta) = \begin{cases} \frac{1}{3} + 2\eta, & if \quad \eta \in \left[0, \frac{1}{3}\right) \\ \eta, & if \quad \eta \in \left(\frac{1}{3}, 4\right] \end{cases} \quad ;$$

and

$$f(\eta)=g(\eta)=\begin{cases}\dfrac{1+\eta}{3}, & if \quad \eta\in\left[0,\dfrac{1}{3}\right)\\ 2, & if \quad \eta\in\left[\dfrac{1}{3},4\right]\end{cases}.$$

Then

$f(X)=g(X)=[0.33,0.55)$ and $S(X)=T((X)=[0.33,4]$

so that the condition (b1) is satisfied.

Clearly $\eta=0$ and $\eta=2$ are the coincidence points for the maps g ,T.

At $\eta=0$, $g(0)=T(0)$ but $gT(0)=g\left(\frac{1}{3}\right)=2\neq\frac{1}{3}=T\left(\frac{1}{3}\right)=Tg(0)$.

At $\eta=2$, $g(2)=T(2)$ and $gT(2)=g(2)=2=T(2)=gT(2)$.

As a result, while g,T are OWC mappings, but they are not weakly compatible.

Take a sequence as $\eta_p=2-\frac{1}{p}$ for $p\geq 1$.

Now $\lim\limits_{p\to\infty} f\eta_p=\lim\limits_{p\to\infty} f\left(2-\frac{1}{p}\right)=\lim\limits_{p\to\infty} 2=2$

and $\lim\limits_{p\to\infty} S\eta_p=\lim\limits_{p\to\infty} S\left(2-\frac{1}{p}\right)=\lim\limits_{j\to\infty} 2-\frac{1}{p}=2.$

Also $fS\eta_p=fS\left(2-\frac{1}{p}\right)=f\left(2-\frac{1}{p}\right)=2$ as $p\to\infty$

and $Sf\eta_p=Sf\left(2-\frac{1}{p}\right)=S(2)=2=S(2)$ as $p\to\infty$

This implies the pair of map (f,S) is sub-compatible and reciprocally continuous. Also the pair (g,T) is occasionally weakly compatible.

Consider a sequence $\eta_p=\frac{1}{p}$ for $p\geq 1$.

Then $\lim\limits_{p\to\infty} f\eta_j = \lim\limits_{p\to\infty} f\left(\frac{1}{p}\right) = \lim\limits_{p\to\infty} \frac{1-\frac{1}{p}}{3} = \lim\limits_{p\to\infty} \frac{1}{3} - \frac{1}{3p} = \frac{1}{3}$

and $\lim\limits_{p\to\infty} S\eta_j = \lim\limits_{j\to\infty} S\left(\frac{1}{p}\right) = \lim\limits_{p\to\infty} \frac{1}{3} + \frac{2}{p} = \frac{1}{3}.$

Also $fS\eta_p = fS\left(\frac{1}{p}\right) = f\left(\frac{1}{3} + \frac{2}{p}\right) = 2$ as $p \to \infty$

and $Sf\eta_p = Sf\left(\frac{1}{p}\right) = S\left(\frac{1}{3} - \frac{1}{3p}\right) = \frac{1}{3} + 2\left(\frac{1}{3} - \frac{1}{3p}\right) = 1 - \frac{2}{3p} = 1$ as $p \to \infty$

so that $\lim\limits_{p\to\infty} d\left(fS\eta_p, Sf\eta_p\right) = d(2,1) = |2-1|^2 = 1 \neq 0.$

Demonstrating that the pair (f, S) is not compatible.

Thus the pair (f,S) and (g,T) satisfy all the conditions of the Theorem 3.1.

Further $f(2) = g(2) = S(2) = T(2) = 2,$

therefore 2 is the unique common fixed point for f,g,S and T.

We now prove another generalization Theorem of 2.5.

3.3. Theorem:

Let (X,d) be a complete b- metric space and the self mappings f,g,S, and T meet the conditions

(b1) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$

(b2) $d(fu, gv) \leq \frac{q}{k^4} \max\left\{d(Su, Tv), d(fu, Su), d(gv, Tv), \frac{1}{2}\left(d(Su, gv) + d(fu, Tv)\right)\right\},$

holds for every $u, v \in X$ with $q \in (0,1)$.

