Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:15
https://doi.org/10.28919/jmcs/6691
ISSN: 1927-5307

VERTEX LABELED GRAPH ENERGY

KAVITA PERMI[1,*], Y. R. L. INDRANI[2], K. N. PRAKASHA[3]

[1]Department of Mathematics, Presidency University, Bengaluru 560059, India
[2]Department of Mathematics, Nrupathunga University, Bengaluru 560001, India
[3]Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru 570002, India

Abstract. Graph energy is defined in [3], as the sum of absolute value of eigen values of the adjacency matrix of the given graph Γ. In this paper, we introduce vertex labeled graphs $V_L(\Gamma)$, find their energy and some bounds. We also find $V_LE(\Gamma)$ of different graphs.

Keywords: vertex labeled graph; vertex labeled matrix; vertex labeled energy.

2010 Mathematics Subject Classification: 05C50.

1. Introduction

A. Rosa [9] introduced labeling in 1966 . Labeling of a graph $\Gamma = (V,E)$ is a function f from the set of vertices V into the set of non-negative integers that assigns a label for each edge $uv\varepsilon E$, depending on labels to the vertices $f(u)$ and $f(v)$. Gutman [3] has defined the energy of graph as the sum of absolute value of eigen values of the adjacency matrix of the graph Γ. If λ_i is the eigen value of adjacency matrix of Γ then graph energy is $E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$. In Chemical science, for a molecule, the total π-electron energy plays a significant role. We consider simple

*Corresponding author
E-mail address: kavitapermi@presidencyuniversity.in
Received August 25, 2021

undirected graph with no self loops and no multiple edges. We refer [5] for standard definitions and terminology in graph theory, .

2. Vertex Labeled Graph Energy

P. G. Bhat and et al [2] have introduced binary labeled graph energy. Motivated by this , we define the vertex labeled matrix and find vertex labeled graph energy.

Let Γ be a simple graph of order n. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set and E be the edge set. Let the vertex be labeled by its degree. Let $l(v_i)$ and $l(v_j)$ be the labels to the vertices v_i, v_j respectively then the vertex labeled matrix $V_L(\Gamma)$ is defined as

$$m_{ij} = \begin{cases} l(v_i) + l(v_j), & \text{if there is a path between } v_i \text{ and } v_j \\ 0, & if\ v_i = v_j \text{ or there is no path between } v_i \text{ and } v_j \end{cases}$$

Let $\xi_1, \xi_2, ...\xi_n$ be the eigen values of vertex labeled matrix $V_L(\Gamma)$ then the vertex labeled graph energy is defined as

$$V_L E(\Gamma) = \sum_{i=1}^{n} |\xi_i| \tag{1}$$

The spectrum $V_L Spec(\Gamma)$ is the arrangement of distinct eigen values $\xi_1 > \xi_2 > \cdots > \xi_r$ with multiplicities $m_1, m_2, \ldots, m_r$, written as

$$V_L Spec(\Gamma) = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

3. Properties of Vertex Labeled Graph Energy

Theorem 3.1 If eigen values of $V_L(\Gamma)$ are $\xi_1 > \xi_2 > \cdots > \xi_r$, then

(1) $\sum_{i=1}^{n} \xi_i = 0$

(2) $\sum_{i=1}^{n} \xi_i^2 = 2\sum_{i=1}^{n} \left[l(v_i) + l(v_j)\right]^2 = 2B$

where $B = \sum_{i=1}^{n} \left[l(v_i) + l(v_j)\right]^2$

Proof. (1) In the vertex labeled graph matrix $V_L(\Gamma)$, diagonal entries are zero, therefore their sum is zero.

