Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7906-7920
https://doi.org/10.28919/jmcs/6698
ISSN: 1927-5307

MULTI LINEAR OPERATOR ON MULTI NORMED LINEAR SPACE

SUJOY DAS*, RANAJOY ROY

Department of Mathematics, Suri Vidyasagar College, Suri, Birbhum, West Bengal, Pin-731101, INDIA

Abstract. In this paper, for the first time, notion of linear operator is introduced on multi normed linear space. Boundedness and continuity of multi linear operators are studied along with their various properties. Norm of a multi linear operator is defined and some of its basic properties are investigated.

Keywords: multi linear operator; continuous multi linear operator; bounded multi linear operator; norm of a multi linear operator.

2010 AMS Subject Classification: 46B20, 47A05.

1. INTRODUCTION

Multiset, which is considered to be the generalization of a set, is an important concept both in mathematics and in computer science ([11], [12], [21]). If repeated occurrences of any object is allowed in a classical set then the mathematical structure is called a multiset (mset, for short), ([20], [22]). We formalize multiset as a collection of elements, each considered with certain multiplicity. It is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We note that each multiplicity k_i is a positive integer.

In classical set theory, an element can appear only once in a set; it assumes that all mathematical objects occur without repetition. Thus, there is only one number zero, one field of

*Corresponding author
E-mail address: sujoy_ math@yahoo.co.in
Received August 25, 2021

real numbers, etc. So, two mathematical objects are either equal or they are different. But, in the physical world there is enormous repetition. For instance, there are many oxygen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate.

Wayne D. Blizard studied thoroughly about multiset theory, real valued multisets and negative membership of the elements of multisets ([1], [2],[3],[4]). After that, K. P. Girish and S. J. John developed the concepts of multiset topologies, multiset relations, multiset functions, ([13], [14],[15]). Different aspects and applications of multi sets in various directions was studied by different authors from time to time. For a short list of reference one can see ([23], [17], [18], [5], [6], [16], [19]).

In our previous papers ([7], [8], [9], [10]), we have introduced the notions of multi metric space, multi metric topology, convergence in multi metric space, complete multi metric space, multi linear (vector) space and multi normed linear space along with their various properties and several examples and counter examples. An analogue of Cantor's intersection theorem and Banach's fixed point theorem are established in multi set settings. In the present paper, we are going to introduce multi linear operator on multi normed linear space. Continuity and boundedness of multi linear operator, norm of a multi linear operator are studied along with their various properties.

2. PRELIMINARIES

Definition 2.1. [13] A **multi set** M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$ where N represents the set of non-negative integers.
Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We represent the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i in the mset M denoted by $x_i \in^{m_i} M, i = 1, 2, ..., n$. However those elements which are not included in the mset M have zero count.

Example 2.2. [13] Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3. [13] Let M and N be two msets drawn from a set X. Then, the following are defined:

(i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.

(ii) $M \subset N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.

(iii) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ for all $x \in X$.

(iv) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ for al $x \in X$.

(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.

(vi) $P = M \ominus N$ if $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Let M be an mset drawn from a set X. The **support set** of M, denoted by M^*, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an **empty mset** if for all $x \in X, C_M(x) = 0$. The cardinality of an mset M drawn from a set X is denoted by $Card(M)$ or $|M|$ and is given by $Card(M) = \sum_{x \in X} C_M(x)$.

Definition 2.4. [7] **Multi point:** Let M be a multi set over a universal set X. Then a multi point of M is defined by a mapping $P_x^k : X \longrightarrow \mathbb{N}$ such that $P_x^k(x) = k$ where $k \leq C_M(x)$.

x and k will be referred to as the **base** and the **multiplicity** of the multi point P_x^k respectively.

Collection of all multi points of an mset M is denoted by M_{pt}.

Definition 2.5. [7] The **mset generated by a collection** B of multi points is denoted by $MS(B)$ and is defined by $C_{MS(B)}(x) = Sup\{k : P_x^k \in B\}$.

An mset can be generated from the collection of its multi points. If M_{pt} denotes the collection of all multi points of M, then obviously $C_M(x) = Sup\{k : P_x^k \in M_{pt}\}$ and hence $M = MS(M_{pt})$.

Definition 2.6. [7] (i) The **elementary union** between two collections of multi points C and D is denoted by $C \sqcup D$ and is defined as $C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D \;\; and \;\; k = max\{l, m\}\}$.

(ii) The **elementary intersection** between two collections of multi points C and D is denoted by $C \sqcap D$ and is defined as $C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D \;\; and \;\; k = min\{l, m\}\}$.

(iii) For two collections of multi points C and D, C is said to be an **elementary subset** of D, denoted by $C \sqsubset D$, iff $P_x^l \in C \Rightarrow \exists m \geq l$ such that $P_x^m \in D$.

Definition 2.7. [7] Let $m\mathbb{R}^+$ denotes the multi set over $\mathbb{R}^+$ (set of non-negative real numbers) having multiplicity of each element equal to w, $w \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called **non-negative multi real points**.

Definition 2.8. [7] Let P_a^i and P_b^j be two multi real points of $m\mathbb{R}^+$. We define $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.

Definition 2.9. [7] **(Addition of multi real points)** We define $P_a^i + P_b^j = P_{a+b}^k$ where $k = Max\{i, j\}, P_a^i, P_b^j \in (m\mathbb{R}^+)_{pt}$.

Definition 2.10. [7] **(Multiplication of multi real points)** We define multiplication of two multi real points in $m\mathbb{R}^+$ as follows:

$P_a^i \times P_b^j = P_0^1$,if either P_a^i or P_b^j equal to P_0^1.

$= P_{ab}^k$, otherwise where k =Max $\{i, j\}$.

Definition 2.11. [7] **Multi Metric:** Let $d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$($M$ being a multi set over a Universal set X having multiplicity of any element atmost equal to w) be a mapping which satisfy the following:

(M1) $d(P_x^l, P_y^m) \geq P_0^1, \forall P_x^l, P_y^m, \in M_{pt}$

(M2) $d(P_x^l, P_y^m) = P_0^1$ iff $P_x^l = P_y^m, \forall P_x^l, P_y^m \in M_{pt}$

(M3) $d(P_x^l, P_y^m) = d(P_y^m, P_x^l), \forall P_x^l, P_y^m \in M_{pt}$

(M4) $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n), \forall P_x^l, P_y^m, P_z^n \in M_{pt}$.

(M5) For $l \neq m$, $d(P_x^l, P_y^m) = P_0^k, \Leftrightarrow x = y$ and $k =$ Max$\{l, m\}$.

Then d is said to be a multi metric on M and (M, d) is called a Multi metric (or an M-metric) space.

Example 2.12. [7] Let M be a multi set over X having multiplicity of any element atmost equal to w. We define $d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that

$d(P_x^l, P_y^m) = P_0^1$ if $P_x^l = P_y^m$

$= P_0^{Max\{l,m\}}$ if $x = y$ and $l \neq m$

$= P_1^j$ if $x \neq y \; \forall \; P_x^l, P_y^m \in M_{pt}$, [$1 \leq j \leq w$ *is some fixed positive integer*].

Then d is an M-metric on M.

Definition 2.13. [10] **Multi vector space:** Let V be vector space over a field K. A multiset X over V is said to be a multi vector space or a multi linear space or Mvector space of V over K if every element of X has the same multiplicity and the support X^* of X is a subspace of V. The multiplicity of every element of X will be denoted by w_X.

