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COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS IN CONNECTION WITH (p,q) CHEBYSHEV POLYNOMIAL

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Abstract. In this present work, authors are introduced a new subclass of bivalent functions $\mathfrak{S}_{\Sigma}(\alpha, x, p, q)$ with respect to symmetric conjugate points in the open unit disc $\mathbb{U}$ related to (p,q) polynomials. Further the initial bounds of the subclass and the well known Fekete-Szeg*ö* inequality are determined.

Keywords: (p, q)-Chebyshev polynomials; bi-univalent functions; subordination.

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1. Introduction

Let R=$(-\infty, \infty)$be the set of real numbers, $\mathscr{C}$ be the set of complex numbers and

$$N := 1, 2, 3... = N_0 \backslash \{0\}$$

be the set of positive integers.

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Let $\mathscr{A}$ denote the family of normalized analytic functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}) \tag{1.1}$$

in the open disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathscr{S}$ denote the class of functions in $\mathscr{A}$ which are also univalent in $\mathbb{U}$.

The well-known Koebe one-quarter theorem [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathscr{A}$ contains a disc of radius 1/4. Hence every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For example, functions in the class Σ are given below [8]:

$$\frac{z}{1-z}, \quad -log(1-z), \quad \frac{1}{2} log\left(\frac{1+z}{1-z}\right).$$

In 1967, Lewin [5] introduced the class Σ of bi-univalent functions and shown that $|a_2| < 1.51$. In 1969, Netanyahu [7] showed that $max_{f\in\Sigma}|a_2| = 4/3$ and Suffridge [9] have given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, in 1980, Brannan and Clunie [1] improved the result as $|a_2| \leq \sqrt{2}$. In 1985, Kedzier-awski [3] proved this conjecture for a special case when the function f and f^{-1} are starlike. In 1984, Tan [10] proved that $|a_2| \leq 1.485$ which is the best estimate for the function in the class of bi-univalent functions.

For any integer $n \geq 2$ and $0 < q < p \leq 1$, the (p,q)-Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x,s,p,q) = (p^n + q^n)xU_{n-1}(x,s,p,q) + (pq)^{n-1} sU_{n-2}(x,s,p,q)$$

with the initial values $U_0(x,s,p,q) = 1$, $U_1(x,s,p,q) = (p+q)x$ and 's' is a variable. By Assuming various values of x,s,p and q we get some interesting polynomials as follows:

- When $x = \frac{x}{2}$, s = s, p = p and q = q, the (p, q)- Chebyshev polynomials of the second kind becomes (p, q)-Fibonacci polynomials.
- When x = x, s = -1, p = 1 and q = 1, the (p, q)- Chebyshev polynomials of the second kind becomes Second kind of Chebyshev polynomials.
- When $x = \frac{x}{2}$, s = 1, p = 1 and q = 1, the (p, q)- Chebyshev polynomials of the second kind becomes Fibonacci polynomials.
- When $x = \frac{1}{2}$, s=1, p=1 and q=1, the (p, q)- Chebyshev polynomials of the second kind becomes Fibonacci numbers.
- When x = x, s = 1, p = 1 and q = 1, the (p, q)- Chebyshev polynomials of the second kind becomes Pell polynomials.
- When x = 1, s = 11, p = 1 and q = 1, the (p, q)- Chebyshev polynomials of the second kind becomes Pell numbers.
- When $x = \frac{1}{2}$, s = 2y, p = 1 and q = 1, the (p, q)- Chebyshev polynomials of the second kind becomes Jacobsthal polynomials.
- When $x = \frac{1}{2}$, s=2, p=1 and q=1, the (p, q)- Chebyshev polynomials of the second kind becomes Jacobsthal numbers.

Recently Kızılateş et al.[4] defined (p, q)-Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the (p, q)- Chebyshev polynomials of the second kind is as follows:

$$G_{p,q}(z) = \frac{1}{1 - xpz\tau_p - xqz\tau_q - spqz^2\tau_{p,q}}$$

$$= \sum_{n=0}^{\infty} U_n(x,s,p,q)z^n \qquad (z \in \mathbb{U})$$

where the Fibonacci operator τ_q was introduced by Mason [6], $\tau_q f(z) = f(qz)$. Similarly, $\tau_{p,q} f(z) = f(pqz)$.

