Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 2, 386-393
ISSN: 1927-5307

MONOCHROMATIC 4-CONNECTED SUBGRAPHS IN CONSTRAINED 2-EDGE-COLORING OF K_n

HONGPING LUO, YINGLI KANG, AND SHILI WEN*

Department of Mathematics, Zhejiang Normal University
Jinhua 321004, P.R. China

Abstract. Bollobás and Gyárfás conjectured that for $n \geq 4k-3$ every 2-edge-coloring of K_n contains a monochromatic k-connected subgraph with at least $n-2k+2$ vertices. It was proved that the conjecture holds for $k = 2, 3$. In this paper, we prove that if each monochromatic k-connected ($k = 2, 3$) subgraph has at most $n - 2k + 2$ vertices in 2-edge-colored K_n ($n \geq 13$), then there exists a monochromatic 4-connected subgraph with at least $n - 6$ vertices.

Keywords: monochromatic subgraph, k-connected subgraph, 2-edge-coloring.

2000 AMS Subject Classification: 05C15, 05C35

1. Introduction

It is easy to see that for any graph G, either G or its complement $\overline{G}$ is connected. This is equivalent that there exists a connected monochromatic subgraph of every 2-edge-coloring of K_n. Bollobás and Gyárfás [1] conjectured that for $n > 4(k-1)$ every 2-edge-coloring of K_n contains a monochromatic k-connected subgraph with at least $n - 2k + 2$ vertices.

*Corresponding author
E-mail addresses: 499569218@qq.com (H. Luo), xiaokangyouxiang@163.com(Y. Kang),
wenshili@yahoo.cn(S. Wen)
Received Dec 12, 2011

Liu et al. [9] proved that the conjecture holds when $n \geq 13k - 15$. Jin et al. [8] characterized all the 2-edge-colorings of K_n where there is a monochromatic k-connected subgraph with ai least $n - 2k + 2$ vertices for $n \geq 13k - 15$. Fujita et al. [7] proved that every 2-edge-coloring of K_n contains a monochromatic k-connected subgraph with at least $n - 2k + 2$ vertices when $n \geq 6.5(k - 1)$. In fact this conjecture is a part of the question due to Bollobás: when we colored the edges of K_n with at most r colors, how large a k-connected subgraph are we guaranteed to find using only at most s colors.

Let ϕ be an r-edge-coloring of K_n. Given a subgraph H of K_n, we write $c_\phi(H)$ for the number of colors in H. Denote by

$$M(\phi, n, r, s, k) = \max\{|V(H)| : H \subseteq K_n,\ H \text{ is } k\text{-connected, and } c_\phi(H) \leq s\}$$

the order of the largest k-connected subgraph of K_n using at most s colors. Let $m(n, r, s, k) = \min_\phi\{M(\phi, n, r, s, k)\}$, where ϕ runs over all the r-edge-colorings of K_n. Thus the question of Ballobás asks for the value of $m(n, r, s, k)$.

When $s = k = 1$, the question asks for the order of monochromatic component in edge colored graph K_n see [3, 5, 6]. Bollobás and Gyárfás [1] gave some bounds for the case $s = 1$. Liu et al. [9, 10] gave some bounds for the parameter $m(n, r, s, k)$ for some r, s and k. Note that only a few cases are determined exactly. Besides of the connectivity of monochromatic subgraphs in edge colored K_n, other propositions should be interesting too. For example, Gyárfás and Sárközy [4, 5] considered the order of monochromatic double stars in edge colored K_n. Burr [2] proved that each 2-edge-colored K_n contains a monochromatic spanning broom.

Bollobás and Gyárfás [1] present a 2-edge-coloring of K_n where each monochromatic k-connected subgraph has order at most $n-2k+2$. They also proved that $m(n, 2, 1, 2) = n-2$ when $n \geq 5$. Liu et al. [9] proved that $m(n, 2, 1, 3) = n - 4$ for $n \geq 9$. Without loss of generality, throughout the paper, we use red and blue to color the edges of K_n. For convenience, denote by R and B the spanning graphs of K_n which contains all the red and blue edges respectively.

2. Main results

First we present some known results, which also appeared in [1, 9, 10].

Lemma 2.1. Let G be a graph and $v \in V(G)$ with $d(v) \geq k$. If $G - v$ is k-connected, then G is also k-connected.