Example 2.14. [10] Let $\mathbb{R}^3$ be the Euclidean 3-dimensional pace over $\mathbb{R}$. Let $X = \{5/(a,b,0) : a,b \in \mathbb{R}\}$. Then X is a multi vector space of $\mathbb{R}^3$ over $\mathbb{R}$.

Definition 2.15. [10] **Multivectors:** Let X be an Mvector space over a vector space V_k. Then every multi point of X ie. every element of X_{pt} will be called a multivector of X.

Definition 2.16. [10] **Multi scalar field**: Let K be a field. Then a multi set L over K is called a multi scalar field or Mscalar field if every element of K has the same multiplicity and the support L^* of L is a subfield of K.

Multi points of L will be referred to as **multi scalars** or **Mscalars** of L.

Multiplicity of each element of L will be denoted by w_L.

Example 2.17. [10] In Example 2.14, $P^1_{(1,1,0)}, P^2_{(1,1,0)}, P^4_{(1,5,0)}$ etc. are Mvectors of the given Mvector space.

Definition 2.18. [10] Let X be an Mvector space over V_K. Then an Mvector P^k_x of X will be called a null Mvector if its base $x = \theta(\theta$ being the null vector of X^* ie $V_K)$.

It will be denoted by Θ^k. An Mvector P^k_x will be called non null if $x \neq \theta$.

Definition 2.19. [10] Let X be an Mvector space over a vector space V_K, L be an Mscalar field over K such that $w_L \leq w_X$, $P^l_x, P^m_y \in X_{pt}$ and $P^i_a \in L_{pt}$.

Then we define $P^l_x + P^m_y = P^1_\theta \; iff \; x = -y \; and \; l = m$

$$= P^{l \vee m}_{x+y} \;\; otherwise.$$

and $P^i_a . P^l_x = P^1_\theta \; iff \; P^i_a = P^1_0 \; or \; P^l_x = P^1_\theta$

$= P^{i \vee l}_{ax} \; otherwise$, where 0 is the null element of K.

Definition 2.20. [10] **Multi linear combination:** Let X be an Mvector space over a vector space V_K and L be an Mscalar field over K such that $w_L \leq w_X$. Then an Mvector $P^l_x \in X_{pt}$ is said to be a multi linear combination or Mlinear combination of the Mvectors $P^{l_1}_{x_1}, P^{l_2}_{x_2}, \ldots\ldots\ldots\ldots, P^{l_n}_{x_n} \in X_{pt}$ if P^l_x can be expressed as $P^l_x = P^{i_1}_{a_1}.P^{l_1}_{x_1} + P^{i_2}_{a_2}.P^{l_2}_{x_2} + \ldots\ldots\ldots\ldots\ldots + P^{i_n}_{a_n}.P^{l_n}_{x_n}$ for some Mscalars $P^{i_1}_{a_1}, P^{i_2}_{a_2}, \ldots\ldots\ldots\ldots, P^{i_n}_{a_n} \in L_{pt}$.

Definition 2.21. [10] **Multi linearly dependent and multi linearly independent:** Let X be an Mvector space over a vector space V_K and L be an Mscalar field over K such that $w_L \leq w_X$. Then a finite collection of Mvectors $\{P^{l_1}_{x_1}, P^{l_2}_{x_2}, \ldots\ldots\ldots\ldots, P^{l_n}_{x_n}\}$ of X is said to be multi linearly dependent or Mlinearly dependent or ML.D if there exist Mscalars $P^{i_1}_{a_1}, P^{i_2}_{a_2}, \ldots\ldots\ldots\ldots, P^{i_n}_{a_n} \in L_{pt}$

with $a_i \neq 0$ for some $i = 1, 2, ..., n$ such that $P^{i_1}_{a_1}.P^{l_1}_{x_1} + P^{i_2}_{a_2}.P^{l_2}_{x_2} + ... + P^{i_n}_{a_n}.P^{l_n}_{x_n} = \Theta^l$. The collection of Mvectors $\{P^{l_1}_{x_1}, P^{l_2}_{x_2}, ..., P^{l_n}_{x_n}\}$ of X is said to be multi linearly independent or Mlinearly independent or ML.Id if the relation $P^{i_1}_{a_1}.P^{l_1}_{x_1} + P^{i_2}_{a_2}.P^{l_2}_{x_2} + ... + P^{i_n}_{a_n}.P^{l_n}_{x_n} = \Theta^l$ holds only when $a_i = 0 \; \forall \; i = 1, 2, ..., n$.

An arbitrary multiset $G \subset X$ is said to be ML.D if there exists a finite collection of Mvectors of G, which is ML.D. An arbitrary multiset $G \subset X$ is Ml.Id if it is not ML.D.

Definition 2.22. [10] **Linear span:** Let X be an Mvector space over a vector space V_K, L be an Mscalar field over K such that $w_L \leq w_X$ and $S = \{P^{l_1}_{x_1}, P^{l_2}_{x_2}, ..., ^{l_n}_{x_n}\}$ be a collection of Mvectors of X. Then the linear span of S denoted by LS(S) is defined as

LS(S)= $\{P^{i_1}_{a_1}.P^{l_1}_{x_1} + P^{i_2}_{a_2}.P^{l_2}_{x_2} + ... + P^{i_n}_{a_n}.P^{l_n}_{x_n} : P^{i_1}_{a_1}, P^{i_2}_{a_2}, ..., P^{i_n}_{a_n} \in L_{pt}\}$.

$MS[LS(S)]$ will be referred to as the multi linear span or Mlinear span of S.

Definition 2.23. [10] An Mvector space X over V_K is said to be finite dimensional if there is a finite set of ML.Id Mvectors in X that also generates M i.e., there exists a finite set $S = \{P^{l_1}_{x_1}, P^{l_2}_{x_2}, ..., ^{l_n}_{x_n}\}$ of Mvectors of X which is ML.Id and MS[LS(S)] = X.

The number of elements of such a set S is called the dimension of X and is denoted by $Dim(X)$.

Notation: Through out this paper we shall consider **V** as a vector space over $\mathbb{R}/\mathbb{C}$; **X** as an Mvector space over V_K with $w_X \leq w$ (w being the multiplicity of every element of $m\mathbb{R}^+$) and **L** as an Mscalar field over K with support $L^* = K$ and $w_l \leq w_X$.

Definition 2.24. [10] A mapping $\| \; \| : X_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ will be called a multi norm or Nmorm on X if it satisfies the following:

(N1) $\|P^l_x\| \geq P^1_0 \; \forall \; P^l_x \in X_{pt}$.

(N2) $\|P^l_x\| = P^k_0$ iff $x = \theta$ and $l = k$.

(N3) $\|P^i_a P^l_x\| = P^i_{|a|} \|P^l_x\| \; \forall \; P^i_a \in L_{pt}, P^l_x \in X_{pt}$.

(N4) $\|P^l_x + P^m_y\| \leq \|P^l_x\| + \|P^m_y\| \; \forall \; P^l_x, P^m_y \in X_{pt}$.

An Mvector space X with an Mnorm $\| \; \|$ on X is called a multi normed linear space or Mnormed linear space and is denoted by $(X, \| \; \|)$. (N1), (N2), (N3) and (N4) are called norms or axioms.

