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ON RIGHT BASES OF ORDERED LA-Γ-SEMIGROUPS

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Abstract. In this paper, the notions of left and right bases of an ordered LA-Γ-semigroup are introduced and described. The structure of an ordered LA-Γ- semigroup with left identity containing right bases will be studied. Moreover, we show that every right base of an ordered LA-Γ-semigroup with left identity has one element and the compliment of the union of all right bases of an ordered LA-Γ-semigroup with left identity is the maximal proper left Γ-ideal.

Keywords: ordered LA-Γ-semigroups; left Γ-ideals; right bases; left bases; maximal proper left Γ-ideals.

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1. INTRODUCTION

The notion of an left almost semigroup (abbreviated as an LA-semigroup) was first introduced by Kazim and Naseerudin [9]. This algebraic structure is closely related to a commutative

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semigroup because a commutative *LA*-semigroup is a semigroup. For examples of some results of *LA*-semigroups, we can see in [6, 7, 15]. The notion of a Γ-semigroup was introduced by Sen [11]. Later, Shah and Rehmen [12] introduced the concept of an *LA*-Γ-semigroup analogous to a Γ-semigroup. Moreover, ordered *LA*-semigroups were studied by the authors in [13]. Next, Khan et al. [10] introduced the concept of an ordered *LA*-Γ-semigroup. Also, some results of *LA*-Γ-semigroups and ordered *LA*-Γ-semigroups can be seen in [1, 2, 8]. The notions of right bases and left bases of semigroups were introduced by Tamura [14]. Later, Fabrici [5] examined the structure of a semigroup containing right bases. Changpas and Kummoon [4] extended results in [5] from semigroups to Γ-semigroups. The aim of this paper is to extend the results obtained by Changpas and Kummoon to ordered *LA*-Γ-semigroups. First, we now recall some definitions and results used throughout the paper.

Definition 1.1. ([12]) Let S and Γ be non-empty sets. The algebraic structure (S,Γ) is called an *LA*-Γ-*semigroup* if there exists a mapping $S\times\Gamma\times S\to S$, written (a,γ,b) and denoted by $a\gamma b$ such that S satisfied the left invertive law

$$(a\gamma b)\beta c=(c\gamma b)\beta a$$

for all $a,b,c\in S$ and $\gamma,\beta\in\Gamma$.

Definition 1.2. ([12]) An element e of an *LA*-Γ-semigroup S is called a *left identity* if $e\gamma a=a$ for all $a\in S$ and $\gamma\ \in\Gamma$.

Lemma 1.1. ([12]) If S is an *LA*-Γ-semigroup with left identity e, then $S\Gamma S=S$ and $S=e\Gamma S=S\Gamma e$.

Proposition 1.2. ([1]) Let S be an *LA*-Γ-semigroup.

(1) Every *LA*-Γ- semigroup with left identity satisfy the equalities

$$a\gamma(b\beta c)=b\gamma(a\beta c)\text{ and }(a\gamma b)\beta(c\alpha d)=(d\gamma c)\beta(b\alpha a)$$

for all $a,b,c,d,\in S$ and $\gamma,\beta,\alpha,\in\Gamma$.

(2) An *LA*-Γ- semigroup S is Γ-medial, i.e.,

$$(a\gamma b)\beta(c\gamma d)=(a\gamma c)\beta(b\alpha d)=(a\gamma c)\beta(b\alpha d)$$

for all $a,b,c,d \in S$ and $\gamma,\beta,\alpha \in \Gamma$.

Definition 1.3. ([10]) An *ordered LA-Γ-semigroup* is the algebraic structure $(S,\Gamma,\leq)$ in which the following conditions hold.

(1) (S,Γ) is an LA-Γ- semigroup.

(2) $(S,\leq)$ is a poset (i.e. reflexive, anti-symmetric and transitive).

(3) For all a,b and $x \in S, a \leq b$ implies $a\alpha x \leq b\alpha x$ and $x\alpha a \leq x\alpha b$ for all $\alpha \in \Gamma$.

Throughout this paper, unless stated otherwise, S stand for an ordered LA-Γ- semigroup. For non-empty subsets A and B of an ordered LA-Γ- semigroup S, we defined

$$A\Gamma B = \{a\gamma b | a \in A, b \in B \text{ and} \gamma \in \Gamma\} \text{ and } (A] = \{b \in S | t \leq a, \text{ for some } a \in A\}.$$

In particular, we write $B\Gamma a$ instead for $B\Gamma\{a\}$, $a\Gamma B$ instead for $\{a\}\Gamma B$ and $a \cup B\Gamma a$ instead for $\{a\} \cup B\Gamma a$. For $A = \{a\}$, we write $(a]$ instead for $(\{a\}]$.

