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DISTRIBUTIONAL HENSTOCK-KURZWEIL INTEGRAL FOR FOURTH ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper existence of solution of fourth order nonlinear boundary value problems involving distributional Henstock-Kurzweil integral is obtained using fixed point theorem and distributional derivative.

Keywords: distributional Henstock - Kurzweil integral; nonlinear boundary value problems; distributional derivatives fixed point theorem.

AMS Subject Classification: 26A39, 45A05.

1. Introduction

In 1989, Lee proved, if F is continuous function and point wise differentiable almost everywhere on $[a,b]$ then F is generalized absolutely continuous [ACG*]. A primitive F of the Henstock - Kurzweil integrable function f is generalized absolutely continuous, see [11] , [5] , [12] . If we consider

$$F(x) = \sum_{n=1}^{\infty} \frac{sinn^2x}{n^2}, x \in [0,1]$$

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$F(x)$ is continuous but not differentiable almost everywhere. It is clear that F is not [ACG*]. In spite of that $F(t)$ has distributional derivative. It means that there exist $F' = f$ ("$'$" denotes the distributional derivative). So f is distributional Henstock - Kurzweil intigrable (D_{HK}) but f is not Henstock - Kurzweil integrable. Also by Chew T. S. and F. Flordelia generalized the classic Caratheodory's theorem of existence on the Cauchy problem $u' = f(t,u)$ with $u(0) = 0$. In this paper, we consider the fourth order differential equations with boundary conditions(NBVP).

$$\begin{aligned} u^{''''}(t) &= -f(t,u(t)), t \in [0,1] \\ u(0) &= u(1) = 0 \\ au^{''}(\xi_1) - bu^{'''}(\xi_1) &= 0 \\ cu^{''}(\xi_2) + du^{'''}(\xi_2) &= 0 \end{aligned} \tag{1}$$

where u'', u''', u'''' stands for the distributional derivative of the function $u \in C^2[0,1]$ and $C^2[0,1]$ denotes the space where the function $u''' : [0,1] \to R$ are continuous, f is distribution means generalized function, $\xi_1, \xi_2 \in [0,1]$ and a,b,c,d are the positive constants. The space $C[0,1]$ is considered with the uniform norm $\|.\|_\infty$. In 1991, NBVP (1) has been studied in [**?**] for ordinary derivative. We obtain the existence of solution of NBVP (1) under weak conditions, This point makes the integrand of the fourth order differential equations more extensive.

The rest of the paper is arranged as under: In section 2, we introduce basic results and fundamental properties of the distributional Henstock - Kurzweil integral. In section 3, we apply Schauder's fixed point theorem to obtain existence of solution of NBVP (1).

2. Preliminaries

The following is space of test functions
$\mathscr{D} = \left\{ \phi : \mathscr{R} \to \mathscr{R} \mid \phi \in C^\infty and\ \phi\ has\ compact\ support\ in\ \mathscr{R} \right\}$, where the compact support of ϕ is the closure of set on which $\phi \neq 0$. That is

$$Supp(\phi) = \{\overline{x \in \mathscr{R} \, : \, \phi(x) \neq 0}\}.$$

A sequence $\{\phi_n\} \subset \mathscr{D}$ convergence to $\phi \in \mathscr{D}$, if there exist a compact set K such that all ϕ_n

have support in K and for each $m \geq 0$, sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$. Here ϕ is test function. If $\phi \in \mathscr{D}$, $\mathscr{D}'$ is the dual space of $\mathscr{D}$ that is if $f \to \mathscr{D}'$ then $f : \mathscr{D} \to \mathscr{R}$, means $< f, \phi > \in \mathscr{R}$ for $\phi \in \mathscr{D}$.

For all $f \in \mathscr{D}'$ define the distributional derivative f' of f to be a distribution satisfying $< f', \phi >= -< f, \phi' >$, where $\phi \in \mathscr{D}$ here ϕ' is the ordinary derivative of ϕ. In this paper all derivatives will be the distributional derivatives.