Hence $\sum_{i=1}^{n} \xi_i = 0$

(2) The sum of squares of eigen values of $V_L(\Gamma)$ is the sum of eigen values of $[V_L(\Gamma)]^2$

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{m} u_{ij}u_{ji}$$
$$= 0 + 2\sum_{i<j}(u_{ij})^2$$
$$= 2\sum_{i=1}^{n}\left[l(v_i)+l(v_j)\right]^2$$
$$= 2B$$
□

Theorem 3.2 Let c_0, c_1 and c_2 be the coefficients of characteristic polynomial of $V_L(\Gamma)$ matrix then we have

(1) $c_0 = 1$

(2) $c_1 = 0$

(3) $c_2 = -B$

Proof. (1)By definition $\Gamma(\xi, x) = det[\xi I - B]$

Therefore $c_0 = 1$

(2) $c_1 = (-1)^1 \times trace(\Gamma) = -1 \times 0 = 0$

(3) By definition $c_2 = \sum_{1\leq i<j\leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1\leq i<j\leq n}(a_{ii}a_{jj} - a_{ij}a_{ji})$

$$= \sum_{1\leq i<j\leq n} a_{ii}a_{jj} - \sum_{1\leq i<j\leq n} a_{ij}^2 = 0 - B = -B$$
□

We find the bounds for $V_L(\Gamma)$ using McClelland's inequalities [7].

Theorem 3.3 Let Γ be a graph with n vertices, then the upper bound for $V_L(\Gamma)$ is

$$E\left[V_L(\Gamma)\right] \leq \sqrt{2nB}$$

Proof. Let $\xi_1 > \xi_2 > \cdots > \xi_r$ be the eigen values of $V_L(\Gamma)$, then by Using Cauchy-Schwarz inequality we have

$$\left[\sum_{i=1}^{n} u_i v_i\right]^2 \leq \left[\sum_{i=1}^{n} u_i^2\right]\left[\sum_{i=1}^{n} v_i^2\right].$$

Choose $u_i = 1, v_i = \left|\xi_i\right|$ and by Theorem - 3.1

$$\left[\sum_{i=1}^{n}\left|\xi_i\right|\right]^2 \leq \left[\sum_{i=1}^{n} 1\right]\left[\sum_{i=1}^{n}\left|\xi_i\right|^2\right] = n\sum_{i=1}^{n}\xi_i^2$$

$$[(V_L(\Gamma))\,E(G)]^2 \leq n2B.$$

Hence

$$E\left[V_L(\Gamma)\right] \leq \sqrt{2nB}$$

□

The lower bounds for $E\left[V_L(\Gamma)\right]$ is as follows

Theorem 3.4 Let Γ be a graph with n vertices. If $\tau = \left|detV_L(\Gamma)\right|$ of Γ then the lower bound is

$$E\left[V_L(\Gamma)\right] \geq \sqrt{2B + n(n-1)\tau^{\frac{2}{n}}}.$$

Proof. By definition we have,

$$\left[E\left(V_L(\Gamma)\right)\right]^2 = \left[\sum_{i=1}^{n} |\xi_i|\right]^2 = \left[\sum_{i=1}^{n} |\xi_i|\right]\left[\sum_{j=1}^{n} |\xi_j|\right]$$

$$= \sum_{i=1}^{n} |\xi_i|^2 + \sum_{i \neq j} |\xi_i||\xi_j|.$$

From the inequality of arithmetic and geometric means

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\xi_i||\xi_j| \geq \left[\prod_{i \neq j} |\xi_i||\xi_j|\right]^{\frac{1}{n(n-1)}}.$$

Therefore

$$\left[E\left(V_L(\Gamma)\right)\right]^2 \geq \sum_{i=1}^{n} |\xi_i|^2 + n(n-1)\left[\prod_{i \neq j} |\xi_i||\xi_j|\right]^{\frac{1}{n(n-1)}}$$

$$\geq \sum_{i=1}^{n} |\xi_i|^2 + n(n-1)\left[\prod_{i=1}^{n} |\xi_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$= \sum_{i=1}^{n} |\xi_i|^2 + n(n-1)\left|\prod_{i=1}^{n} \xi_i\right|^{\frac{2}{n}}$$