Example 2.25. [10] Let $(V, \| \ \|)$ be a normed linear space over $K = \mathbb{R}/\mathbb{C}$ and X be an Mvector space over V with $w_X = w$. Let $\| \ \|_m : X_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that $\|P_x^l\|_m = P_{\|x\|}^l \ \forall \ P_x^l \in X_{pt}$. Then $\| \ \|_m$ is an Mnorm over X and $(X, \| \ \|_m)$ is an Mnormed linear space.

Note 2.26. [10] Corresponding to every normed linear space, there exists a Mnormed linear space.

Theorem 2.27. [10] Let $(X, \| \ \|)$ be an Mnormed linear space over a vector space V_K. Then $d : X_{pt} \times X_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ defined by $d(P_x^l, P_y^m) = \|P_x^l - P_y^m\| \ \forall \ P_x^l, P_y^m \in X_{pt}$ is a multi metric on X.

Definition 2.28. [10] **Mnorm subspace:** Let $(X, \| \ \|_X)$ be an Mnormed linear space over V_K and $Y \subset X$ is an Msubspace of X. Then $\| \ \|_Y : Y_{pt} \longrightarrow (\mathbb{R}^+)_{pt}$ defined by $\|P_x^l\|_Y = \|P_x^l\|_X \ \forall \ P_x^l \in Y_{pt}$ is an Mnorm on Y. This Mnorm is known as the relative Mnorm on Y induced by $\| \ \|_X$. The Mnormed linear space $(Y, \| \ \|_Y)$ is called a an Mnorm subspace or simply an Msubspace of the Mnormed linear space $(X, \| \ \|_X)$.

Definition 2.29. [10] Let $(X, \| \ \|)$ be an Mnormed linear space over a vector space V_K and $r > 0$. We define the following:

(i) $B(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| < P_r^1\}$ as an **open ball** with center P_x^l and radius P_r^1.

(ii) $\overline{B}(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| \leq P_r^1\}$ as a **closed ball** with center P_x^l and radius P_r^1.

(iii) $S(P_x^l, P_r^1) = \{P_y^m \in X_{pt} : \|P_x^l - P_y^m\| = P_r^1\}$ as a **sphere** with center P_x^l and radius P_r^1.

$MS[B(P_x^l, P_r^1)], MS[\overline{B}(P_x^l, P_r^1)]$ and $S(P_x^l, P_r^1)$ are respectively called an Mopen ball, an Mclosed ball and an Msphere with center P_x^l and radius P_r^1.

Definition 2.30. [10] **Convergence of sequence:** A sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \| \ \|)$ over V_K is said to be convergent and converges to an Mvector P_x^l if $\|P_{x_n}^{l_n} - P_x^l\| \longrightarrow P_0^1$ as $n \longrightarrow \infty$ which means, for any $\varepsilon > 0$, $\exists \ n_0 \in \mathbb{N}$ such that $\|P_{x_n}^{l_n} - P_x^l\| < P_\varepsilon^1 \ \forall \ n \geq n_0$ ie. $n \geq n_0 \ \Rightarrow P_{x_n}^{l_n} \in B(P_x^l, P_\varepsilon^1)$. We denote this by $P_{x_n}^{l_n} \longrightarrow P_x^l$ as $n \longrightarrow \infty$ or by $\lim_{n \to \infty} P_{x_n}^{l_n} = P_x^l$. P_x^l is said to be the limit of $\{P_{x_n}^{l_n}\}$ as $n \longrightarrow \infty$.

Definition 2.31. [10] **Boundedness:** (i) In an Mnormed linear space $(X, \| \ \|)$, a multi subset $Y \subset X$ is said to be bounded if $\exists \ r > 0$ such that $\|P_x^l\| < P_r^1 \ \forall \ P_x^l \in Y_{pt}$.

(ii) If a sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \| \ \|)$ is bounded if $\exists \ r > 0$ such that $\|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| < P_r^1 \ \forall \ m, n \in \mathbb{N}$.

Definition 2.32. [10] **Cauchy sequence:** If a sequence $\{P_{x_n}^{l_n}\}$ of Mvectors in an Mnormed linear space $(X, \| \|)$ is said to be Cauchy if for any $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| < P_{\varepsilon}^{1}$ $\forall\, m,n \geq n_0$ ie. $\|P_{x_n}^{l_n} - P_{x_m}^{l_m}\| \longrightarrow P_0^1$ as $m,n \longrightarrow \infty$.

Definition 2.33. [10] **Completeness:** An Mnormed linear space $(X, \| \|)$ is said to be complete if every Cauchy sequence of Mvectors in $(X, \| \|)$ converges to an Mvector of X.

Theorem 2.34. [10] In an Mnormed linear space $(X, \| \|)$, if $P_{x_n}^{l_n} \longrightarrow P_x^l$ and $P_{y_n}^{k_n} \longrightarrow P_y^k$, then $P_{x_n}^{l_n} + P_{y_n}^{k_n} \longrightarrow P_x^l + P_y^k$.

Theorem 2.35. [10] In an Mnormed linear space $(X, \| \|)$ over a vector space V_K, if $\{P_{x_n}^{l_n}\}$ be a sequence of Mvectors such that $P_{x_n}^{l_n} \longrightarrow P_x^l$ and $\{P_{a_n}^{k_n}\}$ be a sequence of Mscalars such that $P_{a_n}^{k_n} \longrightarrow P_a^k$, then $P_{a_n}^{k_n}.P_{x_n}^{l_n} \longrightarrow P_a^k.P_x^l$.

Theorem 2.36. [10] In an Mnormed linear space $(X, \| \|)$ over a vector space V_K, if $\{P_{x_n}^{l_n}\}, \{P_{y_n}^{m_n}\}$ are Cauchy sequences of Mvectors and $\{P_{a_n}^{k_n}\}$ is a Cauchy sequence of Mscalars, then $\{P_{x_n}^{l_n} + P_{y_n}^{m_n}\}, \{P_{a_n}^{k_n}.P_{x_n}^{l_n}\}$ are Cauchy sequences of Mvectors.

Theorem 2.37. [10] If M be an Msubspace of an Mnormed linear space $(X, \| \|)$, then $\overline{M}$ is also an Msubspace of $(X, \| \|)$.

3. Multi Linear Operator on Multi Normed Linear Space

Definition 3.1. Multi linear operator: Let X and Y be Mnormed linear spaces on mvector spaces V_K and W_K respectively where $K = \mathbb{R}/\mathbb{C}$, L be an Mscalar field over K with $L^* = K$ and $w_L \leq w_X, w_Y$. Then $T : X_{pt} \longrightarrow Y_{pt}$ is said to be a multi linear operator if

(L1) T is additive i.e., $T(P_x^l + P_u^m) = T(P_x^l) + T(P_u^m)$, $\forall\, P_x^l, P_u^m \in X_{pt}$.

(L2) T is homogeneous i.e., $T(P_a^i.P_x^l) = P_a^i.T(P_x^l)$, $\forall\, P_a^i \in L_{pt}, P_x^l \in X_{pt}$.

The properties (L1) and (L2) can be put in combined form as

$T(P_a^i.P_x^l + P_b^j.P_u^m) = P_a^i.T(P_x^l) + P_b^j.T(P_x^m)$, $\forall\, P_a^i, P_b^j \in L_{pt}$ and $P_x^l, P_u^m \in X_{pt}$.

Example 3.2. (1) The identity operator $I : X_{pt} \longrightarrow X_{pt}$ defined as $I(P_x^l) = P_x^l$, $\forall\, P_x^l \in X_{pt}$ is a multi linear operator.