Let (a,b) an open interval in $\mathscr{R}$ define,

$\mathscr{D}((a,b)) = \left\{ \phi : (a,b) \to \mathscr{R} \mid \phi \in C_c^\infty \; and \; \phi \; has \; compact \; support \; (a,b) \right\}$.

The space of distributions on (a,b) is denoted by $\mathscr{D}'((a,b))$ that is the dual space of $\mathscr{D}(a,b)$.

Let $C[a,b]$ be the space of continuous function on $[a,b]$.

Let $B_c = \{F \in C[a,b] \mid F(a) = 0\}$. We denote B_c as a Banach space with uniform norm $\|F\|_\infty = \max_{t \in [a,b]} \mid F(t) \mid$.

Next we introduce the definition and properties of $\mathscr{D}_{\mathscr{H}\mathscr{K}}$ integral.

Definition 2.1. *A distribution f in $\mathscr{D}'((a,b))$ is said to be the distributionally Henstock - Kurzweil integrable on $[a,b]$ if there exist $F \in \mathscr{B}_c$ such that $F' = f$ that is f is the distributional derivative of F.*

We will refer to $(\mathscr{D}_{\mathscr{H}\mathscr{K}}) \int$ denote as "$\int$", the space of distribution Henstock - Kurzweil integrable ($[\mathscr{D}_{\mathscr{H}\mathscr{K}}]$ integrable) defined by

$$\mathscr{D}_{\mathscr{H}\mathscr{K}} = \left\{ f \in \mathscr{D}'((a,b)) \mid f = F' \; for \; some F \in \mathscr{B}_c \right\}.$$

Using this definition, if $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ then for all $\phi \in \mathscr{D}((a,b))$

$< f, \phi >=< F', \phi >= -< F, \phi' >= -\int_a^b F\phi'$.

Definition 2.2. *Let $Q = [a,b] \times [c,d] \; \mathscr{R}^2 \; \widetilde{\mathscr{B}_c} = \{\mathscr{H} \in C(Q) : \mathscr{H}(a,y)\} = \mathscr{H}(x,c) = 0 \, for \; every$ $x \in [a,b], y \in [c,d]$.*

We obtain ∂_1 *and* ∂_2 the distributional derivative of x *and* y respectively, hence define

$\mathscr{D}_{\mathscr{H}\mathscr{K}}(Q) = \left\{ f \in \mathscr{D}'(Q) : f = \partial F, F \in \widetilde{B}_c \right\}$.

Note that if $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}(Q)$ then corresponding continuous function F is unique ($F \in \widetilde{B}_c$ *and* $f = \partial F$).

Definition 2.3. *If* $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}(Q)$, $x \in [a,b]$ *and* $y \in [c,d]$ *we define*
$\int\limits_a^x f(\xi_1,\cdot)d\xi_1 = \partial_2 F_f(x,\cdot)$ *in* $D'((c,d))$ *and*
$\int\limits_c^y f(\cdot,\xi_2)d\xi_2 = \partial_1 F_f(\cdot,y)$ *in* $\mathscr{D}'((a,b))$.

It is obtained that $\int\limits_a^x f(s,\cdot)ds \in \mathscr{D}_{\mathscr{H}\mathscr{K}}((c,d))$, $\int\limits_c^y f(\cdot,t)dt \in \mathscr{D}_{\mathscr{H}\mathscr{K}}((a,b))$.

Following are the basic properties of the distributional Henstock - Kurzweil integral which are used to obtain solution of NBVP (1) in the next section.

Lemma 2.1. *Fundamental theorem of calculus (* [13] *Theorem 4)*

(a): *Let* $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$, *and define* $F(t) = \int\limits_a^t f$ *then* $F \in \mathscr{B}_c$ *and* $F' = f$

(b): *Let* $F \in C[a,b]$, *then* $\int\limits_a^t F' = F(t) - F(a)\ \forall t \in [a,b]$.

We define Alexiewicz norm as

$$\|f\| = \|F\|_\infty = Sup|\int\limits_0^t f(s)ds|, \quad f \in \mathscr{D}_{\mathscr{H}\mathscr{K}} \text{ and } F \in \mathscr{B}_c.$$

We say that sequence $\{f_n\} \subset \mathscr{D}_{\mathscr{H}\mathscr{K}}$ converges strongly to $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ if $\|f_n - f\| \to 0$ as $n \to \infty$ and we have following result.