$$= 2B + n(n-1)\tau^{\frac{2}{n}}.$$

Hence

$$E\left[V_L(\Gamma)\right] \geq \sqrt{2B + n(n-1)\tau^{\frac{2}{n}}}$$

□

4. Vertex Labeled Graph Energy of Different Class of Graphs

Theorem 4.1. Let K_n be the vertex labeled complete graph with $n \geq 2$ then

$$V_L E(K_n) = 4n^2 - 8n + 4.$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of the vertex labeled complete graph K_n with $n \geq 2$, then the vertex labeled matrix is given by

$$V_L(K_n) = \begin{bmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1 & 2(n-1) & 0 & \dots & 2(n-1) & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 0 & 2(n-1) \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$[\lambda - (2n^2 - 4n + 2)][\lambda + (2n-2)]^{n-1} = 0.$$

The spectrum of $V_L(K_n)$ is

$$V_L spec(K_n) = \begin{pmatrix} 2n^2 - 4n + 2 & -(2n-2) \\ 1 & n-1 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_L E(K_n) &= \left| 2n^2 - 4n + 2 \right| (1) + \left| -(2n-2) \right| (n-1) \\ &= 4n^2 - 8n + 4. \end{aligned}$$

□

Theorem 4.2. For the star graph $K_{1,n-1}$ we have

$$V_L E(K_{1,n-1}) = (2n-4) + 4\sqrt{12n^2 - 51n + 64}$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of vertex labeled star graph $K_{1,n-1}$, then

$$V_LE(K_{1,n-1}) = \begin{bmatrix} 0 & n & n & \dots & n & n \\ n & 0 & 2 & \dots & 2 & 2 \\ n & 2 & 0 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 2 & 2 & \dots & 0 & 2 \\ n & 2 & 2 & \dots & 2 & 0 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda^2 - (2n-4)\lambda - (11n^2 - 47n + 60)](\lambda + 2)^{n-2} = 0.$$

The spectrum is

$$V_Lspec(K_{1,n-1}) = \begin{pmatrix} \frac{(2n-4)-\sqrt{48n^2-204n+256}}{2} & \frac{(2n-4)+\sqrt{48n^2-204n+256}}{2} & -2 \\ 1 & 1 & n-2 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_LE(K_{1,n-1}) &= \left| \frac{(2n-4)-\sqrt{48n^2-204n+256}}{2} \right| (1) + \left| \frac{(2n-4)+\sqrt{48n^2-204n+256}}{2} \right| (1) \\ &+ \left| -2 \right| (n-2) \end{aligned}$$

Hence

$$V_LE(K_{1,n-1}) = (2n-4) + 4\sqrt{12n^2 - 51n + 64}$$

□

Theorem 4.3. For the Wheel graph W_n with $n \geq 4$ we have

$$V_LE(W_n) = (6n-12) + 4\sqrt{27n^2 - 110n + 152}$$

Proof. Let $V = \{ v_1, v_2, \cdots, v_n\}$ be the vertex set of vertex labeled Wheel graphW_n with $n \geq 4$, then

$$V_L(W_n) = \begin{bmatrix} 0 & n+2 & n+2 & \dots & n+2 & n+2 \\ n+2 & 0 & 6 & \dots & 6 & 6 \\ n+2 & 6 & 0 & \dots & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n+2 & 6 & 6 & \dots & 0 & 6 \\ n+2 & 6 & 6 & \dots & 6 & 0 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda^2 - (6n-12)\lambda - (18n^2 - 74n + 116)][\lambda + 6]^{n-2} = 0.$$

The spectrum of $V_L(W_n)$ is

$$V_L spec(W_n) = \begin{pmatrix} \frac{(6n-12)-\sqrt{108n^2-440n+608}}{2} & \frac{(6n-12)+\sqrt{108n^2-440n+608}}{2} & -6 \\ 1 & 1 & n-2 \end{pmatrix}.$$

The vertex labeled graph energy is

$$V_L E(W_n) = \left| \frac{(6n-12)-\sqrt{108n^2-440n+608}}{2} \right| (1) + \left| \frac{(6n-12)+\sqrt{108n^2-440n+608}}{2} \right| (1) + \left| -6 \right| (n-2)$$