Definition 1.4. ([2]) A non-empty subset A of an ordered LA-Γ-semigroup S is called an *LA-Γ-subsemigroup* of S if $A\Gamma A \subseteq A$.

Definition 1.5. ([10]) A non-empty subsets A of an ordered LA-Γ- semigroup S is called a *left (resp. right) Γ-ideal* of S if

(1) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$);

(2) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 1.6. A proper left Γ-ideal M of an ordered LA-Γ-semigroup S is said to be *maximal* if for any left Γ-ideal A of $S, M \subseteq A \subseteq S$ implies $M = A$ or $A = S$.

Lemma 1.3. ([10]) Let S be an ordered LA-Γ-semigroup, then the following are true.

(1) $A \subseteq (A]$, for all $A \subseteq S$.

(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$, for all subsets A,B of S.

(4) $(A] = ((A]]$, for all $A \subseteq S$.

(5) For every left (resp. right) Γ- ideal T of $S, (T] = T$.

(6) $((A]\Gamma(B]] = (A\Gamma B]$, for all subsets A,B of S.

(7) $(A\cup B] = (A]\cup(B]$, for all subsets A,B of S.

Lemma 1.4. Let S be an ordered LA-Γ-semigroup with left identity and A_i be a left Γ-ideal of S for each $i\in I$, then the following statements hold ;

(1) If $\bigcap\limits_{i\in I} A_i \neq \varnothing$ then $\bigcap\limits_{i\in I} A_i$ is a left Γ-ideal of S.
(2) $\bigcup\limits_{i\in I} A_i$ is a left Γ-ideal of S.

Proof. (1) Assume that $\bigcap\limits_{i\in I} A_i \neq \varnothing$. We will show that $S\Gamma\bigcap\limits_{i\in I} A_i \subseteq (\bigcap\limits_{i\in I} A_i)$. First, let $x\in S\Gamma(\bigcap\limits_{i\in I} A_i)$. Then $x = s\gamma a$ for some $s = S, \gamma\in\Gamma$ and $a\in\bigcap\limits_{i\in I} A_i$. Since $a\in\bigcap\limits_{i\in I} A_i$, we obtain $a\in A_i$ for all $i\in I$. Since A_i is a left Γ- ideal of S for all $i\in I$, then $x = s\gamma a\in S\Gamma A_i\subseteq A_i$, for all $i\in I$. So $x\in\bigcap\limits_{i\in I} A_i$. Thus $S\Gamma(\bigcap\limits_{i\in I} A_i)\subseteq\bigcap\limits_{i\in I} A_i$. Next, let $x\in\bigcap\limits_{i\in I} A_i$ and $y\in S$ such that $y\le x$. Since $x\in\bigcap\limits_{i\in I} A_i$, we obtain $x\in A_i$ where A_i is a left Γ-ideal of S for all $i\in I$. So $y\in A_i$ for all $i\in I$. Thus $y\in\bigcap\limits_{i\in I} A_i$. Therefore $\bigcap\limits_{i\in I} A_i$ is a left Γ- ideal of S.

(2) To show that $\bigcup\limits_{i\in I} A_i$ is a left Γ-ideal of S, we let $x\in(S\Gamma\bigcup\limits_{i\in I} A_i)$. Then $x = s\gamma a$ for some $s\in S, \gamma\in\Gamma$ and $a\in\bigcup\limits_{i\in I} A_i$. Since $a\in\bigcup\limits_{i\in I} A_i$, we obtain $a\in A_i$ for some $i\in I$. Since A_i is a left Γ-ideal of S for all $i\in I$, so $x = s\gamma a\in S\Gamma A_i\subseteq A_i\subseteq\bigcup\limits_{i\in I} A_i$. Thus $x\in\bigcup\limits_{i\in I} A_i$. Hence $S\Gamma(\bigcup\limits_{i\in I} A_i)\subseteq\bigcup\limits_{i\in I} A_i$. Next, let $x\in\bigcup\limits_{i\in I} A_i$, and $y\in S$ such that $y\le x$. Since $x\in\bigcup\limits_{i\in I} A_i$, we obtain $x\in A_i$for some $i\in I$, where A_i is a left Γ-ideal of S for all $I\in I$. So $y\in A_i$ for some $i\in I$. Thus $y\in\bigcup\limits_{i\in I} A_i$. Therefore $\bigcup\limits_{i\in I} A_i$ is a left Γ-ideal of S. □

Definition 1.7. Let A be a non-empty subset of an ordered LA-Γ-semigroup S. Then, the intersection of all left Γ-ideals of S containing A is the *smallest left Γ-ideal of S generated by A* and is denoted by $(A)_L$.