Lemma 2.2. *(*[13] *Theorem 2)* $\mathscr{D}_{\mathscr{H}\mathscr{K}}$ *is Banach space with the Alexiewicz norm.*

Lemma 2.3. *(* [13] *Definition 6) Integration by parts.*

If $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *and g is a function of bounded variation that is* $g \in \mathscr{B}v$ *then*

$$fg = \mathscr{D}\mathscr{H},\ \mathscr{H}(t) = F(t)g(t) - \int\limits_a^t F dg.$$

Moreover $fg \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *and* $\int\limits_a^b fg = F(b)g(b) - \int\limits_a^b F dg$. *Here (*$\int = \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *integral).*

We introduce a partial ordering on distributional Henstock - Kurzweil for $f,g \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ then $(g \leq f)$ iff $f-g$ is positive on $[a,b]$ by the definition if $f,g \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ then $\int\limits_I f \geq \int\limits_I g$ for $I=[c,d] \subset [a,b]$ when $f \geq g$ see [].

Lemma 2.4. [2]*(Dominated convergence theorem for the distributional Henstock - Kurzweil integral) Let* $\{f_n\}_{n=0}^{\infty}$ *be a sequence which belongs to* $\mathscr{D}_{\mathscr{H}\mathscr{K}}$ *such that* $f_n \to f$ *as* $n \to \infty$ *in* $\mathscr{D}'$. *Suppose that there exist* $f_-, f_+ \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *satisfying* $f_- \leq f_n \leq f_+$ *for all* $n \in \mathscr{N}$ *then* $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *and* $\lim_{n\to\infty} \int\limits_a^b f_n = \int\limits_a^b f$

Lemma 2.5. [2]*(Fubini theorem) For all* $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}(Q)$, *we have*
$\int_Q f = \int\limits_c^d (\int\limits_a^b f(\cdot,\eta)d\eta) = \int\limits_a^b (\int\limits_c^d f(\xi,\cdot)d\xi)$.

Lemma 2.6. *If* $f \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ *and* $\{f_n\}_{n=0}^{\infty}$ *be a sequence in* $\mathscr{D}_{\mathscr{H}\mathscr{K}}$ *such that* $f_n \to f$ *as* $n \to \infty$ *in* $\mathscr{D}'$, *then define* $F_n(x) = \int\limits_a^x f_n$ *and* $F(x) = \int\limits_a^x f$. *If* g *is a function which is bounded variation and* $F_n \to F$ *as* $n \to \infty$ *on* $[a,b]$ *then* $\int\limits_a^g f_n g \to \int\limits_a^g fg$ *as* $n \to \infty$.

3. MAIN RESULTS

In this section we obtain the existence of solution of NBVP (1). Assume that f satisfies the following:

- **(Z1):** $f(t,u)$ is distributional Henstock - Kurzweil integrable that is ($\mathscr{D}_{\mathscr{H}\mathscr{K}}$ integrable) with respect to t for all $u \to C^2[0,1]$
- **(Z2):** $f(t,u)$ is continuous with respect to u for all $t \in [0,1]$, i.e. for each $t \in [0,1]$, $\|f(t,u_n) - f(t,u)\| \to 0$ as $\|(u_n - u)\|_{c^2} \to 0$ for $u_n \to C^2[0,1]$.
- **(Z3):** There exist $f_-, f_+ \in \mathscr{D}_{\mathscr{H}\mathscr{K}}$ such that $f_-(\cdot) \leq f(\cdot,u) \leq f_+(\cdot)$ for all $u \in C^2[0,1]$