(b3) The pair (g, T) is occasionally weakly compatible

(b4) the pair (f, S) is faintly compatible and reciprocally continuous.

Then the mappings f,g,S and T have a unique common fixed point.

Proof:

By above (b4) the pair (f,S) is faintly compatible then $\exists$ one more sequence $\{t_n\}$ in X such that $\lim_{j\to\infty} ft_j = \partial = \lim_{j\to\infty} St_j$ for ∂ in X satisfying $\lim_{j\to\infty} d(fSt_j, Sft_j) = 0$.

And above (C4) (f,S) reciprocally continuous then $\lim_{j\to\infty} d(fSt_j, f\partial) = 0$ and $\lim_{j\to\infty} d(Sft_j, S\partial) = 0$.

This implies $d(f\partial, S\partial) = 0$.

This gives $f\partial = S\partial$.

Now we claim $f\partial = \lambda$.

Putting $u = \partial$ and $v = u_{2j+1}$ in (b2), we get

$$d(f\partial, gu_{2j+1}) \le \frac{q}{k^4}\max\left\{d(S\partial, Tu_{2j+1}), d(f\partial, S\partial), d(gu_{2j+1}, Tu_{2j+1}), \frac{1}{2}\left(d(S\partial, gu_{2j+1}) + d(f\partial, Tu_{2j+1})\right)\right\}$$

take the sup limit as $j \to \infty$ on both the sides and by Lemma(2.6),which gives

$$d(f\partial, \lambda) \le \lim_{j\to\infty} Sup\, d(f\partial, gu_{2j+1})$$

$$\frac{q}{k^4}\max\left\{\begin{array}{l}\lim_{j\to\infty} Sup\, d(S\partial, Tu_{2j+1}), \lim_{j\to\infty} Sup\, d(f\partial, S\partial), \lim_{j\to\infty} Sup\, d(gu_{2j+1}, Tu_{2j+1}), \\ \frac{1}{2}\left(\lim_{j\to\infty} Sup\, d(S\partial, gu_{2j+1}) + \lim_{j\to\infty} Sup\, d(f\partial, Tu_{2j+1})\right)\end{array}\right\}$$

$$\le \frac{q}{k^4}\max\left\{k^2 d(f\partial, \lambda), 0, 0, \frac{k^2}{2}\left(d(f\partial, \lambda) + d(f\partial, \lambda)\right)\right\}$$

$$= \frac{q}{k^4} k^2 d(f\partial, \lambda)$$

$$= \frac{q}{k^2} d(f\partial, \lambda)$$

$$\frac{d(f\partial, \lambda)}{k^2} \le \frac{q}{k^2} d(f\partial, \lambda)$$

as $0 < q < 1$, so $f\partial = \lambda$. Hence $f\partial = \lambda = S\partial$.

By (b4) (f,S) is faintly compatible so $f\partial = S\partial$, this implies $fS\partial = Sf\partial$ hence $f\lambda = S\lambda$.

Now we assert $f\lambda = \lambda$.

Substituting $u=\lambda$ and $v=u_{2j+1}$ in (b2), we get

$$d(f\lambda, gu_{2j+1}) \le \frac{q}{k^4}\max\left\{d(S\lambda, T\alpha_{2j+1}), d(f\lambda, S\lambda), d(gu_{2j+1}, Tu_{2j+1}), \frac{1}{2}\left(d(S\lambda, gu_{2j+1}) + d(f\lambda, Tu_{2j+1})\right)\right\}$$

takethe sup limit as $j \to \infty$ on both the sides and using the Lemma(2.6),which gives

$$d(f\lambda, \lambda) \le \lim_{j\to\infty} Sup\, d(f\lambda, gu_{2j+1})$$

$$\frac{q}{k^4}\max\left\{\begin{array}{l}\lim_{j\to\infty} Sup\, d(S\lambda, Tu_{2j+1}), \lim_{j\to\infty} Sup\, d(f\lambda, S\lambda), \lim_{j\to\infty} Sup\, d(gu_{2j+1}, Tu_{2j+1}),\\ \frac{1}{2}\left(\lim_{j\to\infty} Sup\, d(S\lambda, gu_{2j+1}) + \lim_{j\to\infty} Sup\, d(f\lambda, Tu_{2j+1})\right)\end{array}\right\}$$