Hence

$$V_L E(W_n) = (6n-12) + 4\sqrt{27n^2 - 110n + 152}.$$

□

Theorem 4.4. For the cocktail party graph $K_{n\times 2}$ we have

$$V_L E(K_{n\times 2}) = 16n^2 - 24n + 8.$$

Proof. Let $V = \{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$ be the vertex set of vertex labeled cocktail party graph $K_{n\times 2}$ then

$$V_L(K_{n\times 2}) = \begin{bmatrix} 0 & 4n-4 & 4n-4 & \dots & 4n-4 & 4n-4 \\ 4n-4 & 0 & 4n-4 & \dots & 4n-4 & 4n-4 \\ 4n-4 & 4n-4 & 0 & \dots & 4n-4 & 4n-4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4n-4 & 4n-4 & 4n-4 & \dots & 0 & 4n-4 \\ 4n-4 & 4n-4 & 4n-4 & \dots & 4n-4 & 0 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda - (8n^2 - 12n + 4)][\lambda + (4n-4)]^{2n-1} = 0$$

The spectrum of $V_L(K_{n\times 2})$ is

$$V_L spec(K_{n\times 2}) = \begin{pmatrix} 8n^2 - 12n + 4 & -(4n-4) \\ 1 & 2n-1 \end{pmatrix}.$$

The vertex labeled graph energy is

$$V_LE(K_{n\times 2}) = \left| (8n^2 - 12n + 4) \right| (1) + \left| -(4n-4) \right| (2n-1)$$

Hence

$$V_LE(K_{n\times 2}) = 16n^2 - 24n + 8.$$

□

Theorem 4.5. Let $S_n^0, n \geq 3$ be a crown graph with $2n$ vertices then

$$V_LE(S_n^0) = 8n^2 - 12n + 4.$$

Proof. Let $V = \{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$ be the vertex set of vertex labeled crown graph S_n^0 of order $2n$, then

$$V_L(S_n^0) = \begin{bmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \dots & 2(n-1) & 2(n-1) \\ 2(n-1) & 2(n-1) & 0 & \dots & 2(n-1) & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 0 & 2(n-1) \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$[\lambda^2-(4n^2-6n+2)][\lambda+(2n-2)]^{2n-1}=0.$$

The spectrum is

$$V_Lspec(S_n^0) = \begin{pmatrix} 2n^2-4n+2 & -(2n-2) \\ 1 & 2n-1 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_LE(S_n^0) &= \left| 2n^2-4n+2 \right| (1) + \left| -(2n-1) \right| (2n-1) \\ &= 8n^2-12n+4. \end{aligned}$$

□

Theorem 4.6. For the complete bipartite graph $K_{n,n}$,

$$V_LE(K_{n,n}) = 8n^2-4n$$

Proof. Let $V=\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$ be the vertex set of vertex labeled complete bipartite graph $K_{n,n}$ then

$$V_L(K_{n,n}) = \begin{bmatrix} 0 & 2n & 2n & \dots & 2n & 2n \\ 2n & 0 & 2n & \dots & 2n & 2n \\ 2n & 2n & 0 & \dots & 2n & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n & 2n & 2n & \dots & 0 & 2n \\ 2n & 2n & 2n & \dots & 2n & 0 \end{bmatrix}.$$