Lemma 3.1. *Assume that* $ad+bc+ac(\xi_2-\xi_1) \neq 0$. *The non-linear boundary value problem NBVP (1) is equivalent to the following integral equation*

$$u(t) = c_0 \Big\{ \int\limits_0^1 \left[\frac{t^3}{6} - \frac{t^2}{2}\frac{a\xi_1 - b}{b} + \frac{\xi_2 + d}{c} - \frac{t}{3}(1 - \frac{a\xi_1 - b}{b} + \frac{\xi_2 + d}{c}) \right] \tag{2}$$

$$(1-s)^3 f(s,u(s))ds$$

$$-\int_0^{\xi_1}\left[\frac{t^3}{2a}(a+b)+\frac{t^2}{2}\frac{1}{a(a+b)}-t(\frac{1+a}{2a})(a+b)+\frac{1}{a(a+b)}\right](\xi_1-s)f(s,u(s))ds$$

$$+\int_0^{t}\frac{1}{6}\left[\frac{a\xi_1-b}{a}+\frac{(\xi_2+d}{c})\right]-\frac{1}{3}(1-\frac{a\xi_1-b}{b}+\frac{\xi_2+d}{c})(t-s)^3f(s,u(s))ds\Big\},$$

$t\in[0,1]$,

where ξ_1 *and* ξ_2 *are constants with* $0\le\xi_1\le\xi_2$ *and* $c_0=\frac{1}{ad+bc+ac(\xi_2-\xi_1)}$.

Proof. For all $t\in[0,1], s\in[0,1], u'''\in C[0,1], u''\in C[0,1], u\in[0,1]$ according to the Lemma [**?**], we have

$$u'''(t)-u'''(0)=-\int_0^t f(s,u(s))ds$$

$$u''(t)-u''(0)=-\int_0^t(t-\theta)f(\theta,u(\theta))d\theta+u'''(0)t$$

$$u'(t)-u'(0)=-\int_0^t\frac{1}{2}(t-s)^2f(s,u(s))ds+u'''(0)\frac{t^2}{2}+u''(0)t$$

$$u(t)-u(0)=-\int_0^t\frac{1}{6}(t-s)^3f(s,u(s))ds+\frac{1}{6}u'''(0)t^3+\frac{1}{2}u''(0)t^2+u'(0)t \tag{3}$$

Using boundary conditions, we get

$$-\int_0^1\frac{1}{6}(1-s)^3f(s,u(s))ds+\frac{1}{6}u'''(0)+\frac{1}{2}u''(0)+u'(0)+u(0)=0 \tag{4}$$

$$-\int_0^{\xi_1}(a+b)(\xi_1-s)f(s,u(s))ds+(a\xi_1-b)u'''(0)+au''(0)=0$$

$$-\int_0^{\xi_2}(c-d)(\xi_2-s)f(s,u(s))ds+(c\xi_2+d)u'''(0)+cu''(0)=0$$

$$u(0)=0$$

From above equations, we obtain

$$u'''(0) = \int_0^1 \frac{(1-s)^3 f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)} - \int_0^{\xi_1} \frac{\frac{3}{a}(a+b)(\xi_1-s)f(s,u(s)ds)}{ad+bc+ac(\xi_2-\xi_1)}$$

$$-\int_0^{\xi_2} \frac{\frac{3}{c}(c-d)(\xi_2-s)f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)}$$

$$u''(0) = \int_0^{\xi_1} \frac{\frac{1}{a}(a+b)(\xi_1-s)f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)} - \int_0^{\xi_2} \frac{\frac{1}{c}(c-d)(\xi_2-s)f(s,u(s))ds}{ab+cd+ac(\xi_2-\xi_1)}$$

$$-\int_0^1 \frac{((a\xi_1-b)/a)+((\xi_2+d)/c)(1-s^3)f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)}$$

$$u'(0) = -\int_0^{\xi_1} \frac{((1+a)/2a)[(a+b)+1/a(a+b)](\xi_1-s)f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)}$$

$$-\int_0^{\xi_2} \frac{((1-c)/2c)[(c-d)-(1/c)(c-d)](\xi_2-s)f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)}$$

$$-\int_0^1 \frac{(1/3)[1-((a\xi_1-b)/a)+((\xi_2+d)/c)](1-s)^3 f(s,u(s))ds}{ad+bc+ac(\xi_2-\xi_1)}$$