Lemma 1.5. Let A be a non-empty subset of an ordered LA-Γ-semigroup S with left identity e. Then $(A)_L = (A\cup S\Gamma A]$.

Proof. Let $B = (A \cup S\Gamma A]$. First, we consider

$$\begin{aligned} S\Gamma B &= S\Gamma(A \cup S\Gamma A] = (S]\Gamma(A \cup S\Gamma A] \\ &\subseteq ((S)\Gamma(A \cup S\Gamma A] && \text{by Lemma 1.2(3)} \\ &= (S\Gamma A \cup S\Gamma(S\Gamma A)] \\ &= (S\Gamma A \cup (S\Gamma S)\Gamma(S\Gamma A)] && \text{by Lemma 1.1} \\ &= (S\Gamma A \cup (A\Gamma S)\Gamma(S\Gamma S)] && \text{by Proposition 1.2(1)} \\ &= (S\Gamma A \cup (A\Gamma S)\Gamma S] \\ &= (S\Gamma A \cup (S\Gamma S)\Gamma A] && \text{by left invertive law} \\ &= (S\Gamma A \cup S\Gamma A] = (S\Gamma A] \subseteq (A \cup S\Gamma A] = B \end{aligned}$$

Then $S\Gamma B \subseteq B$. Next, let $a \in B = (A \cup S\Gamma A]$ and $b \in S$ such that $b \leq a$. Since $b \leq a$ and $a \in (A \cup S\Gamma A]$, we obtain $b \in ((A \cup S\Gamma A]] = (A \cup S\Gamma A] = B$. So $b \in B$. Thus B is a left Γ-ideal of S containing A. Next, let C be a left Γ-ideal of S containing A. Since $A \subseteq C$, then $S\Gamma A \subseteq S\Gamma C \subseteq C$. So $A \cup S\Gamma A \subseteq C$. Thus $B = (A \cup S\Gamma A] \subseteq (C] = C$. Hence B is the smallest left Γ-ideal of S containing A. Therefore $(A)_L = (A \cup S\Gamma A]$. □

For an element $a \in S$, we write $(a)_L$ for $(\{a\})_L$ which is called the principal left Γ-ideal of S generated by a. Thus

$$(a)_L = (a \cup S\Gamma a].$$

Corollary 1.6. Let S be an ordered LA-Γ- semigroup with left identity. Then $(S\Gamma a]$ is a left Γ-ideal of S for all $a \in S$.

2. Main Results

We begin this section with the definition of a right base of an ordered LA-Γ- semigroup with left identity as follows:

Definition 2.1. Let S be an ordered LA-Γ-semigroup with left identity. A non-empty subset A of S is called a right base of S if it satisfies the two following conditions:

(1) $S = (A \cup S\Gamma A]$, i.e. $S = (A)_L$;

(2) if B is a subset of A such that $S = (B)_L$, then $B = A$.

For a *left base* of S is defined dually.

Example 2.1. Let $S = \{e, a, b, c, d\}$ and $\Gamma = \{\beta\}$ with the multiplication defined by

β	e	a	b	c	d
e	e	e	e	e	e
a	e	a	a	a	a
b	e	a	c	d	b
c	e	a	d	c	c
d	e	a	b	c	d

and $\leq := \{(e,e), (a,a), (a,b), (a,c), (a,d), (b,b), (c,c), (d,d)\}$

Then S is an ordered LA-Γ- semigroup with left identity d. The right bases of S are $A = \{b\}$, $B = \{c\}$ and $C = \{d\}$. The left bases of S are the same as the right bases of S.

Example 2.2. Let $S = \{1, 2, 3, 4\}$ and $\Gamma = \{\beta\}$ with the multiplication defined by

β	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

and $\leq := \{(1,1), (2,2), (3,3), (4,4), (1,2), (4,2)\}$

Then S is an ordered LA-Γ- semigroup with left identity 3. The right base of S is $A = \{3\}$. The left base of S is the same as the right base of S.