(2) The null operator $\overline{O} : X_{pt} \longrightarrow X_{pt}$ defined as $\overline{O}(P_x^l) = P_\theta^1$, $\forall\, P_x^l \in X_{pt}$ is a multi linear operator where θ is the null element in V_K.

(3) Let $P_a^i \in L_{pt}$. Define $T(P_x^l) = P_a^i.P_x^l = P_{ax}^{i \vee l}$, $\forall\, P_x^l \in X_{pt}$. Then $\forall\, P_x^l, P_u^m \in X_{pt}$, $T(P_x^l + P_u^m) =$

$T(P^{l\vee m}_{x+u}) = P^{i\vee(l\vee m)}_{a(x+u)} = P^{(i\vee l)\vee(i\vee m)}_{ax+au} = P^{i\vee l}_{ax} + P^{i\vee m}_{au} = T(P^l_x) + T(P^m_u)$.

For any $P^j_b \in L_{pt}$, $T(P^j_b.P^l_x) = T(P^{j\vee l}_{bx}) = P^{i\vee(j\vee l)}_{a(bx)} = P^{j\vee(i\vee l)}_{b(ax)} = P^j_b.P^{i\vee l}_{ax} = P^j_b.T(P^l_x)$.

Theorem 3.3. Let X and Y be Mnormed linear spaces over V_K and L be an Mscalar field over K. If $T : X_{pt} \longrightarrow Y_{pt}$ is a Multi linear operator, then

(1) $T(P^l_x - P^m_u) = T(P^l_x) - T(P^m_u)$, $\forall\, P^l_x, P^m_u \in X_{pt}$.

(2) $T(P^k_{\theta_X}) = P^1_{\theta_Y}$ where θ_X and θ_Y are null elements of X and Y respectively.

(3) $T(-P^l_x) = -T(P^l_x)$.

(4) $T(\sum_{r=1}^n P^{i_r}_{a_r}.P^{l_r}_{x_r}) = \sum_{r=1}^n P^{i_r}_{a_r}.T(P^{l_r}_{x_r})$.

Proof. (1) Let $T(P^m_u) = P^n_y \in Y_{pt}$. Then $T(P^l_x - P^m_u) = T(P^l_x + P^m_{-u}) = T(P^l_x + P^1_{-1}.P^m_u) = T(P^l_x) + T(P^1_{-1}.P^m_u) = T(P^l_x) + P^1_{-1}.T(P^m_u) = T(P^l_x) + P^1_{-1}.P^n_y = T(P^l_x) + P^1_{-1}.P^n_y = T(P^l_x) - T(P^m_u)$.

(2) $P^k_{\theta_X} + P^k_{\theta_X} = P^k_{\theta_X} \implies T(P^k_{\theta_X} + P^k_{\theta_X}) = T(P^k_{\theta_X}) \implies T(P^k_{\theta_X}) + T(P^k_{\theta_X}) = T(P^k_{\theta_X}) \implies T(P^k_{\theta_X}) + T(P^k_{\theta_X}) - T(P^k_{\theta_X}) = T(P^k_{\theta_X}) - T(P^k_{\theta_X}) \implies T(P^k_{\theta_X}) + P^1_{\theta_Y} = P^1_{\theta_Y} \implies T(P^k_{\theta_X}) = P^1_{\theta_Y}$.

(3) $P^l_x - P^l_x = P^1_\theta \implies T(P^l_x - P^l_x) = T(P^1_\theta) \implies T(P^l_x) + T(-P^l_x) = P^1_\theta \implies T(-P^l_x) = -T(P^l_x)$.

(4) We shall prove this by method of induction. For $n = 1$, the result is obvious. Let the result be true for $n = k$ ie. $T(\sum_{r=1}^k P^{i_r}_{a_r}.P^{l_r}_{x_r}) = \sum_{r=1}^k P^{i_r}_{a_r}.T(P^{l_r}_{x_r})$.

Now $T(\sum_{r=1}^{k+1} P^{i_r}_{a_r}.P^{l_r}_{x_r}) = T(\sum_{r=1}^k P^{i_r}_{a_r}.P^{l_r}_{x_r} + P^{i_{k+1}}_{a_{k+1}}.P^{l_{k+1}}_{x_{k+1}}) = T(\sum_{r=1}^k P^{i_r}_{a_r}.P^{l_r}_{x_r}) + T(P^{i_{k+1}}_{a_{k+1}}.P^{l_{k+1}}_{x_{k+1}}) = \sum_{r=1}^k P^{i_r}_{a_r}.T(P^{l_r}_{x_r}) + P^{i_{k+1}}_{a_{k+1}}.T(P^{l_{k+1}}_{x_{k+1}}) = T(\sum_{r=1}^{k+1} P^{i_r}_{a_r}.P^{l_r}_{x_r}) = \sum_{r=1}^{k+1} P^{i_r}_{a_r}.T(P^{l_r}_{x_r})$.

Definition 3.4. A multi linear operator $T : X_{pt} \longrightarrow Y_{pt}$ is said to be **continuous** at $P^{l_0}_{x_0} \in X_{pt}$ if for every sequence $\{P^{l_n}_{x_n}\}$ in Xpt with $P^{l_n}_{x_n} \longrightarrow P^{l_0}_{x_0}$ as $n \longrightarrow \infty$, we have $T(P^{l_n}_{x_n}) \longrightarrow T(P^{l_0}_{x_0})$ as $n \longrightarrow \infty$ ie. $\|T(P^{l_n}_{x_n}) - T(P^{l_0}_{x_0})\| \longrightarrow P^1_0$ as $n \longrightarrow \infty$.

If T is continuous at every point of X_{pt}, then T is said to be a **continuous multi linear operator**.

Example 3.5. The Multi linear operators given in Example 3.2 are continuous. (1) and (2) are obviously continuous. For (3), since $P^{l_n}_{x_n} \longrightarrow P^{l_0}_{x_0}$ as $n \longrightarrow \infty$, for $0 < \eta < \varepsilon$, $\exists\, n_0 \in \mathbb{N}$ such that $\|P^{l_n}_{x_n} - P^{l_0}_{x_0}\|_X < P^1_{\eta/|a|}$ $\forall\, n \geq n_0$ $[assuming |a| \neq 0]$.

Now $\forall\, n \geq n_0, \|T(P^{l_n}_{x_n}) - T(P^{l_0}_{x_0})\| = \|P^i_a.P^{l_n}_{x_n} - P^i_a.P^{l_0}_{x_0}\| = \|P^i_a.(P^{l_n}_{x_n} - P^{l_0}_{x_0})\| = P^i_{|a|}.\|(P^{l_n}_{x_n} - P^{l_0}_{x_0})\| < P^i_{|a|}.P^1_{\eta/|a|} = P^1_\eta < P^1_\varepsilon \implies T(P^{l_n}_{x_n}) \longrightarrow T(P^{l_0}_{x_0})$ as $n \longrightarrow \infty \implies$ T is continuous.

If $a = 0$, $\|T(P^{l_n}_{x_n}) - T(P^{l_0}_{x_0})\| = P^k_0$ [for some $1 \leq k \leq w_Y$] $< P^1_\varepsilon$.

Theorem 3.6. Let $T : X_{pt} \longrightarrow Y_{pt}$ be a multi linear operator. If T is continuous at some $P^{l_0}_{x_0} \in X_{pt}$, then T is continuous at every element of X_{pt}.