Further more from equations (3) and (4), we conclude that

$$u(t) = \left[\frac{t^3}{6} - \frac{t^2}{2}(\frac{a\xi_1-b}{b} + \frac{\xi_2+d}{c}) - \frac{t}{3}(1 - \frac{a\xi_1-b}{b} + \frac{\xi_2+d}{c})\right](1-s)^3 f(s,u(s))ds$$

$$-\int_0^{\xi_1} \frac{t^3}{2a}(a+b) + \frac{t^2}{2}\frac{1}{a(a+b)}$$

$$-\int_0^{\xi_1}\left[\frac{t^3}{2a}(a+b) + \frac{t^2}{2}\frac{1}{a(a+b)} - t(\frac{1+a}{2a})(a+b) + \frac{1}{a(a+b)}\right](\xi_1-s)f(s,u(s))ds$$

$$-\int_0^{\xi_1}\left[\frac{t^3}{2c}(c-d) - \frac{t^2}{2}\frac{1}{c(c+d)} - t(\frac{1-c}{2a})(c-d) - \frac{1}{c(c-d)}\right](\xi_2-s)f(s,u(s))$$

$$+\int_0^t \frac{1}{6}\left[\frac{a\xi_1-b}{a} + \frac{\xi_2+d}{c} - \frac{1}{3}(1 - \frac{a\xi_1-b}{b} + \frac{\xi_2+d}{c})\right](t-s)^3 f(s,u(s))ds\},$$

$t \in [0,1]$

By taking the derivative of both sides we obtain the result. □

Lemma 3.2. *(Theorem (6.15)* [10]*) The compact operator $\mathscr{T}: M \to M$ has at least one fixed point when M is bounded, closed, convex, non empty subset of a Banach space X over $\mathscr{R}$.*

Theorem 3.1. *If f satisfies $(Z_1)-(Z_3)$ and $ab+bc+ac(\xi_2-\xi_1) \neq 0$, then there exists at least one solution of the NBVP (1).*

Proof. Consider $F_u(t) = \int_0^t f(s,u(s))ds$, $F_-(t) = \int_0^t f_-(s)ds$, $F_+(t) = \int_0^t f_+(s)ds$

Similarly, we also consider

$$\mathscr{H}_u(t) = \int_0^t F_u(s)ds.$$

Since $f_-(s) \leq f(s,u(s)) \leq f_+(s) \forall u \in C^2[0,1]$ we can get

$$F_+(t) \geq F_u(t) \geq F_-(t), \mathscr{H}_+(t) \geq \mathscr{H}_u(t) \geq \mathscr{H}_-(t).$$

Hence we write

$$\int_0^t \mathscr{H}_+(s)ds \geq \int_0^t \mathscr{H}_u(s)ds \geq \int_0^t \mathscr{H}_-(s)ds.$$

and

$$M_1 = \max_{t\in[0,1]} |\int_0^t f_-(s)ds| + \max_{t\in[0,1]} |\int_0^t f_+(s)ds|$$

$$M_2 = \max_{t\in[0,1]} |\int_0^t F_-(s)ds| + \max_{t\in[0,1]} |\int_0^t F_+(s)ds|$$

$$M_3 = \max_{t\in[0,1]} |\int_0^t \mathscr{H}_-(s)ds| + \max_{t\in[0,1]} |\int_0^t \mathscr{H}_+(s)ds|.$$

$$M_4 = |u(0)| + 2|u'(0)| + 5/2|u''(0)| + 8/3|u'''(0)|$$

Obviously, for each $t \in [0,1]$ and $u \in C_2[0,1]$ we obtain

$$|F_u(t) \leq \max_{0\leq t\leq 1} |F_1(t)| + \max_{0\leq t\leq 1} |F_+(t)| = M_1$$

$$M_2 = |\mathscr{H}_u(t)| \leq \max_{0\leq t\leq 1} |\mathscr{H}_-(t) + \max_{0\leq t\leq 1} |\mathscr{H}_+(t)|$$

$$M_3 = |\int_0^t \mathscr{H}_u(s)ds| \leq \max_{0\leq t\leq 1} |\int_0^t \mathscr{H}_-(s)ds| + \max_{0\leq t\leq 1} |\int_0^t \mathscr{H}_+(s)ds|.$$