First, we have the following useful lemma:

Lemma 2.1. Let A be a right base of an ordered LA-Γ- semigroup with left identity and let $a, b \in A$. If $a \in (S\Gamma b]$, then $a = b$.

Proof. Assume that $a, b \in A$ such that $a \in (S\Gamma b]$, and suppose that $a \neq b$. Let $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$, we have $b \in B$. We will show that $(A)_L \subseteq (B)_L$. Let $x \in (A)_L = (A \cup S\Gamma A]$. Then $x \leq z$ for some $z \in A \cup S\Gamma A$. Let $z \in A$. If $z \neq a$, then $z \in B \subseteq (B \cup S\Gamma A]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, then $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in (B)_L$. If $z = a$, then by assumption, we have $z = a \in (S\Gamma B] \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, then $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in (B)_L$. Next, let $z \in S\Gamma A$, then $z = s\gamma c$ for some $s \in S, \gamma \in \Gamma$ and $c \in A$. If $c = a$, then $z = s\gamma c \in S\Gamma(S\Gamma b]$. Since $(S\Gamma b]$ is a left Γ-ideal of S for all $b \in S$, so $z \in (S\Gamma b] \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, we have $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in (B)_L$. If $c \neq a$, then $z = s\gamma c \in S\Gamma B \subseteq (B \cup S\Gamma B]$. Since $x \leq z$ and $z \in (B \cup S\Gamma B]$, then $x \in ((B \cup S\Gamma B]] = (B \cup S\Gamma B]$. So $x \in (B)_L$. Hence $(A)_L \subseteq (B)_L$. By $S = (A)_L \subseteq (B)_L \subseteq S$, we have that $(B)_L = S$. This is a contradiction. Therefore, $a = b$. □

Definition 2.2. Let S be an ordered LA-Γ-semigroup with left identity. Define a *quasi-order* on S by, for any $a, b \in S$,

$$a \leq_L b :\Leftrightarrow (a)_L \subseteq (b)_L.$$

We write $a <_L b$ if $a \leq_L b$ but $a \neq b$, i.e., $(a)_L \subset (b)_L$.

From Definition 2.2, the relation $\leq_L$ is not a partial order. By Example 2.1, we have $(b)_L \subseteq (c)_L$ and $(c)_L \subseteq (b)_L$. So $b \leq_L c$ and $c \leq_L b$. But $b \neq c$. Thus $\leq_L$ is not a partial order on S.

Lemma 2.2. Let S be an ordered LA-Γ- semigroup with left identity. For any $a, b \in S$, if $a \leq b$, then $a \leq_L b$.

Proof. Let $a, b \in S$ such that $a \leq b$. We will show that $a \leq_L b$, i.e., $(a)_L \subseteq (b)_L$. Suppose that $x \in (a)_L$. Since $x \in (a \cup S\Gamma a]$, then $x \leq y$ for some $y \in a \cup S\Gamma a$. We have $y = a$ or $y \in S\Gamma a$. If $y = a$, then $x \leq a \leq b$, we have $x \leq b$ for some $b \in b \cup S\Gamma b$. So $x \in (b \cup S\Gamma b]$, and $x \in (b)_L$. If $y \in S\Gamma a$, then $y = s\gamma a$ for some $s \in S, \gamma \in \Gamma$. Since $a \leq b$, then $s\gamma a \leq s\gamma b$ and $s\gamma b \in S\Gamma b \subseteq b \cup S\Gamma b$. So $y = s\gamma a \in (b \cup S\Gamma b]$. Since $x \leq y$ and $y \in (b \cup S\Gamma b]$, then $x \in ((b \cup S\Gamma b]] = (b \cup S\Gamma b]$. So $x \in (b)_L$. Thus $(a)_L \subseteq (b)_L$, i.e., $a \leq_L b$. □

The following theorem characterizes when a non-empty subset of an ordered LA-Γ-semigroup is a right base of the ordered LA-Γ-semigroup.

Theorem 2.3. *A* non-empty subset A of an ordered LA-Γ-semigroup with left identity is a right base of S if and only if A satisfies the two following conditions:

(1) for any $x \in S$ there exists $a \in A$ such that $x \leq_L a$;

(2) for any two distinct elements $a, b \in A$ neither $a \leq_L b$ nor $b \leq_L a$.