Definition 1. *For $0<\alpha\leq 1$, a function $s\in\sigma$ is belong to the class $\mathfrak{S}_{\Sigma}(\alpha,x,p,q)$ if it satisfies the following conditions*

$$\left\{\frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}}+\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'}-\frac{2\alpha z^2s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})'+(1-\alpha)(s(z)-\overline{s(-\overline{z})})}\right\}\prec G_{p,q}(z) \tag{1.3}$$

and

$$\left\{\frac{2wr'(w)}{r(w)-\overline{r(-\overline{w})}}+\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'}-\frac{2\alpha w^2r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})'+(1-\alpha)(r(w)-\overline{r(-\overline{w})})}\right\}\prec G_{p,q}(w) \tag{1.4}$$

where $r=s^{-1}$.

By setting $\alpha=0$, $\mathfrak{S}_{\Sigma}(\alpha,x,p,q)$=$\mathfrak{S}_{\Sigma}(0,x,p,q)$ which holds the following conditions

$$\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'}\prec G_{p,q}(z)\quad \textit{and}\quad \frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'}\prec G_{p,q}(z),$$

where r is the extension of f^{-1}.

2. Estimation of Initial Coefficients & Fekete-Szegö Inequality

Theorem 1. *A function $f\in\Sigma$ of the form* (1.1) *is said to be in the class $\mathfrak{S}_{\Sigma}(\alpha,x,p,q)$, then*

$$|a_2|\leq\frac{u_1(x,s,p,q)}{2}\left[\frac{\sqrt{u_1(x,s,p,q)}(m_2+n_2)}{\sqrt{(3-2\alpha)u_1^2(x,s,p,q)-2(2-\alpha)^2}}\right] \tag{2.1}$$

and

$$|a_3|\leq\frac{u_1(x,s,p,q)}{4}\left[\frac{(m_2-n_2)}{(3-2\alpha)-\frac{u_1(x,s,p,q)(m_1^2+n_1^2)}{2(2-\alpha)^2}}\right]. \tag{2.2}$$

Proof. Suppose that $f\in\mathfrak{S}_{\Sigma}(\alpha,x,p,q)$, then from (1.3) and (1.4)

$$\left\{\frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}}+\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'}-\frac{2\alpha z^2s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})'+(1-\alpha)(s(z)-\overline{s(-\overline{z})})}\right\}=G_{p,q}(\phi(z)) \tag{2.3}$$

and for its inverse map $g = f^{-1}$, we have

$$\left\{\frac{2wr'(w)}{r(w)-\overline{r(-\overline{w})}}+\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'}\right. \tag{2.4}$$
$$\left. -\frac{2\alpha w^2 r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})'+(1-\alpha)(r(w)-\overline{r(-\overline{w})})}\right\} = G_{p,q}(\varphi(w)).$$

For some analytic functions ϕ and φ such that $\phi(0)=\varphi(0)=0$ and $|\phi(z)|=|\varphi(w)|<1$ for all $z,w\in\mathbb{U}$. It is well known that if

$$|\phi(z)| = |m_1 z + m_2 z^2 + m_3 z^3 + ...| < 1$$

and

$$|\varphi(w)| = |n_1 w + n_2 w^2 + n_3 w^3 + ...| < 1$$

where $z,w\in\mathbb{U}$, then $|m_k| = |n_k| < 1 \quad (\forall\, k\in N)$.

From (2.3) and (2.4),

$$\left\{\frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}}+\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'}-\frac{2\alpha z^2 s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})'+(1-\alpha)(s(z)-\overline{s(-\overline{z})})}\right\}$$
$$= U_0(x,s,p,q)+U_1(x,s,p,q)\phi(z)+U_2(x,s,p,q)\phi^2(z)+\cdots$$

and

$$\left\{\frac{2wr'(w)}{r(w)-\overline{r(-\overline{w})}}+\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'}-\frac{2\alpha w^2 r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})'+(1-\alpha)(r(w)-\overline{r(-\overline{w})})}\right\}$$
$$= U_0(x,s,p,q)+U_1(x,s,p,q)\varphi(w)+U_2(x,s,p,q)\varphi^2(w)+\cdots.$$