The characteristic equation is

$$[\lambda-(4n^2-2n)][\lambda+2n]^{2n-1}=0$$

The spectrum is

$$V_Lspec(K_{n,n}) = \begin{pmatrix} 4n^2-2n & -2n \\ 1 & 2n-1 \end{pmatrix}.$$

The Vertex labeled graph energy is

$$V_LE(K_{n,n}) \quad = \left| 4n^2-2n \right| (1) + \left| -2n \right| (2n-1)$$

Hence

$$V_LE(K_{n,n}) = 8n^2-4n$$

□

Theorem 4.7. For double star graph $S_{n,n}$

$$V_LE(S_{n,n}) = (6n-6)+4\sqrt{53n^2-198n+245}.$$

Proof. Let $S_{n,n}$ be the double star graph then

$$V_L(S_{n,n}) = \begin{bmatrix} 0 & n+1 & n+1 & n+1 & \dots & 2n & n+1 & n+1 & n+1 \\ n+1 & 0 & 2 & 2 & \dots & n+1 & 2 & 2 & 2 \\ n+1 & 2 & 0 & 2 & \dots & n+1 & 2 & 2 & 2 \\ n+1 & 2 & 2 & 0 & \dots & n+1 & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2n & n+1 & n+1 & n+1 & \dots & 0 & n+1 & n+1 & n+1 \\ n+1 & 2 & 2 & 2 & \dots & n+1 & 0 & 2 & 2 \\ n+1 & 2 & 2 & 2 & \dots & n+1 & 2 & 0 & 2 \\ n+1 & 2 & 2 & 2 & \dots & n+1 & 2 & 2 & 0 \end{bmatrix}.$$

The characteristic equation becomes

$$\left[\lambda^2 - (2n^2+n-3)\lambda - (6n^2+2n-4)\right][\lambda+2n][\lambda+2]^{2n-3} = 0.$$

The spectrum is

$$V_L spec(S_{n,n}) = \begin{pmatrix} \frac{(6n-6)-\sqrt{212n^2-792n+980}}{2} & \frac{(6n-6)-\sqrt{212n^2-792n+980}}{2} & -2n & -2 \\ 1 & 1 & 1 & 2n-3 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_LE(S_{n,n}) &= \left| \frac{(6n-6)-\sqrt{212n^2-792n+980}}{2} \right| (1) + \left| \frac{(6n-6)+\sqrt{212n^2-792n+980}}{2} \right| (1) \\ &\quad + |-2n|(1) + |-2|(2n-3) \end{aligned}$$

Hence

$$V_LE(S_{n,n}) = (6n-6) + 4\sqrt{53n^2-198n+245}.$$

□

Theorem 4.8. For the Friendship graph F_n with $2n+1$ vertices,

$$V_LE(F_n) = (8n-4) + 8\sqrt{26n^2-54n+49}.$$

Proof. Let F_n be a Friendship graph with $2n+1$ vertices then

$$V_L(F_n)=\begin{bmatrix} 2n+2 & 2n+2 & 2n+2 & \dots & 2n+2 & 2n+2 \\ 2n+2 & 0 & 4 & \dots & 4 & 4 \\ 2n+2 & 4 & 0 & \dots & 4 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n+2 & 4 & 4 & \dots & 0 & 4 \\ 2n+2 & 4 & 4 & \dots & 4 & 0 \end{bmatrix}.$$

The characteristic equation becomes

$$\left[\lambda^2-4(2n-1)\lambda-(88n^2-200n+192)\right][\lambda+4]^{2n-1}=0.$$

The spectrum is

$$\begin{pmatrix} \frac{4(2n-1)-\sqrt{208n^2-432n+392}}{2} & \frac{4(2n-1)+\sqrt{208n^2-432n+392}}{2} & -4 \\ 1 & 1 & 2n-1 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_LE(S_{n,n}) &= \left| \frac{4(2n-1)-\sqrt{208n^2-432n+392}}{2} \right|(1)+\left| \frac{4(2n-1)+\sqrt{208n^2-432n+392}}{2} \right|(1) \\ &+\left| -4 \right|(2n-1) \end{aligned}$$

Hence

$$V_LE(F_n)=(8n-4)+8\sqrt{26n^2-54n+49}.$$

□

Theorem 4.9. For the complement of double star graph $\overline{S_{n,n}}$,

$$V_LE(\overline{(S_{n,n})})=\sqrt{64n^4-288n^3+1236n^2-3944n+4852}+(8n^2-18n+10).$$