$$\frac{d(f\lambda, \lambda)}{k^2} \le \lim_{j\to\infty} Sup\, d(f\lambda, gu_{2j+1})$$

$$\le \frac{q}{k^4}\max\left\{k^2 d(f\lambda, \lambda), k^2 d(f\lambda, S\lambda), k^2 d(\lambda, \lambda), \frac{k^2}{2}\left(d(f\lambda, \lambda) + d(f\lambda, \lambda)\right)\right\}$$

$$\frac{d(f\lambda, \lambda)}{k^2} \le \frac{q}{k^2}\ d(f\lambda, \lambda)$$

as $0<q<1$,so $f\lambda=\lambda.$

Which gives $f\lambda = S\lambda = \lambda.$--------------(1)

Since $\eta = f\eta \in f(X) \subseteq T(X)$ by (b1) implies $f\lambda = Tu$ for some $u \in X$.

Now we claim $\lambda = gu$.

Putting $\alpha=\lambda$ and $\beta = u$ in (b2), we get

$$d(f\lambda, gu) \le \frac{q}{k^4}\max\left\{d(S\lambda, Tu), d(f\lambda, S\lambda), d(gu, Tu), \frac{1}{2}\left(d(S\lambda, gu) + d(f\lambda, Tu)\right)\right\}$$

take he sup limit as $j \to \infty$ on both the sides and by Lemma(2.6),which gives

$$d(\lambda, gu) \leq \lim_{j\to\infty} Sup\, d(f\lambda, gu)$$

$$\frac{q}{k^4}\max\left\{\begin{array}{l}\lim_{j\to\infty} Sup\, d(S\lambda, Tu), \lim_{j\to\infty} Sup\, d(f\lambda, S\lambda), \lim_{j\to\infty} Sup\, d(gu, Tu), \\ \frac{1}{2}\left(\lim_{j\to\infty} Sup\, d(S\lambda, gu) + \lim_{j\to\infty} Sup\, d(f\lambda, Tu)\right)\end{array}\right\}$$

$$\leq \frac{q}{k^4}\max\left\{0,0,k^2 d(gu,\lambda), \frac{k^2}{2}\left(d(\lambda, gu)+0\right)\right\}$$

$$= \frac{q}{k^2} d(gu,\lambda)$$

$$d(gu,\lambda) \leq q d(gu,\lambda)$$

as $0 < q < 1$, so $gu = \lambda$.

This gives $\lambda = gu = Tu$.

By (b3) we have the pair (g,T) is occasionally weakly compatible then $\exists$ $u \in X$ when $gu = Tu$ which gives $gTu = Tgu$

$\Rightarrow$ this gives $T\lambda = g\lambda$.

Now we show that $g\lambda = \lambda$.

Putting $\alpha = \lambda$, $\beta = \lambda$ in (b2) we get

$$d(f\lambda, g\lambda) \leq \frac{q}{k^4}\max\left\{d(S\lambda, T\lambda)\beta, d(f\lambda, S\lambda), d(g\lambda, T\lambda), \frac{1}{2}\left(d(S\lambda, g\lambda) + d(f\lambda, T\lambda)\right)\right\}$$

take the sup limit as $j \to \infty$ on both the sides and by Lemma(2.6),which gives

$$\frac{d(\lambda, g\lambda)}{k^2} \leq \lim_{j\to\infty} Sup d(\lambda, g\lambda) \leq \frac{q}{k^4}\max\left\{k^2 d(\lambda, g\lambda), 0, 0, \frac{k^2}{2}\left(d(\lambda, g\lambda) + d(\lambda, g\lambda)\right)\right\}$$

$$\frac{d(\lambda, g\lambda)}{k^2} \leq \frac{q}{k^4}\max\left\{k^2 d(\lambda, g\lambda), k^2 d(\lambda, g\lambda)\right\}$$

$$d(\lambda, g\lambda) \leq q d(\lambda, g\lambda)$$

as $0 < q < 1$, so $g\lambda = \lambda$.

$\Rightarrow g\lambda = T\lambda = \lambda.$---------------(2)

We get from (1) and (2)

$f\lambda = S\lambda = g\lambda = T\lambda = \lambda.$

Therefore λ is the required common fixed point.

Uniqueness follows easily.