Lemma 2.2. Let G and H be k-connected graphs. If $|V(G) \cap V(H)| \geq k$, then $G \cup H$ is also k-connected.

Lemma 2.3. For $n \geq 4k - 3$, $m(n, 2, 1, k) = n - 2k + 2$, $k = 2, 3$.

Second we will prove the following theorem.

Theorem 2.4. *Let K_n $(n \geq 13)$ be 2-edge-colored. If each monochromatic k-connected $(k = 2, 3)$ subgraph has at most $n-2k+2$ vertices in K_n, then there exists a monochromatic 4-connected subgraph with at least $n - 6$ vertices.*

Proof. We use red and blue to color the edges of K_n. From Theorem 2.3. we can assume that G_1 is a monochromatic 3-connected graph with $n - 4$ vertices and G_2 is a 2-connected graph with $n - 2$ vertices in K_n. Let $C_1 = V(K_n) \backslash V(G_1)$, and then $|C_1| = 4$. Let $C_1 = \{v_1, v_2, v_3, v_4\}$.

Case 1 The graphs G_1 and G_2 have the same color. Without loss of generality, let $G_i \subseteq R$ $(i = 1, 2)$.

Since G_1 is a red 3-connected graph, we have that G_1 is a red 2-connected graph. Note that $G_1 \subseteq G_2$. Otherwise, since $|V(G_2)| = n - 2$, we have that $|V(G_1) \cap V(G_2)| \geq n - 6$. By Lemma 2.2, the graph $G_1 \cup G_2$ is a red subgraph with at least $n - 1$ vertices, a contradiction. Note that there are two vertices, say $v_3, v_4 \in C_1$, each of which sends two red edges to G_1, i.e., $V(G_2) = V(G_1) \cup \{v_3, v_4\}$. Otherwise, by Lemma 2.1, there exists a red 2-connected subgraph with at least $n - 1$ vertices, a contradiction. Then it is easy to see that each vertex of $\{v_1, v_2\}$ sends at most one red edge to G_1.

If G_1 is 4-connected, then we are done. Now we suppose that G_1 isn't a 4-connected subgraph, then there exists a cut set C of G_1 with at most 3 vertices. Let A_1 be the union of vertices of some components of $G_1 - C$ and $B_1 = V(G_1) \backslash A_1$ such that $|A_1| \geq |B_1|$ and

$|B_1|$ as large as possible. Choose the cut set C that maximize the set $|B_1|$. It is easy to see that all edges between A_1 and B_1 are blue. This forms a complete bipartite graph in blue. Let $G_3 = B[A_1 \cup B_1 \cup \{v_1, v_2\}]$ and $G_4 = B[A_1 \cup B_1 \cup C_1]$.

Case 1.1 $|B_1| \geq 4$.

Then $B[A_1, B_1]$ is a blue 4-connected complete bipartite graph with at least $n-7$ vertices. Since each vertex of C_1 sends at least $n-|C|-|C_1|-2 \geq 5$ blue edges to $B[A_1, B_1]$, by Lemma 2.1, we know that G_3 is a blue 4-connected subgraph with at least $n-3$ vertices.

Case 1.2 $|B_1| = 3$.

Then $|A_1| = n - |C| - |C_1| - |B_1| \geq 4$ $(n \geq 13)$. Note that each vertex of $\{v_1, v_2\}$ sends at most one red edges to G_1. If the red edges between $\{v_1, v_2\}$ and $V(G_1)$ are non-adjacent, then it is easy to see that G_3 is a blue 4-connected subgraph with at least $n-5$ vertices. If the red edges between $\{v_1, v_2\}$ and $V(G_1)$ are adjacent, then there exists a vertex $u \in V(G_1)$ such that both uv_1 and uv_2 are red edges. Then we have that the graph $G_3 - u$ is a blue 4-connected subgraph with at least $n-6$ vertices.

Case 1.3 $|B_1| = 2$.

Then $|A_1| = n - |C| - |C_1| - |B_1| \geq 4$.

Case 1.3.1 There are at least one vertex of $\{v_1, v_2\}$, say v_1, that sends one red edge to C.