If $\mathscr{B} = \{u \in C^2[0,1] : \|u\|_{C^2} \leq l\}, M_1 + M_2 + M_3 + M_4 > 0 = l,$

define the operator

$$\begin{aligned}
\mathscr{T}u(t) = &\left[\frac{t^3}{6} - \frac{t^2}{2}(\frac{a\xi_1 - b)}{b} + \frac{\xi_2 + d}{c} - \frac{t}{3}(1 - \frac{a\xi_1 - b}{b} + \frac{\xi_2 + d}{c}\right] \\
&(1-s)^3 f(s,u(s))ds - \int_0^{\xi_1} \frac{t^3}{2a}(a+b) + \frac{t^2}{2}\frac{(1}{a(a+b)} \\
&- \int_0^{\xi_1} \left[\frac{t^3}{2a}(a+b) + \frac{t^2}{2}\frac{1}{a(a+b)} - t(\frac{1+a}{2a})(a+b) + \frac{1}{a(a+b)}\right](\xi_1 - s)f(s,u(s))ds \\
&- \int_0^{\xi_1} \left[\frac{t^3}{2c}(c-d) - \frac{t^2}{2}\frac{1}{c(c+d)} - t(\frac{1-c}{2a})((c-d) - \frac{1}{c(c-d)}\right](\xi_2 - s)f(s,u(s)) \\
&+ \int_0^t \frac{1}{6}\left[\frac{a\xi_1 - b}{a} + \frac{\xi_2 + d}{c} - \frac{1}{3}(1 - \frac{a\xi_1 - b}{b} + \frac{\xi_2 + d}{c}\right](t-s)^3 f(s,u(s))ds\}, \\
&t \in [0,1]
\end{aligned}$$

Now we prove the theorem in three steps:

Step 1. Let $\mathscr{T} : B \to B$ and for all $u \in B$,

$$\begin{aligned}
\|\mathscr{T}u\|_{C^2} &= \|u'\|_\infty + \|u''\|_\infty + \|u'''\|_\infty \\
&= \max_{t\in[0,1]} |u(0) + u'(0) + \frac{1}{2}u''(0)t^2 + \frac{1}{2}u'''(0)t^3 - \int_0^t \frac{1}{6}(t-s)^3 f(s,u(s))ds| \\
&\quad + \max_{t\in[0,1]} |u'(0) + u''(0)t + \frac{1}{2}u'''(0)t^2 - \int_0^t \frac{1}{2}(t-s)^2 f(s,u(s))ds| \\
&\quad + \max_{t\in[0,1]} |u''(0) + u'''(0)t - \int_0^t (t-s)f(s,u(s))ds| + \max_{t\in[0,1]} |u'''(0) - \int_0^t f(s,u(s))ds|
\end{aligned}$$

$$\|\mathscr{T}u\|_{C^2} \leq (|u(0)|+2|u'(0)|+\frac{5}{2}|u''(0)|+\frac{8}{3}|u'''(0)|)+\max_{t\in[0,1]}|\int_0^t \frac{1}{6}(t-s)^3 f(s,u(s))ds|$$

$$+\max_{t\in[0,1]}|\int_0^t \frac{1}{2}(t-s)^3 f(s,u(s))ds|+\max_{t\in[0,1]}|\int_0^t \frac{1}{2}(t-s)^2 f(s,u(s))ds|$$

$$+\max_{t\in[0,1]}|\int_0^t (t-s)f(s,u(s))ds|+\max_{t\in[0,1]}|\int_0^t f(s,u(s))ds|$$

For all $t\in[0,1]$, we have

$$\max_{t\in[0,1]}|\int_0^t \frac{1}{6}(t-s)^3 f(s,u(s)ds| = \max_{t\in[0,1]}|\int_0^t \frac{1}{6}(t-s)^3 dF_u(s)ds|$$