Proof. Let $P_x^l \in X_{pt}$ be arbitrary, $\{P_{x_n}^{l_n}\}$ be a sequence in X_{pt} converging to P_x^l and $P_{u_n}^{k_n} = P_{x_n}^{l_n} - P_x^l + P_{x_0}^{l_0}$ $\forall\, n \in \mathbb{N}$. Then $P_{u_n}^{k_n}$ is a sequence in X_{pt} converging to $P_{x_0}^{l_0}$.
$\therefore$ by continuity of T at $P_{x_0}^{l_0}$, $T(P_{u_n}^{k_n}) \longrightarrow T(P_{x_0}^{l_0})$ as $n \longrightarrow \infty \implies T(P_{x_n}^{l_n} - P_x^l + P_{x_0}^{l_0}) \longrightarrow T(P_{x_0}^{l_0})$ as $n \longrightarrow \infty \implies T(P_{x_n}^{l_n}) - T(P_x^l) + T(P_{x_0}^{l_0}) \longrightarrow T(P_{x_0}^{l_0})$ as $n \longrightarrow \infty \implies T(P_{x_n}^{l_n}) - T(P_x^l) \longrightarrow P_\theta^1$ as$n \longrightarrow \infty \implies T(P_{x_n}^{l_n}) \longrightarrow T(P_x^l) \implies$ T is continuous at P_x^l. Since $P_x^l \in X_{pt}$ is arbitrary, the result follows.

Definition 3.7. A multi linear operator $T : X_{pt} \longrightarrow Y_{pt}$ is said to be **bounded** if $\exists\, r > 0$ such that $\|T(P_x^l)\| \leq P_r^1 \|P_x^l\|$ $\forall\, P_x^l \in X_{pt}$.

Theorem 3.8. Let $T : X_{pt} \longrightarrow Y_{pt}$ be a multi linear operator. If T is bounded, then T is continuous.

Proof. Let $P_{x_0}^{l_0} \in X_{pt}$ and $\{P_{x_n}^{l_n}\}$ be a sequence in X_{pt} converging to $P_{x_0}^{l_0}$. Also since T is bounded, $\exists\, r > 0$ such that $\|T(P_x^l)\| \leq P_r^1 \|P_x^l\|$ $\forall\, P_x^l \in X_{pt}$. Let $\varepsilon > 0$ be arbitrary. Since $P_{x_n}^{l_n} \longrightarrow P_{x_0}^{l_0}$, $\exists\, n_0 \in \mathbb{N}$ such that $\|P_{x_n}^{l_n} - P_{x_0}^{l_0}\| < P_{\varepsilon/r}^1$ $\forall\, n \geq n_0 \implies \|T(P_{x_n}^{l_n}) - T(P_{x_0}^{l_0})\| = \|T(P_{x_n}^{l_n} - P_{x_0}^{l_0})\| \leq P_r^1 \|P_{x_n}^{l_n} - P_{x_0}^{l_0}\| < P_r^1 . P_{\varepsilon/r}^1 = P_\varepsilon^1$ $\forall\, n \geq n_0 \implies T(P_{x_n}^{l_n}) \longrightarrow T(P_{x_0}^{l_0})$.

Theorem 3.9. Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed linear spaces over $K = \mathbb{R}/\mathbb{C}$ and $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ are two multi normed linear spaces over $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ respectively. Let $T : V \longrightarrow W$ be a linear operator. Then $T_m : X_{pt} \longrightarrow Y_{pt}$ such that $T_m(P_x^l) = P_{T(x)}^l$ is a multi linear operator.

Proof. For $P_a^i, P_b^j \in L_{pt}$ [L_{pt} being an Mscalar field over K such that $w_L \leq w_X, w_Y$] and $P_x^l, P_u^m \in X_{pt}$, $T_m(P_a^i . P_x^l + P_b^j . P_u^m) = T_m\{(P_{ax+bu}^{(i\vee l)\vee(j\vee m)})\} = P_{T(ax+bu)}^{(i\vee l)\vee(j\vee m)} = P_{aT(x)+bT(u)}^{(i\vee l)\vee(j\vee m)} = P_a^i . P_{T(x)}^l + P_b^j . P_{T(u)}^m$ $= P_a^i . T_m(P_x^l) + P_b^j . T_m(P_u^m)$.

Theorem 3.10. Let $T_m : X_{pt} \longrightarrow Y_{pt}$ be a multi linear operator on X where $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed linear spaces over $K = \mathbb{R}/\mathbb{C}$; $(X, \|\ \|_X)$, $(Y, \|\ \|_Y)$ are two multi normed linear spaces over $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ respectively and $X^* = V$. We denote $\widetilde{b}(P_x^l)$ as the base of the multi vector $P_x^l \in X_{pt}\ or\ Y_{pt}$. Also let $T_l : V \longrightarrow W$ such that $T_l(x) = \widetilde{b}\{T_m(P_x^l)\}$ $\forall\, x \in V$. Then $\{T_l : 1 \leq l \leq w_X\}$ is a family of normed linear operators on $(V, \|\ \|_V)$.
If we define $T_m^* : X_{pt} \longrightarrow Y_{pt}$ *such that* $T_m^*(P_x^l) = P_{T_l(x)}^{\widetilde{m}\{T_m(P_x^l)\}}$ [$\widetilde{m}P_x^l$ being the multiplicity of $P_x^l \in X_{pt} or\ Y_{pt}$], then T_m^* is a multi normed linear operator on X with $T_m^* = T_m$.

Proof. Clearly $\forall\, l,\ T_l$ is well defined. Now for $x,u\in V$ and $a,b\in K$, $T_l(ax+bu)=\widetilde{b}[T_m(P^l_{ax+bu})]$ $=\widetilde{b}[T_m(P^l_a.P^l_x+P^l_b.P^l_u]=\widetilde{b}[P^l_a.T_m(P^l_x)+P^l_b.T_m(P^l_u)]=\widetilde{b}(P^l_a.P^i_y+P^l_b.P^j_v)$ [where $T_m(P^l_x)=P^i_y$ and $T_m(P^l_u)=P^i_y$]$=\widetilde{b}(P^{i\vee j\vee l}_{ay+bv})=ay+bv=a.\widetilde{b}(T_m(P^l_x))+b.\widetilde{b}(T_m(P^l_u))=a.T_l(x)+b.T_l(u)$.

The second part is obvious.

Theorem 3.11. $T: X_{pt}\longrightarrow Y_{pt}$ is continuous at a point of $X_{pt}\Longrightarrow$ T is continuous everywhere in X_{pt}.

Proof. Let T be continuous at $P^{l_0}_{x_0}\in X_{pt},\ P^l_x\in X_{pt}$ be arbitrary and $\{P^{l_n}_{x_n}\}$ be a sequence in X_{pt} such that $P^{l_n}_{x_n}\longrightarrow P^l_x$. If $\forall\, n\in\mathbb{N},\ P^{l_n}_{x_n}-P^l_x+P^{l_0}_{x_0}=P^{k_n}_{u_n}$, then $\{P^{k_n}_{u_n}\}$ is a sequence in X_{pt} converging to $P^{l_0}_{x_0}$.

So by continuity of T at $P^{l_0}_{x_0}$, for any $\varepsilon>0$, $\exists\, m\in\mathbb{N}$ such that $\|T(P^{k_n}_{u_n})-T(P^{l_0}_{x_0})\|=\|T(P^{l_n}_{x_n}-P^l_x+P^{l_0}_{x_0})-T(P^{l_0}_{x_0})\|=\|T(P^{l_n}_{x_n})-T(P^l_x)+T(P^{l_0}_{x_0})-T(P^{l_0}_{x_0})\|=\|T(P^{l_n}_{x_n})-T(P^l_x))\|<P^1_\varepsilon\ \forall\ n\geq m\implies T(P^{l_n}_{x_n})\longrightarrow T(P^l_x)\implies$ T is continuous at P^l_x.