Proof. Let $\overline{S_{n,n}}$ the complement of double star graph then

$$V_L(\overline{S_{n,n}}) = \begin{bmatrix} 0 & 3n-3 & 3n-3 & 3n-3 & \dots & 2n-2 & 3n-3 & 3n-3 & 3n-3 \\ 3n-3 & 0 & 4n-4 & 4n-4 & \dots & 3n-3 & 4n-4 & 4n-4 & 4n-4 \\ 3n-3 & 4n-4 & 0 & 4n-4 & \dots & 3n-3 & 4n-4 & 4n-4 & 4n-4 \\ 3n-3 & 4n-4 & 4n-4 & 0 & \dots & 3n-3 & 4n-4 & 4n-4 & 4n-4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2n-2 & 3n-3 & 3n-3 & 3n-3 & \dots & 0 & 3n-3 & 3n-3 & 3n-3 \\ 3n-3 & 4n-4 & 4n-4 & 4n-4 & \dots & 3n-3 & 0 & 4n-4 & 4n-4 \\ 3n-3 & 4n-4 & 4n-4 & 4n-4 & \dots & 3n-3 & 4n-4 & 0 & 4n-4 \\ 3n-3 & 4n-4 & 4n-4 & 4n-4 & \dots & 3n-3 & 4n-4 & 4n-4 & 0 \end{bmatrix}.$$

The characteristic equation becomes

$$\left[\lambda^2-(8n^2-18n+10)\lambda-(188n^2-896n+1188)\right][\lambda+(2n-2)][\lambda+(4n-4)]^{(2n-3)}=0.$$

The spectrum is

$$V_Lspec(\overline{S_{n,n}}) = \begin{pmatrix} \frac{(8n^2-18n+10)-\sqrt{64n^4-288n^3+1236n^2-3944n+4852}}{2} & 1 \\ \frac{(8n^2-18n+10)+\sqrt{64n^4-288n^3+1236n^2-3944n+4852}}{2} & 1 \\ -(2n-2) & 1 \\ -(4n-4) & 2n-3 \end{pmatrix}.$$

The vertex labeled graph energy is

$$\begin{aligned} V_LE(\overline{S_{n,n}}) &= \left|\frac{(8n^2-18n+10)-\sqrt{64n^4-288n^3+1236n^2-3944n+4852}}{2}\right|(1) \\ &+ \left|\frac{(8n^2-18n+10)-\sqrt{64n^4-288n^3+1236n^2-3944n+4852}}{2}\right|(1) \\ &+ \left|-(2n-2)\right|(1)+\left|-(4n-4)\right|(2n-3) \end{aligned}$$

Hence

$$V_LE(\overline{S_{n,n}}) = \sqrt{64n^4-288n^3+1236n^2-3944n+4852}+(8n^2-18n+10).$$

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] C. Adiga, M. Smitha, On maximum degree energy of a graph, J. Math. Comput. Sci. 4(8) (2009), 385-396.

[2] P. G. Bhat, S. D'Souza, Energy of Binary Labeled Graphs, Trans. Comb. 2(3) (2013), 53-67.

[3] I. Gutman, The energy of a graph, Ber. Math. Stat. Sekt. Forschungsz. Graz 103 (1978), 1-22.

[4] I. Gutman, Boris Furtula , The Total *pi*-Electron Energy Saga , Croatica Chem. Acta 90(3) (2017), 359-368.

[5] F. Harary, Graph Theory, Addison-Wesley Publishing, 1969.

[6] Kavita Permi, Manasa H S, Geetha M C, The minimum covering signless laplacian energy of graph, J. Phys.: Conf. Ser. 1597 (2020), 012031.

[7] B. J. McClelland, Properties of the latent roots of a matrix, The estimation of π-electron energies, J. Chem. Phys. 54 (1971), 640-643.

[8] K. N. Prakasha, P. S. K. Reddy, I. N. Cangul, Partition Laplacian energy of a graph, Adv. Stud. Contemp. Math. 27(4) (2017), 477-494.

[9] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Inter-nat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349-355.