Then there are at most one vertex v_2 of $\{v_1, v_2\}$ that sends one red edge to $V(G_1)\backslash C$. And all edges between v_1 and $A_1 \cup B_1$ are blue. If v_2 sends one red edge to B_1, then G_3 a blue 4-connected graph with at least $n-5$ vertices. If v_2 sends one red edge to A_1, then there exists a vertex u of A_1 such that uv_2 is red. Then the graph $G_3 - u$ is a blue 4-connected graph with at least $n-6$ vertices.

Case 1.3.2 Each vertex of $\{v_1, v_2\}$ sends one red edge to B_1.

Then all edges between $\{v_1, v_2, B_1\}$ and A_1 are blue. We have that the graph G_3 is a blue 4-connected subgraph with at least $n-5$ vertices.

Case 1.3.3 There exists only one vertex of $\{v_1, v_2\}$, say v_1, that sends one red edge to A_1.

Then v_2 sends at most one red edge to B_1. Let $u \in A_1$ such that uv_1 is red. It's easy to see that all the edges between B_1 and v_2 are blue. Then the graph $G_3 - u$ is a blue 4-connected graph with at least $n-6$ vertices.

Case 1.3.4 Each vertex of $\{v_1, v_2\}$ sends one red edge to A_1.

Then all the edges between B_1 and $\{v_1, v_2\}$ are blue edges. If there exists a vertex $u \in A_1$ such that both uv_1 and uv_2 are red edges, then the graph $G_3 - u$ is a blue 4-connected graph with at least $n-6$ vertices.

Suppose that there exist two vertices $u_1, u_2 \in A_1$ such that v_1u_1 and v_2u_2 are red edges. Then it is easy to see that the graph $G_3 - \{u_1, u_2\}$ is a blue 4-connected subgraph. We know that each vertex of $\{v_3, v_4\}$ sends two red edges to G_1. If there exists a vertex of $\{v_3, v_4\}$ and a vertex of $\{u_1, u_2\}$, say v_3 and u_1, such that u_1v_3 is blue, then the graph $G_4 - u_2 - v_4$ is a 4-connected subgraph with at least $n-5$ vertices. If there isn't a vertex of $\{v_3, v_4\}$ such that u_iv_3 $(i = 1, 2)$ is blue, then each vertex of $\{v_3, v_4\}$ sends at least four red edges to $G_3 - u_1 - u_2$. By Lemma 2.1, the graph $G_4 - u_1 - u_2$ is 4-connected with at least $n-5$ vertices.

Case 1.4 $|B_1| = 1$.

Then $|A_1| = n - |B_1| - |C| - |C_1| \geq 5$. It is easy to see that there are at most six red edges between C_1 and G_1 in all.

Case 1.4.1 There are at most two red edges between A_1 and C_1.

Then there exists at most one vertex v of A_1 that sends at most three blue edges to $B_1 \cup C_1$. It's easy to see that the graph $G_4 - v$ is a 4-connected subgraph with at least

$n-4$ vertices.

Case 1.4.2 There are three red edges between A_1 and C_1.

Suppose that there exists a vertex of C_1, say v_3, that sends two red edges to different vertices of A_1. Then there exists a vertex of C_1, say v_1, that sends one red edges to the vertex u of A_1. If uv_3 is a red edge, then the graph $G_4 - v_3$ is a 4-connected subgraph with at least $n-4$ vertices. If uv_3 is a blue edge, then the graph $G_4 - v_1 - v_3$ is a 4-connected subgraph with at least $n-5$ vertices. Suppose that there exist three vertices of C_1, say v_1, v_2, v_3, each of which sends one red edge to A_1. If each vertex of A_1 sends at least four blue edges to $B_1 \cup C_1$, then the graph G_4 is a 4-connected subgraph with with at least $n-3$ vertices. If there exists a vertex u of A_1 such that u sends three red edges to $B_1 \cup C_1$, then the graph $G_4 - u$ is a 4-connected subgraph with at least $n-4$ vertices. If there exist two vertices u_1, u_2 of A_1 such that u_1v_1, u_1v_2, u_2v_3 are red edges, then the graph $G_4 - u_1 - v_3$ is a 4-connected subgraph with at least $n-5$ vertices.

Case 1.4.3 There are four red edges between A_1 and C_1.