Theorem 3.12. $T: X_{pt}\longrightarrow Y_{pt}$ is continuous $\Longrightarrow$ T is bounded.

Proof. If possible let T be not bounded. Then $\forall\, n\in\mathbb{N},\ \exists\, P^{l_n}_{x_n}\in X_{pt}$ such that $T(P^{l_n}_{x_n})>P^1_n\|P^{l_n}_{x_n}\|$. Let $\|P^{l_n}_{x_n}\|=P^{i_n}_{a_n}\ \forall n\in\mathbb{N}$. Then $a_n>0\forall n\in\mathbb{N}$ since $a_n=0$ for some $n=m\in\mathbb{N}\implies\|P^{l_m}_{x_m}\|=P^{i_m}_0\Longrightarrow x_m=\theta\ and\ l_m=i_m\Longrightarrow\|T(P^{l_m}_{x_m})\|=\|T(P^{i_m}_\theta)\|\ngtr P^1_m\|P^{l_m}_{x_m}\|=P^{i_m}_0$. So $\forall\, n\in\mathbb{N},\ \|P^{l_n}_{x_n}\|>P^{i_n}_{n.a_n}$ and $a_n>0$.

We consider $\forall\ n\in\mathbb{N},\ P^{k_n}_{u_n}=P^{k_n}_{\frac{x_n}{na_n}}$ where $k_n=l_n\vee i_n$. Then $\|P^{k_n}_{u_n}\|=\|P^{l_n}_{x_n}.P^{i_n}_{\frac{1}{na_n}}\|=P^{i_n}_{\frac{1}{na_n}}.\|P^{l_n}_{x_n}\|=P^{i_n}_{\frac{1}{na_n}}.P^{i_n}_{a_n}=P^{i_n}_{\frac{1}{n}}\longrightarrow P^1_0$ as $n\longrightarrow\infty\implies\|P^{k_n}_{u_n}-P^1_\theta\|\longrightarrow P^1_0$. But $\|T(P^{k_n}_{u_n})\|=\|T(P^{k_n}_{\frac{x_n}{na_n}})\|=\|T(P^{l_n}_{x_n}.P^{i_n}_{\frac{1}{na_n}})\|=\|P^{i_n}_{\frac{1}{na_n}}.T(P^{l_n}_{x_n})\|=P^{i_n}_{\frac{1}{na_n}}.\|T(P^{l_n}_{x_n})\|>P^{i_n}_{\frac{1}{na_n}}.P^{i_n}_{na_n}=P^{i_n}_1$, which contradicts the fact that T is continuous.

Lemma 3.13. Let $X,\|\ \|$ be a multi normed linear space over V_K and $P^{l_1}_{x_1},P^{l_2}_{x_2},........P^{l_n}_{x_n}$ be linearly independent mvectors of X. Then for any set of mscalars $P^{i_1}_{a_1},P^{i_2}_{a_2},..........P^{i_n}_{a_n}$, $\exists\ c>0$ such that $\|P_{a_1}.P^{l_1}_{x_1}+P^{i_2}_{a_2}.P^{l_2}_{x_2}+..........,P^{i_n}_{a_n}.P^{l_n}_{x_n}\|\geq P^1_c(P^{i_1}_{|a_1|}+P^{i_2}_{|a_2|}+..........+P^{i_n}_{|a_n|})$.

Proof. Let $s=|a_1|+|a_2|+..........+|a_n|$. If $s=0$, then $a_i=0\ \forall\ i=1,2,....,n$ and the result holds.

Let $s>0$. Now we have to prove $\exists\ c>0$ such that $\|P^{p_n}_{a_1.x_1+a_2.x_2+......+a_n.x_n}\|\geq P^1_c.P^{j_n}_s$ where $p_n=\ Max\ \{i_1,i_2,.....,i_n,l_1,l_2,......,l_n\}$, $j_n=\ Max\ \{i_1,i_2,.....,i_n\}$ i.e., we have to prove $\|P^{p_n}_{\frac{a_1.x_1+a_2.x_2+......+a_n.x_n}{s}}\|\geq P^{j_n}_c$ ie. $\|P^{p_n}_{\frac{b_1.x_1+b_2.x_2+......+b_n.x_n}{s}}\|\geq P^{j_n}_c$.........(1), where $b_i = \frac{a_i}{s}\ \forall\ i = 1,2,....,n$, so that $\sum_{i=1}^{n} b_i = 1$.

It is now sufficient to prove the existance of $c > 0$ such that (1) is true for any set of mscalars $P_{b_1}^{i_1}, P_{b_2}^{i_2}, ...P_{b_n}^{i_n}$ with $\sum_{i=1}^{n} b_i = 1$.

If possible, let this is not true ie. for every $m \in \mathbb{N}$ there is a sequence $\{y_m\}$ in V_k such that $y_m = b_1^{(m)}x_1 + b_2^{(m)}x_2 + ... + b_n^{(m)}x_n$ with $\sum_{i=1}^{n} |b_i^{(m)}| = 1$ and $\|P_{y_m}^{p_n}\| < P_{\frac{1}{m}}^{1}\ \forall\ m \in \mathbb{N}$. Since $\frac{1}{m} \longrightarrow 0\ as\ m \longrightarrow \infty$, it follows that $P_{y_m}^{p_n} \longrightarrow P_{\theta}^{p_n}\ as\ m \longrightarrow \infty$. Since $\sum_{i=1}^{n} |b_i^{(m)}| = 1\ \forall\ m \in \mathbb{N}$, we have $|b_i^{(m)}| \leq 1\ \forall\ i = 1,2,....,n\ and\ m \in \mathbb{N}$. Hence for each fixed $i = 1,2,....,n$, the sequence $\{b_i^{(m)}\} = \{b_i^{(1)}, b_i^{(2)}, ..., b_i^{(m)}, ...\}$ is bounded. So by Bolzano Weierstrass theorem, $\{b_1^{(m)}\}$ has a subsequence converging to c_1, and let $\{y_{1,m}\}$ be the corresponding subsequence of $\{y_m\}$. By the same reason, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$, say for which the corresponding subsequence of scalars $\{b_2^{(m)}\}$ converges to c_2, say. This process continues till we reach the n-th stage. At the n-th stage, we obtain a subsequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2}, ...\}$ of $\{y_m\}$ whose terms are of the form $y_{n,m} = d_1^{(m)}x_1 + d_2^{(m)}x_2 + ... + d_n^{(m)}x_n$ with $\sum_{i=1}^{n} |d_i^{(m)}| = 1$ and $d_i^{(m)} \longrightarrow c_i$ as $m \longrightarrow \infty$. Let $y = c_1x_1 + c_2x_2 + ... + c_nx_n \in V_K$.