Proof. Assume that A is a right base of S. Then $S = (A)_L$. First, to show that the condition (1) holds, we let $x \in S$. Then $x \in (A \cup S\Gamma A]$. Since $x \in (A \cup S\Gamma A]$, we have $x \leq y$ for some $y \in A \cup S\Gamma A$. Then $y \in A$ or $y \in S\Gamma A$. If $y \in A$, and $x \leq y$, by Lemma 2.2, $x \leq_L y$. If $y \in S\Gamma A$, then $y = s\gamma a$ for some $s \in S, \gamma \in \Gamma$ and $a \in A$. Since $y = s\gamma a \in S\Gamma a \subseteq (S\Gamma a] \subseteq (a)_L$ and $S\Gamma y \subseteq S\Gamma(S\Gamma a] \subseteq (S\Gamma a] \subseteq (a)_L$, then $y \cup S\Gamma y \subseteq (a)_L$. So $(y)_L = (y \cup S\Gamma y] \subseteq ((a)_L] = (a)_L$, i.e., $y \leq_L a$. Since $x \leq y$, by Lemma 2.2, $x \leq_L y$. So $x \leq_L y \leq_L a$. Thus $x \leq_L a$. Hence the condition (1) holds. Next, to show that the condition (2) holds. Let $a, b \in A$ such that $a \neq b$. Suppose $a \leq_L b$. Let $B = A \setminus \{a\}$, then $B \subset A$. Since $a \neq b$, we have $b \in B$. Let $x \in S$, by condition (1), there exists $c \in A$ such that $x \leq_L c$. Since $c \in A$, there are two cases to consider. If $c = a$, then $x \leq_L c \leq_L b$, and so $x \leq_L b$. By $x \in (x)_L \subseteq (b)_L \subseteq (B)_L$. Thus $s \subseteq (B)_L$ and $S = (B)_L$. This is a contradiction. If $c \neq a$, then $c \in B$. So $x \in (x)_L \subseteq (c)_L \subseteq (B)_L$. Thus $S \subseteq (B)_L$ and $S = (B)_L$. This is a contradiction. The case $b \leq_L a$ proved similarly. Hence the condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. We will show that A is a right base of S. First, to show that $S = (A)_L$. Clearly, $(A)_L \subseteq S$. Let $x \in S$, by condition (1), there exists $a \in A$ such that $x \leq_L a$, i.e. $(x)_L \subseteq (a)_L$. Then $x \in (x)_L \subseteq (a)_L \subseteq (A)_L$. So $S \subseteq (A)_L$. Thus $S = (A)_L$. Next, to show that A is a minimal subset of S with the property $S = (A)_L$. Let $B \subset A$ such that $S = (B)_L$. Then there exists $a \in A$ and $a \notin B$. Since $a \in A \subseteq S = (B \cup S\Gamma B] = (B] \cup (S\Gamma B]$, then $a \in (B]$ or $a \in (S\Gamma B]$. If $a \in (B]$, then $a \leq y$ for some $y \in B$, by Lemma 2.2, $a \leq_L y$. This is a contradiction. Thus $a \notin (B]$, and so $a \in (S\Gamma B]$. Since $a \in (S\Gamma B]$, we have $a \leq c$ for some $c \in S\Gamma B$. Let $c = s\gamma b$ for some $s \in S, \gamma \in \Gamma$ and $b \in B$. Since $a \leq c$ and $c = s\gamma v \in S\Gamma b \subseteq b \cup S\Gamma b$, we have $a \in (b \cup S\Gamma b]$. Since $\{a\} \subseteq (b \cup S\Gamma b]$ and $(b \cup S\Gamma b]$ is a left Γ- ideal of S, we have $S\Gamma a \subseteq S\Gamma(b \cup S\Gamma b] \subseteq (b \cup S\Gamma b]$. Thus $a \cup S\Gamma a \subseteq (b \cup S\Gamma b]$, and so $(a)_L = (a \cup S\Gamma a] \subseteq ((b \cup S\Gamma b]] = (b \cup S\Gamma b] = (b)_L$. Hence $a, b \in A$. This is a contradiction. Therefore A is a right base of S. □

Theorem 2.4. A right base A of an ordered LA-Γ-semigroup S with left identity is a left Γ-ideal of S if and only if $A = S$.