Now we justify our Theorem with the following example.

3.2. Example:

Suppose $X = [0, 5.5]$ is a $b-$metric space with $d(u,v) = |u-v|^2$ where $u, v \in X$.

The self maps defined as below

$$T(\eta) = S(\eta) = \begin{cases} 2+9\eta \;\; if \;\; \eta \in \left[0, \frac{1}{6}\right) \\ \eta \;\; if \;\; \eta \in \left[\frac{1}{6}, 5\right] \\ 5.5-\eta \;\; if \;\; \eta \in (5, 5.5] \end{cases} \quad ;$$

and

$$f(\eta) = g(\eta) = \begin{cases} 2-6\eta \quad if \quad \eta \in \left[0, \frac{1}{6}\right) \\ 5, \quad if \;\; \eta \in \left[\frac{1}{6}, 5.5\right] \end{cases} .$$

Then

$f(X) = g(X) = (1,2] \cup 5$ and $S(X) = T((X) = [0,5]$

Here the condition (C1) is satisfied.

Clearly $\eta = 0$ and $\eta = 5$ points are the coincidence points for the maps g ,T.

At $\eta = 0$, $g(0) = T(0)$ but $gT(0) = g(2) = 5 \neq 2 = T(2) = Tg(0)$.

At $\eta = 5$, $g(5) = T(5)$ and $gT(5) = g(5) = 5 = T(5) = gT(5)$.

Therefore the pair (g,T) is occasionally weakly compatible but not weakly compatible mappings.

Take a sequence as $\eta_p = 5 - \frac{1}{p}$ for $p \geq 1$.

Now $\lim_{p\to\infty} f\eta_p = \lim_{j\to\infty} f\left(5 - \frac{1}{p}\right) = \lim_{p\to\infty} 5 = 5$

and $\lim_{p\to\infty} S\eta_j = \lim_{p\to\infty} S\left(5 - \frac{1}{p}\right) = \lim_{p\to\infty} 5 - \frac{1}{p} = 5.$

Also $fS\eta_p = fS\left(5 - \frac{1}{p}\right) = f\left(5 - \frac{1}{j}p\right) = 5 = f(5)$ as $p \to \infty$

and $Sf\eta_p = Sf\left(5 - \frac{1}{p}\right) = S(5) = 5 = S(5)$ as $p \to \infty$

This means that the mappings f,S are sub-compatible, reciprocally continuous and g,T are occasionally weakly compatible mappings.

Consider a sequence $\eta_p = \frac{1}{7j}$ for $p \geq 0$.

Then $\lim_{p\to\infty} f\eta_p = \lim_{j\to\infty} f\left(\frac{1}{7p}\right) = \lim_{j\to\infty} 2 - 6\left(\frac{1}{7p}\right) = 2$

and $\lim_{p\to\infty} S\eta_j = \lim_{p\to\infty} S\left(\frac{1}{j}\right) = \lim_{p\to\infty} 2 + \frac{9}{7p} = 2.$

Also $fS\eta_p = fS\left(\frac{1}{7p}\right) = f\left(2 + \frac{9}{7p}\right) = 5$ as $p \to \infty$

and $Sf\eta_p = Sf\left(\frac{1}{7p}\right) = S\left(2 - \frac{6}{7p}\right) = 2 - \frac{6}{7p} = 2$ as $p \to \infty$

so that $\lim_{j\to\infty} d\left(fS\eta_p, Sf\eta_p\right) = d(5,2) = |5-2|^2 = 9 \neq 0.$

Therefore the pair (f, S) is not compatible.

Thus the pair (f,S) and (g,T) satisfy all the conditions of the Theorem 3.2.

Further $f(5) = g(5) = S(5) = T(5) = 5,$

therefore 5 is the unique common fixed point for f,g,S and T.

CONCLUSION

This work is focused to improve J.R. Roshan and others results mentioned in Theorem 2.5 under weaker conditions by using the pair (f,S) as subcompatible, reciprocally continuous and (g,T) as occasionally weakly compatible instead of compatible mappings. We also proved another Theorem by taking the pair (f,S) as faintly compatible, reciprocally continuous and (g,T) as occasionally weakly compatible instead of compatible mappings. We note that none of the mappings in our two Theorems are continuous. At the end of the our discussion we justified our results with suitable examples.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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