Then $\|P_{y_{n,m}}^{p_{n,m}} - P_{y}^{p_{n,m}}\| = \|P_{d_1^{(m)}x_1 + d_2^{(m)}x_2 + ...+ d_n^{(m)}x_n}^{p_{n,m}} - P_{c_1x_1 + c_2x_2 + ...+ c_nx_n}^{p_{n,m}}\|$
$= \|P_{(d_1^{(m)} - c_1)x_1 + (d_2^{(m)} - c_2)x_2 + ...+ (d_n^{(m)} - c_n)x_n}^{p_{n,m}}\|$
$= \|P_{d_1^{(m)} - c_1}^{1} P_{x_1}^{p_{n,m}} + P_{d_2^{(m)} - c_2}^{1} P_{x_2}^{p_{n,m}} + ... + P_{d_n^{(m)} - c_n}^{1} P_{x_n}^{p_{n,m}}\|$
$\leq \|P_{d_1^{(m)} - c_1}^{1} P_{x_1}^{p_{n,m}}\| + \|P_{d_2^{(m)} - c_2}^{1} P_{x_2}^{p_{n,m}}\| + ... + \|P_{d_n^{(m)} - c_n}^{1} P_{x_n}^{p_{n,m}}\|$
$= P_{|d_1^{(m)} - c_1|}^{1} \|P_{x_1}^{p_{n,m}}\| + P_{|d_2^{(m)} - c_2|}^{1} \|P_{x_2}^{p_{n,m}}\| + ... + P_{|d_n^{(m)} - c_n|}^{1} \|P_{x_n}^{p_{n,m}}\| \longrightarrow P_0^1$ as $m \longrightarrow \infty$ since $d_i^{(m)} \longrightarrow c_i$ as $m \longrightarrow \infty$ and $\sum_{i=1}^{n} |c_i| = \sum_{i=1}^{n} |lim_{m\to\infty} d_i^{(m)}| = lim_{m\to\infty} \sum_{i=1}^{n} |d_i^{(m)}| = 1$.

So, $c_i \neq 0$ for some $i = 1,2,....,n$ and as $x_1, x_2, ..., x_n$ are linearly independent in V_K
$[\because\ P_{x_i}^{l_i}, i = 1,2,...,n$ are multi linearly independent in $(X, \|\ \|)]$,
it follows that $y = c_1x_1 + c_2x_2 + ... + c_nx_n \neq \theta$. Now $P_{y_m}^{p_n} \longrightarrow P_{\theta}^{p_n}$ and $\{P_{y_{n,m}}^{p_{n,m}}\}$ is a subsequence of $\{P_{y_m}^{p_n}\}$, but $P_{y_{n,m}}^{p_{n,m}} \longrightarrow P_{y}^{p_{n,m}}$ where $y \neq \theta$, a contradiction, which proves the lemma.

Theorem 3.14. If $\{P_{x_n}^{l_n}\}$ be a sequence of mvectors in an Mnormed linear space $(X, \|\ \|)$ such that $P_{x_n}^{l_n} \longrightarrow P_x^l$, then every subsequence $P_{x_{n_k}}^{l_{n_k}}$ of $\{P_{x_n}^{l_n}\}$ converges to P_x^l and conversely.

Proof. The proof is straight forward and hence omitted.

Theorem 3.15. Let $T : X_{pt} \longrightarrow Y_{pt}$ be a multi linear operator $(X, \|\ \|_x)$ and $(Y, \|\ \|_y)$ are two multi normed linear spaces. If X is finite dimensional, then T is bounded and hence continuous.

Proof. Let dimension of X be n and $\{P^{l_1}_{x_1}, P^{l_2}_{x_2}, ...P^{l_n}_{x_n}\}$ be a basis of X.

Let $P^i_a = Max \{\|T(P^{l_1}_{x_1})\|_y, \|T(P^{l_2}_{x_2})\|_y, ..., \|T(P^{l_n}_{x_n})\|_y\}$ and

$P^l_x = \sum_{k=1}^n P^{i_k}_{a_k} P^{l_k}_{x_k} \in X_{pt}$. Then by linearity of T,

$\|T(P^l_x)\|_y = \|\sum_{k=1}^n P^{i_k}_{a_k} T(P^{l_k}_{x_k})\|_y \leq \sum_{k=1}^n P^{i_k}_{|a_k|} \|T(P^{l_k}_{x_k})\|_y \leq P^i_a \sum_{k=1}^n P^{i_k}_{|a_k|} ..(1)$. By Lemma 3.13, $\exists\, c > 0$ such that $\|P^l_x\|_x = \|\sum_{k=1}^n P^{i_k}_{a_k} P^{l_k}_{x_k}\|_x \geq P^1_c \sum_{k=1}^n P^{i_k}_{|a_k|} \Longrightarrow \sum_{k=1}^n P^{i_k}_{|a_k|} \leq P^1_{\frac{1}{c}} \|P^l_x\|_x$. $\therefore$ from (1), $\|T(P^l_x)\|_y \leq P^i_a P^1_{\frac{1}{c}} \|P^l_x\|_x = P^1_{\frac{a}{c}} \|P^l_x\|_x \Longrightarrow \|T(P^l_x)\|_y < P^1_r \|P^l_x\|_x$ where $0 < \frac{a}{c} < r$. $\therefore T$ is bounded.

Definition 3.16. Let $T : X_{pt} \longrightarrow Y_{pt}$ be a bounded multi linear operator. Then $\exists\, r > 0$ such that $\|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}$. Let $s = Inf\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$

We define $\|T\|$ as $\|T\| = P^1_s \; if \; s \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \;\forall\, P^l_x \in X_{pt}\}$

$$= P^w_s \; if \; s \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \;\forall\, P^l_x \notin X_{pt}\}.$$

Note 3.17. Since $\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$ is bounded below (0 being a lower bound), the Infimum s exists. If $s \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$, then $\|T(P^l_x)\| \leq \|T\|\|P^l_x\| \;\forall\, P^l_x \in X_{pt}$.

If $s \notin \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$, then for $\varepsilon > 0$ arbitrary, $\exists\, r_0 \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \;\forall\, P^l_x \in X_{pt}\}$ such that $s + \varepsilon > r_0$. Now, $\forall\, P^l_x \in X_{pt}, \|T(P^l_x)\| \leq P^1_{r_0} \|P^l_x\| < P^1_{s+\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, $\forall P^l_x \in X_{pt},\; \|T(P^l_x)\| \leq P^1_s \|P^l_x\| \Rightarrow \|T(P^l_x)\| \leq \|T\|\|P^l_x\| \;\forall\, P^l_x \in X_{pt}$.

Theorem 3.18. $\|T\| = P^1_{s_0}$ where $s_0 = Inf\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\|, \forall\, P^l_x \in X_{pt} \; s.t \; \|P^l_x\| = P^i_1\}$.

Proof. Let $\|T\| = P^1_s$ where $s = Inf\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$. Since $\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt} \; s.t \; \|P^l_x\| = P^i_1\} \subset \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}, Inf\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt} s.t. \|P^l_x\| = P^i_1\} \geq Inf\{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \forall\, P^l_x \in X_{pt}\} \Rightarrow s_0 \geq s$ Let $r_0 \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$.