Proof. Assume that A is a left Γ-ideal of S. Then $S = (A)_L = (A \cup S\Gamma A] = (A \cup A] = (A] = A$. So $S = A$. Conversely, assume that $A = S$. To show that A is a left Γ- ideal of S. First, we have

$S\Gamma A = S\Gamma S = S = A$, so $S\Gamma A = A$. Next, let $a \in A$ and $b \in S$ such that $b \leq a$. Since $b \in S$ and $S = A$, then $b \in A$. Thus A is a left Γ-ideal of S. □

Theorem 2.5. The right bases of an ordered LA-Γ- semigroup S with left identity have the same cardinality.

Proof. Assume that A and B are right bases of S. Let $a \in A$. Since B is a right base of S, by condition (1) of Theorem 2.3, there exists $b \in B$ such that $a \leq_L b$. Since A is a right base of S, there exists $a^{'} \in A$ such that $b \leq_L a^{'}$. So $a \leq_L b \leq_L a^{'}$ i.e., $a \leq_L a^{'}$. By condition (2) of Theorem 2.3, we have $a = a^{'}$. Thus $(a)_L = (b)_L$. Define a mapping $\varphi : A \to B; \varphi(a) = b$ for all $a \in A$. We will show that φ is well-defined. Let $a_1, a_2 \in A$ such that $a_1 = a_2$, $\varphi(a_1) = b_1$, and $\varphi(a_2) = b_2$, for some $b_1, b_2 \in B$. Then $(a_1)_L = (b_1)_L$ and $(a_2)_L = (b_2)_L$. Since $a_1 = a_2$, then $(a_1)_L = (a_2)_L$, so $(a_1)L = (a_2)_L = (b_1)_L = (b_2)_L$. We have $b_1 \leq_L b_2$ and $b_2 \leq_L b_1$, by condition (2) of Theorem 2.3, $b_1 = b_2$. Thus $\varphi(a_1) = \varphi(a_2)$. Therefore φ is well-defined. Next, we will show that φ is one-to-one. Let $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2)$. Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. We have $(a_1)_L = (a_2)_L = (b)_L$. Since $(a_1)_L = (a_2)_L$, so we have $a_1 \leq_L a_2$ and $a_2 \leq_L a_1$. Thus $a_1 = a_2$. Therefore φ is one-to-one. Finally, we will show that φ is onto. Let $b \in B$, then there exists $a \in A$ such that $b \leq_L a$. Similarly, there exists $b^{'} \in B$ such that $a \leq_L b^{'}$. Then $b \leq_L a \leq_L b^{'}$, i.e., $b \leq_L b^{'}$. By condition (2) of Theorem 2.3, $b = b^{'}$. So $(b)_L \subseteq (a)_L$ and $(a)_L \subseteq (b)_L$. Thus $(a)_L = (b)_L$. Therefore φ is onto, and the proof is completes. □

Theorem 2.6. Let A be a right base of an ordered LA-Γ-semigroup of S with left identity and let $a \in A$. If $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$, then b belongs to some right base of S which is different from A.

Proof. Assume that $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$. Setting $B = (A \setminus \{a\}) \cup \{b\}$. Then $B \neq A$. We will show that B is a right base of S using Theorem 2.3. First, let $x \in S$. Since A is a right base of S by Theorem 2.3(1), $x \leq_L c$ for some $c \in A$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_L = (a)_L$. Since $(a)_L = (b)_L$. we have $(c)_L = (b)_L$, i.e., $c \leq_L b$. So $x \leq_L c \leq_L b$. Thus $x \leq_L b$ and $b \in B$. Next, let $b_1, b_2 \in B$ such that $b_1 \neq b_2$ We will show that neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$. Since $b_1 \in B$ and $b_2 \in B$ we have $b_1 = b$ or $b_1 \neq b$ and $b_2 = b$ or $b_2 \neq b$. Then are four cases to consider:

Case 1 : $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. Since A is a right base of S, neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$.

Case 2 : $b_1 \neq b$ and $b_2 = b$. Then $(b_2)_L = (b)_L$. If $b_1 \leq_L b_2$, then $(b)_L \subseteq (b_2)_L = (b)_L = (a)_L$. Thus $b_1 \leq_L a$ and $b_1, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $(A)_L = (b)_L = (b_2)_L \subseteq (b_1)_L$. Thus $a \leq_L b_1$ and $b_1, a \in A$. This is a contradiction.