Thus, we write

$$\left\{\frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}}+\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'}-\frac{2\alpha z^2 s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})'+(1-\alpha)(s(z)-\overline{s(-\overline{z})})}\right\} \tag{2.5}$$
$$= 1+U_0(x,s,p,q)+m_1(z)+\left[U_1(x,s,p,q)m_2+U_2(x,s,p,q)m_1^2\right]z^2+...$$

and

$$\left\{\frac{2wr'(w)}{r(w)-\overline{r(-\overline{w})}}+\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'}-\frac{2\alpha w^2 r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})'+(1-\alpha)(r(w)-\overline{r(-\overline{w})})}\right\} \tag{2.6}$$
$$= 1+U_0(x,s,p,q)+n_1(w)+\left[U_1(x,s,p,q)n_2+U_2(x,s,p,q)n_1^2\right]w^2+\cdots.$$

By equating the coefficients from (2.5) and (2.6)

$$2(2-\alpha)a_2 = u_1(x,s,p,q)m_1 \tag{2.7}$$

$$2(3-2\alpha)a_3 = u_1(x,s,p,q)m_2 + u_2(x,s,p,q)m_1^2 \tag{2.8}$$

$$-2(2-\alpha)a_2 = u_1(x,s,p,q)n_1 \tag{2.9}$$

$$2(3-2\alpha)(2a_2^2 - a_3) = u_1(x,s,p,q)n_1^2. \tag{2.10}$$

From (2.7) and (2.9)

$$m_1 = -n_1 \tag{2.11}$$

and

$$8(2-\alpha)^2 a_2^2 = u_1^2(x,s,p,q)(m_1^2 + n_1^2). \tag{2.12}$$

By using (2.8) and (2.10) we obtain,

$$4(3-2\alpha)a_2^2 = u_1(x,s,p,q)(m_2+n_2) + u_2(x,s,p,q)(m_1^2+n_1^2). \tag{2.13}$$

By using (2.12) in (2.13) we get,

$$\left[4(3-2\alpha) - \frac{8(2-\alpha)^2 u_2(x,s,p,q)}{u_1^2(x,s,p,q)}\right] a_2^2 = u_1(x,s,p,q)(m_2+n_2). \tag{2.14}$$

From (2.13) we acquired the result which is desired in (2.1).

By subtracting (2.10) from (2.8)

$$-4(3-2\alpha)(a_2^2 - a_3) = u_1(x,s,p,q)(m_2-n_2) + u_2(x,s,p,q)(m_1^2 - n_1^2).$$

Using (2.11) and (2.12),

$$4(3-2\alpha)\frac{u_1^2(x,s,p,q)(m_1^2+n_1^2)}{8(2-\alpha)^2} + 4(3-2\alpha)a_3 = u_1(x,s,p,q)(m_2-n_2)$$

$$a_3 = \frac{u_1(x,s,p,q)(m_2-n_2)}{4(3-2\alpha)} + \frac{u_1^2(x,s,p,q)(m_1^2+n_1^2)}{8(2-\alpha)^2}. \tag{2.15}$$

By using (2.11), we obtain the desired result in (2.2). □

Theorem 2. *A function $f \in \Sigma$ of the form* (1.1) *is said to be in the class $\mathfrak{S}_\Sigma(\alpha,x,p,q)$, then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|u_1(x,s,p,q)|}{2(3-2\alpha)}, & \phi \leq \dfrac{1}{4(3-2\alpha)}, \\ 2|u_1(x,s,p,q)||p|, & \phi \geq \dfrac{1}{4(3-2\alpha)}. \end{cases}$$

Proof. From (2.14) and (2.15),

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{[u_1(x,s,p,q)]^3(m_2+n_2)(1-\mu)}{4(3-2\alpha)u_1^2(x,s,p,q)-8(2-\alpha)^2u_2(x,s,p,q)} + \frac{u_1(x,s,p,q)(m_2-n_2)}{4(3-2\alpha)} \\ &= u_1(x,s,p,q)\left[m_2+\left(\phi+\frac{1}{4(3-2\alpha)}\right)+n_2\left(\phi-\frac{1}{4(3-2\alpha)}\right)\right] \end{aligned}$$

where

$$\phi = \frac{u_1^2(x,s,p,q)(1-\mu)}{4(3-2\alpha)u_1^2(x,s,p,q)-8(2-\alpha)^2u_2(x,s,p,q)}.$$

□

Corollary 1. *When $\alpha = 0$,*

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|u_1(x,s,p,q)|}{6}, & \phi \leq \dfrac{1}{12}, \\ 2|u_1(x,s,p,q)||p|, & \phi \geq \dfrac{1}{12}. \end{cases}$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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