Thus

$$\max_{t\in[0,1]}|\frac{1}{6}(t-s)^3 F_u(s)\ |_{s=0}^{s=t}+\int_0^t \frac{1}{2}(t-s)^2 F_u(s)ds| = \max_{t\in[0,1]}|\int_0^t \frac{1}{2}(t-s)^2 d\mathscr{H}_u(s)|$$

$$= \max_{t\in[0,1]}|\frac{1}{2}(t-s)\mathscr{H}_u(s)\ |_{s=0}^{s=t}-\int_0^t \mathscr{H}_u(s)ds|$$

$$= \max_{t\in[0,1]}|\int_0^t \mathscr{H}_u(s)ds|$$

$$\leq M_3$$

Similarly, we have

$$\max_{t\in[0,1]}|\int_0^t (t-s)f(s,u(s))ds| \leq M_2$$

$$\max_{t\in[0,1]}|\int_0^t f(s,u(s))ds| \leq M_1$$

From above, we have

$$\|\mathscr{T}u\|_{C^2} \leq M_1+M_2+M_3+M_4.$$

Hence

$$\mathscr{T}(B)\subseteq B.$$

Step 2. $\mathscr{T}(B)$ **is equi-continuous:** Let $t_1, t_2 \in [0,1], u \in B$,

$$|\mathscr{T}u'(t_1) - \mathscr{T}u'(t_2) = |u''(0)(t_1 - t_2) + \frac{1}{2}u'''(0)(t_1^2 - t_2^2) - \int_0^{t_1} \frac{1}{2}(t_1 - s)^2 f(s,u(s))ds$$

$$+ \int_0^{t_2} 1/1(t-s)^2 f(s,u(s))ds$$

$$\leq |t_1 - t_2|(|u''(0)| + |u'''(0)| + \frac{1}{2}|t_1 - t_2|| \int_0^{t_1} (t_1 + t_2 - 2s) f(s,u(s))ds|$$

$$+ | \int_{t_1}^{t_2} \frac{1}{2}(t_2 - s)^2 f(s,u(s))ds|$$

$$|\mathscr{T}u''(t_1) - \mathscr{T}u''(t_2) = |u'''(0)(t_1 - t_2) - \int_0^{t_1} (t_1 - s) f(s,u(s))ds + \int_0^{t_2} (t_2) f(s,u(s))ds$$

$$\leq |t_1 - t_2|(|u'''(0)| + | \int_0^{t_1} f(s,u(s))ds|) + | \int_{t_1}^{t_2} (t_2 - s) f(s,u(s))ds|$$

Also

$$|\mathscr{T}u'''(t_1) - \mathscr{T}u'''(t_2) = | \int_0^{t_1} f(s,u(s)) + \int_0^{t_2} f(s,u(s))ds|$$

$$\leq \int_{t_1}^{t_2} f(s,u(s))ds$$

Since the functions are bounded and continuous on $[0,1]$ so are uniformly continuous. From above equations $\mathscr{T}(B)$ is equi-continuous on [a,b], $\forall \quad u \in B$. From Step 1, Step 2 and Ascoli - Arzela theorem, $\mathscr{T}(B)$ is relatively compact.

Step 3. $\mathscr{T}$ **is continuous mapping:**

If $u \in B$, $\{u_n\}_{n \in \mathscr{N}}$ be a sequence in B and $u_n \to u$ as $n \to \infty$ by (Z2), we write $f(\cdot, u_n) \to f(\cdot, u)$ as $n \to \infty$.

By (Z3) and Lemma 2.4 we have,

$$\lim_{n\to\infty}\int_0^t f(s,u_n(s))ds = \int_0^t f(s,u(s))ds, \quad t\in[0,1].$$

Applying Lemma 2.6, we obtain

$$\lim_{n\to\infty}\mathscr{T}(u_n) = \mathscr{T}(u).$$

Hence $\mathscr{T}$ is continuous.

Hence by Lemma 3.2 there exist a fixed point of $\mathscr{T}$ which is a solution of equation (2). By Lemma 3.1 the NBVP (1) has at least one solution. □

4. Conclusion

Schauder's fixed point theorem and distributional derivative is used to obtain existence of solution of fourth order nonlinear boundary value problems involving distributional Henstock-Kurzweil integral.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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