Case 3 : $b_1 = b$ and $b_2 \neq b$. Then $(b_1)_L = (b)_L$. If $b_1 \leq_L b_2$, then $(a)_L = (b)_L = (b_1)_L \subseteq (b_2)_L$. Thus $a \leq_L b_2$ and $b_2, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $(b_2)_L \subseteq (b_1)_L = (b)_L = (a)_L$. Thus $b_2 \leq_L a$ and $b_2, a \in A$. This is a contradiction.

Case 4 : $b_1 = b$ and $b_2 = b$. This is impossible.

Therefore B is a right base of S. □

Theorem 2.7. Let U be the union of all right bases of an ordered LA-Γ- semigroup S with left identity. If $S \setminus U \neq \varnothing$, then $S \setminus U$ is a left Γ- ideal of S.

Proof. Assume that $S \setminus U \neq \varnothing$. First, let $x \in S, \gamma \in \Gamma$ and $a \in S \setminus U$. We will show that $x\gamma a \in S \setminus U$. Suppose that $x\gamma a \notin S \setminus U$. Then $x\gamma a \in U$. Thus $x\gamma a \in A$ for some a right base A of S. Let $x\gamma a = b$ for some $b \in A$. Then $b = x\gamma a \in S\Gamma a \subseteq (a \cup S\Gamma a]$. Since $\{b\} \subseteq S\Gamma a \subseteq (S\Gamma a]$ and $(S\Gamma a]$ is a left Γ-ideal of S, then $S\Gamma b \subseteq S\Gamma(S\Gamma a] \subseteq (S\Gamma a]$. So $b \cup S\Gamma b \subseteq (a \cup S\Gamma a]$, and $(b)_L = (b \cup S\Gamma b] \subseteq (a \cup S\Gamma a] = (a)_L$. Thus $(b)_L \subseteq (a)_L$. If $(b)_L = (a)_L$, by Theorem 2.6, we have $a \in U$. This is a contradiction. Hence $(b)_L \subset (a)_L$, i.e., $b <_L a$. Since A is a right base of S, there exists $b^{'} \in A$ such that $a \leq b^{'}$. We have $b <_L a \leq_L b^{'}$, and $b \leq_L b^{'}$ where $b, b^{'} \in A$. This contradicts to the condition (2) of Theorem 2.3. Thus $x\gamma a \in S \setminus U$. Next, let $b \in S \setminus U$ and $c \in S$ such that $c \leq b$. We will show that $c \in S \setminus U$. If $c \in U$, then $c \in B$ for some a right base B of S. Let $d \in B$ such that $b \leq_L d$. Since $c \leq b$, by Lemma 2.2, we have $c \leq_L b$. So $c \leq_L d$ where $c, d \in B$. This is a contradiction. Thus $c \notin U$, i.e., $c \in S \setminus U$. Therefore $S \setminus U$ is a left Γ-ideal of S. □

Theorem 2.8. Let U be the union of all right bases of an ordered LA-Γ-semigroup S with left identity such that $\varnothing \neq U \subset S$. If S contains the maximal left Γ-ideal of S containing every proper left Γ- ideal of S, denoted by L^*, then $S \setminus U = L^*$ if and only if $|A| = 1$ for every right base A of S.

Proof. Assume that S contains a maximal left Γ-ideal of S containing every proper left Γ-ideal of S, say L^*. Let $S\setminus U = L^*$. To show that $U \subseteq (a)_L$ for all $a \in U$, suppose $U \nsubseteq (a)_L$ for some $a \in U$. Then $(a)_L \subset S$ and $(a)_L$ is a proper left Γ- ideal of S. This implies that $a \in (a)_L \subseteq L^* = S\setminus U$, and so $a \in S\setminus U$. This is a contradiction. Hence $U \subseteq (a)_L$ for all $a \in U$. We claim that $S\setminus U \subseteq (a)_L$ for all $a \in U$. Suppose that $S\setminus U \nsubseteq (a)_L$ for some $a \in U$. Then $(a)_L \subset S$ and $(a_1)_L$ is a proper left Γ- ideal of S. This implies that $a_1 \in (a)_L \subseteq L^* = S\setminus U$, and so $a_1 \in S\setminus U$. This is a contradiction. Hence $S\setminus U \subseteq (a)_L$ for all $a \in U$. Since $U \subseteq (a)_L$ and $S\setminus U \subseteq (a)_L$ for all $a \in U$, it follows that $S = (S\setminus U)\cup \subseteq (a)_L \subseteq S$. So $(a)_L = S$ for all $a \in U$. Therefore, $\{a\}$ is a right base of S for all $a \in U$. Next, let A be a right base of S, and let $a, b \in A$. Suppose that $a \neq b$. Since $A \subseteq U, a \in U$, so $S = (a)_L$. Since $a \neq b$ and $b \in S = (a \cup S\Gamma a] = (a] \cup (S\Gamma a]$, then $b \in (a]$ or $b \in (S\Gamma a]$. If $b \in (a]$ then $b \leq a$, by Lemma 2.2, $b \leq_L a$ where $a, b \in A$. This contradicts to condition (2) of Theorem 2.3. Thus $b \in (S\Gamma a]$, by Lemma 2.1, $a = b$. This is a contradiction. Hence $a = b$. Therefore $|A| = 1$.