Then $\forall\, P^l_x \in X_{pt}, \|T(P^l_x)\| \leq P^1_{r_0} \|P^l_x\|$. Let $\|P^l_x\| = P^i_a$ and $x \neq \theta$ so that $a > 0$. Consider P^l_y where $y = a^{-1}x$. Then $\|P^l_y\| = \|P^l_{a^{-1}x}\| = \|P^1_{a^{-1}} P^l_x\| = P^1_{a^{-1}} \|P^l_x\| = P^1_{a^{-1}} P^i_a$. Now $\|T(P^l_x)\| \leq P^1_{r_0} \|P^l_x\| \Rightarrow P^1_{a^{-1}} \|T(P^l_x)\| \leq P^1_{r_0} P^1_{a^{-1}} \|P^l_x\| \Rightarrow \|T(P^1_{a^{-1}} P^l_x)\| \leq P^1_{r_0} \|P^1_{a^{-1}} P^l_x\| \Rightarrow \|T(P^l_{a^{-1}x})\| \leq P^1_{r_0} \|P^l_{a^{-1}x}\| \Rightarrow \|T(P^l_y)\| \leq P^1_{r_0} \|P^l_y\| \Rightarrow r_0 \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt} s.t \; \|P^l_x\| = P^i_1\} \;\Rightarrow r_0 \geq s_0$. Since $r_0 \in \{r > 0 : \|T(P^l_x)\| \leq P^1_r \|P^l_x\| \;\forall\, P^l_x \in X_{pt}\}$ is arbitrary, it follows that $s \geq s_0$.

Example 3.19. (1) For the Identity Operator $I : X_{pt} \longrightarrow X_{pt}$ such that $I(P^l_x) = P^l_x, \|I\| = P^1_1$.

(2) For the Null Operator $\overline{O} : X_{pt} \longrightarrow X_{pt}$ such that $\overline{O}(P^l_x) = P^l_\theta, \|\overline{O}\| = P^{w_x}_0$.

Theorem 3.20. Let $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$ are two normed linear spaces; $(X, \| \ \|_X)$, $(Y, \| \ \|_Y)$ are two multi normed linear spaces on $(V, \| \ \|_V)$, $(W, \| \ \|_W)$ respectively such that $\|P_y^m\|_Y = P^m_{\|y\|_W}$, $\forall \ P_y^m \in Y_{pt}$ and $T : V \longrightarrow W$ be a bounded linear operator. Then $T_M : X_{pt} \longrightarrow Y_{pt}$ such that $T_M(P_x^l) = P^l_{T(x)} \ \forall P_x^l \in X_{pt}$ is also a bounded multi linear operator.

Proof. Since $T : V \longrightarrow W$ is bounded, $\exists r > 0$ such that $\|T(x)\|_W \leq r\|x\|_V, \forall x \in V$.

Then $\|T_M(P_x^l)\|_Y = \|P^l_{T(x)}\|_Y = P^l_{\|T(x)\|_W} \leq P_r\|x\|_V^l = P_r^1 P^l_{\|x\|_V} \Rightarrow T_M$ is bounded.

Theorem 3.21. Let $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$ be two normed linear spaces; $(X, \| \ \|_X)$, $(Y, \| \ \|_Y)$ are two multi normed linear spaces on $(V, \| \ \|_V)$, $(W, \| \ \|_W)$ respectively with $X^* = V$ and $\|P_y^m\|_Y = P^m_{\|y\|_W}$, $\forall \ P_y^m \in Y_{pt}$. Let for $1 \leq l \leq w_X, T_l : V \to W$ such that $T_l(x) = \widetilde{b}[T_M(P_x^l)] \ \forall \ x \in V$. Then $\{T_l : 1 \leq l \leq w_X\}$ is a family of bounded linear operators. More over, if we define $T_M^* : X_{pt} \to Y_{pt}$ such that $T_M^*(P_x^l) = T_M(P_x^l) \ \forall \ P_x^l \in X_{pt}$, then T_M^* is a bounded multi linear operator with $T_M^* = T_M$.

Proof. Since T_M is a bounded multi linear operator, $\exists \ r > 0$ such that

$\|T_M(P_x^l)\|_Y \leq P_r^1 \|P_x^l\|_X \ \forall \ P_x^l \in X_{pt}$. Let $T_M(P_x^l) = P_y^k$. Then

$\|T_M(P_x^l)\|_Y = \|P_y^k\|_Y = P^k_{\|y\|_W} \leq P_r^1 \|P_x^l\|_X$.

$\therefore \ \|T_l(x)\|_W = \|\widetilde{b}\{T_M(P_x^l)\}\|_W = \|\widetilde{b}(P_y^k)\|_W = \|y\|_W$.

4. Conclusions

Theory of operator is an important branch of functional analysis and it has many applications in Mathematics and Sciences. In this paper, an attempt has been made to introduce linear operators on multi normed linear space. There is an ample scope for further research on multi linear operators. Research on multi linear functionals and multi inner product can be of special interest.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] W. D. Blizard, Multiset theory, Notre Dame J. Formal Logic. 30(1) (1989), 36 – 66.

[2] W. D. Blizard, Real-valued multisets and fuzzy sets, Fuzzy Sets Syst. 33(1) (1989), 77 – 97.

[3] W. D. Blizard, Negative membership, Notre Dame J. Formal Logic. 31(3) (1990), 346 – 368.

[4] W. D. Blizard, The development of multiset theory, Modern Logic. 1(4) (1991), 319 – 352.

[5] K. Chakrabarty, Bags with interval counts, Found. Comput. Decision Sci. 25(1) (2000), 23 – 36.

[6] K. Chakrabarty and I. Despi, n^k-bags, Int. J. Intell. Syst. 22(2) (2007), 223 — 236.

[7] S. Das, R. Roy, An introduction to multi metric spaces, Adv. Dyn. Syst. Appl. 16(2) (2021), 605-618.

[8] S. Das, R. Roy, Some topological properties of multi metric spaces, J. Math. Comput. Sci. 11 (2021), 7253-7268.

[9] S. Das, Completeness in multi metric spaces, Communicated.

[10] R. Roy, S. Das, S.K. Samanta On Multi Normed Linear Spaces, Int. J. Math. Trends Technol. 48(2) (2017), 111–119.

[11] G. F. Clements, On multiset k-families, Discrete Math. 69(2) (1988), 153 – 164.

[12] M. Conder, S. Marshall and Arkadii M. Slinko, Orders on multisets and discrete cones, Order. 24 (2007), 277 – 296.

[13] K. P. Girish, S. J. John, Multiset topologies induced by multiset relations, Inform. Sci. 188 (2012), 298 – 313.

[14] K. P. Girish, S. J. John, On multiset topologies, Theory Appl. Math. Computer Sci. 2(1) (2012), 37 – 52.

[15] K. P. Girish, S. J. John, Relations and functions in multiset context, Inf. Sci. 179(6) (2009), 758 – 768.

[16] S. P. Jena, S.K. Ghosh and B.K. Tripathy, On the theory of bags and lists, Inform. Sci. 132(14) (2001), 241 – 254.

[17] Sk. Nazmul, S. K. Samanta, On soft multigroups, Ann. Fuzzy Math. Inform. 10(2) (2015), 271 – 285.

[18] Sk. Nazmul, P. Majumdar, S. K. Samanta, *On multisets and multigroups*, Ann. Fuzzy Math. Inform. 6(30) (2013), 643 – 656.

[19] J. L. Peterson, Computation sequence sets, J. Computer Syst. Sci. 13(1) (1976), 1 – 24.

[20] D. Singh, A note on the development of multiset theory, Modern Logic 4(4) (1994), 405 – 406.

[21] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, Complementation in multiset theory, Int. Math. Forum. 6(38) (2011), 1877 – 1884.

[22] D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, An overview of the applications of multisets, Novi Sad J. Math. 37(2) (2007), 73 – 92.

[23] D. Tokat, I. Osmanoglu, Y. Ucok, On Soft Multi Continuous Functions, J. New Theory, 1(2015), 50 – 58.