Since there are at most six red edges between C_1 and G_1 in all, we have that there exists a vertex $w \in C$ that sends $|C_1|$ blue edges to C_1. There are at most two vertices u_1, u_2 of A_1 each of which sends at most three blue edges to $B_1 \cup C_1$. Then the graph $B[(A_1 \backslash \{u_1, u_2\}) \cup B_1 \cup C_1 \cup \{w\}]$ is a 4-connected subgraph with at least $n-4$ vertices.

Case 1.4.4 There are five red edges between A_1 and C_1.

Then there are at least two vertices of C each of which sends $|C_1|$ blue edges to C_1. There are at most two vertices u_1, u_2 of A_1 each of which sends at most three blue edges to $B_1 \cup C_1$. Then the graph $B[(A_1 \backslash \{u_1, u_2\}) \cup B_1 \cup C_1 \cup \{w_1, w_2\}]$ is a blue 4-connected subgraph with at least $n-3$ vertices.

Case 1.4.5 There are six red edges between A_1 and C_1.

Then There are at most three vertices u_1, u_2, u_3 of A_1 each of which sends at most three blue edges to $B[B_1 \cup C, C_1]$. It's easy to see that each vertex of C sends $|C_1|$ blue edges to C and each vertex of C_1 sends $|B_1|$ blue edges to B_1. Then $B[B_1 \cup C \cup C_1]$ is a 4-connected graph. Note that each vertex of $A_1 \backslash \{u_1, u_2, u_3\}$ sends at least four blue edges to $B[B_1 \cup C \cup C_1]$. By Lemma 2.1, the graph $B[(A_1 \backslash \{u_1, u_2, u_3\}) \cup B_1 \cup C_1 \cup C]$ is a 4-connected subgraph with at least $n-3$ vertices.

Case 2 Suppose that G_1 and G_2 have different colors. Without loss of generality, let $G_1 \subseteq R$ and $G_2 \subseteq B$.

Note that there exists at most one vertex v of C_1 that sends two red edges to G_1. Otherwise, there are at least two vertices v_1, v_2 of C_1 each of which sends two red edges to G_1. By Lemma 2.1, $R[G_1 \cup \{v_1, v_2\}]$ is a red 2-connected subgraph with at least $n-2$ vertices, a contradiction. Then there are at most five red edges between C_1 and G_1. Then There are at most five vertices, say $u_i \in G_1 (i = 1, 2, 3, 4, 5)$, each of which sends at least one red edge to C_1. Since $n \geq 13$, we have that there exists a blue 4-connected subgraph $B[C_1 \cup G_1 \backslash \{u_1, u_2, u_3, u_4, u_5\}$ with at least $n-5$ vertices.

This completes the proof.

References

[1] B. Ballobás, A. Gyárfás, Highly connected monochromatic subgraphs, Discrete Math. 308(2008), 1722-1725.

[2] S.A. Burr, Either a graph or its complement contains a spanning broom, manuscript.

[3] Z. Füredi, Maximum degreen and fractial matchings in uniform hypergraphs, Combinatorica 1(2008), 155-162.

[4] A. Gyárfás, G.N. Sárközy, Size of monchromatic double stars in edge colorings, Graphs and Combinatorics 24(2008), 531-536.

[5] A. Gyárfás, G.N. Sárközy, Size of monchromatic components in local edge colorings, Discrete Math. 308(2008), 2620-2622.

[6] A. Gyárfás, Patition coverings and blocking sets of hypergraphs (in Hungarian), Comm. Coput. Automat. Inst. Hungar. Acad. Sci. 71(1997), 62.

[7] Shinya Fujita, Colton Magnant, Note on highly connected monochromatic subgraphs in 2-colored graphs, The Electronic Journal of Combinatorics 18(2011).

[8] Z.M. Jin, S.L. Wen, Note on 2-edge-colorings with small monochromatic k-connected subgraphs, manuscript.

[9] H. Liu, R. Morris and N. Prince, Highly connected monochromatic subgraphs of multicolored graphs, J. Graph Thory 61(2008), 22-44.

[10] H. Liu, R. Morris and N. Prince, Highly connected multicolred subgraphs of multicolored graphs, Discrete Math. 308(2008), 5096-5121.

[11] H. Liu, Y. Person, Highly connected coloured subgraphs via the regularity lemma, Discrete Math. 309(2009), 6277-6287.