Conversely, assume that every right base of S has only one element. Then $S = (a)_L$ for all $a \in U$. We will show that $S\setminus U = L^*$. Since $\varnothing \neq U \subset S$, then $\varnothing \neq S\setminus U \subset S$. By Theorem 2.7, we have $S\setminus U$ is a proper left Γ- ideal of S. Next, let M be a left Γ-ideal of S such that $S\setminus U \subseteq M \subseteq S$. Suppose that $S\setminus U \neq M$. We have $S\setminus U \subset M$ and there exists $x \in M$ and $x \notin S\setminus U$, i.e., $x \in U$. Then $x \in M \cap U$ and so $M \cap U \neq \varnothing$. Let $a \in M \cap U$. Then $a \in M$ and $a \in U$. Since $a \in M$ and $S\Gamma a \subseteq S\Gamma M \subseteq M$, then $a \cup S\Gamma a \subseteq M$. So $(a)_L = (a \cup S\Gamma a] \subseteq (M] = M$. Since $a \in U$, by assumption we have $S = (a)_L$. So $S = (a)_L \subseteq M \subseteq S$. Thus $S = M$. Hence $S\setminus U$ is a maximal proper left Γ- ideal of S. Finally, let B be a proper left Γ- ideal of S. If $B \nsubseteq S\setminus U$, then exists $b \in B$ such that $b \notin S\setminus U$. So $B \cap U \neq \varnothing$ Let $c \in B \cap U$. It follows that $(c)_L \subseteq B$ and $S = (c)_L$. Then $S = (c)_L \subseteq B \subseteq S$. Thus $S = B$. This is a contradiction. Hence $B \subseteq S\setminus U$. Therefore $S\setminus U = L^*$. □

Theorem 2.9. Let S be an ordered LA-Γ-semigroup with left identity. If e is a left identity of S, then $\{e\}$ is a right base of S.

Proof. Assume that e is a left identity of S. Let $A = \{e\}$. We will show that A is a right base of S using Definition 2.1. First, we will show that $S = (A)_L$. Since e is a left identity of S, by Lemma

1.1, we have $S\Gamma e = S$. So $e \cup S\Gamma e = S$. Thus $(A)_L = (e \cup S\Gamma e] = (S] = S$. Hence $(A)_L = S$. The condition (2) of Definition 2.1 is obvious. Therefore $A = \{e\}$ is a right base of S. □

In Example 2.1 and Example 2.2, it is observed that every right base of S is only one element. However, it turns out that is true in general. The following corollary is combining Theorem 2.5 and Theorem 2.9.

Corollary 2.10. Let S be an ordered LA-Γ- semigroup with left identity. Then every right base of S is one element.

In Example 2.1, the right bases of S are $A = \{b\}, B = \{c\}$ and $C = \{d\}$. We have set S eliminating the union of all right bases of S denoted by $S \setminus U$ and $S \setminus U = \{e, a\}$. Thus $S \setminus U$ is a maximal proper left Γ-ideal of S containing every proper left Γ-ideal of S. From Theorem 2.8 and Corollary 2.10, we conclude the following theorem.

Theorem 2.11. Let U be the union of all right bases of an ordered LA-Γ-semigroup S with left identity. Then $S \setminus U$ is a maximal proper left Γ-ideal of S containing every proper left Γ-ideal of S.

Proof. Let S be an ordered LA-Γ-semigroup with left identity. By Corollary 2.10, we have every right base of S is one element. Since every right base of S is one element, by Theorem 2.8, we have $S \setminus U = L^*$. Therefore $S \setminus U$ is a maximal proper left Γ-ideal of S containing every proper left Γ-ideal of S. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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