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KAMAL TRANSFORM FOR INTEGRABLE BOEHMIANS

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Abstract: In this paper, basic properties of Kamal transform for Integrable Boehmians are demonstrated. Also inversion theorem for Kamal transform is proved.

Key words: Boehmians; generalized functions; integral transform; Kamal transform.

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1. INTRODUCTION

The study of generalized functions has become a major area of research for last five decades. The literature with Schwartz distribution theory, tempered distributions and their applications are available. Dirac introduced the δ -function and the property that the derivative of the Heaviside function is the δ–function.

The impact of generalized functions on the integral transformations has revolutionaries the theory of generalized integral transformations. McBride, Mendez and Zemanian have given their contribution in this field. Mikusinski J. has given algebraic approach to generalized functions in

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[10]. Similar to Mikusinski operators, a very general new class of generalized functions, called as Boehmians was constructed by Mikusinski P. [12]. Several properties of Kamal transform are investigated with applications in [1-3]. The Various integral transforms on Boehmians spaces are developed and studied in [4-9, 13-14].

In this paper we mainly deal with a case of Kamal transform and developed Boehmian space for it.

2. GENERALIZED FUNCTIONS [15-17]:

The straight forward approach for generalized function is given by Temple G. [16] with the fact that different sequences may have the same generalized functions.

Hence it is a need to define an equivalence relation between sequences that represent the same generalized functions.

Suppose $(f_n) and\ (g_n)$ are the sequences of functions so that the integrals $\int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt$ and $\int_{-\infty}^{\infty} g_n(t)\varphi(t)\,dt$ exists for all n and for all φ from a given class of functions $\varphi(t)$.

Assume also that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt\ and\ \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(t)\varphi(t)\,dt$ exists.

The two sequences $(f_n) and\ (g_n)$ are called equivalent with respect to $\varphi(t)$ if and only if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(t)\varphi(t)\,dt$$

for all $\varphi \in \varphi(t)$.

The expression $\int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt$ associates a number with every φ, this quantity is called a functional. We can write this as $\langle f_n,\ \varphi\rangle = \int_{-\infty}^{\infty} f_n(t)\varphi(t)\,dt$.

We denote $\lim_{n\to\infty} \langle f_n, \varphi\rangle\ by\ \langle f, \varphi\rangle$, is well defined, and is a complex number, called as generalized function. The definition of generalized functions as functional is used by Schwartz and Gelfand in their monographs.

3. INTEGRABLE BOEHMIANS

A general construction of Boehmians was studied in [11]. The space of Boehmians with two notions of convergence was well defined in [12]. The integral transforms have been extended to the context of Boehmian spaces. Fourier [5, 6, 7, 13, 14], Hilbert [8, 9] are some of them.

Suppose L^1 is the space of complex valued (Lebesgue) Integrable functions on $(0,\infty)$.

Define the norm on L^1 as $\|g\| = \int_0^{\infty} |g(u)|\, du$.

If $f_1, f_2 \in L^1$ then the convolution product $(f_1 * f_2)(u)$;

$(f_1 * f_2)(u) = \int_0^{\infty} f_1(x) f_2(u-x)\, dx$; is an element of L^1 and we have

$\|f_1 * f_2\| \le \|f_1\| \|f_2\|$.

A sequence of continuous real valued functions $(\emptyset_n) \in L^1$, called a delta sequence if

(i) $\int_0^{\infty} \emptyset_n(x)\, dx = 1, \forall\, n \in \mathbb{N}$.

(ii) $\|\emptyset_n\| < K$, for some $K \in (0,\infty)$ and all $n \in \mathbb{N}$.

(iii) $\lim_{n\to\infty} \int_{|x|>\varepsilon} |\emptyset_n(x)|\, dx = 0, for\ each\ \varepsilon > 0$.

If $(f_n) and\ (g_n)$ are delta sequences then $(f_n * g_n)$ is also a delta sequence.

If $f_1 \in L^1 and\ (\emptyset_n)$ is a delta sequence then $\|f_1 * \emptyset_n - f_1\| \to 0\ as\ n \to \infty$.

The delta sequences have approximate identities or summability kernels as other notions.

A pair of sequences (f_n, g_n) is called quotient of sequences; denoted by f_n/g_n, if $f_n \in L^1\ and\ (g_n)$ is a delta sequence and $f_m * g_n = f_n * g_m$, for all $m, n \in \mathbb{N}$.

Two quotient sequences f_n/g_n and $h_n/\emptyset_n$ are equivalent if $f_n * \emptyset_n = g_n * h_n, \forall\, n \in \mathbb{N}$.

The equivalence class of quotient of sequences is called as an Integrable Boehmian.

The space of all these Integrable Boehmians is denoted by B_L^1.

If we define addition, multiplication by a scalar and convolution on B_L^1 as

$[f_n/g_n] + [h_n/\emptyset_n] = [(f_n * \emptyset_n + h_n * g_n)/(g_n * \emptyset_n)]$,

$k[f_n/g_n] = [kf_n/g_n]$,

$[f_n/g_n] * [h_n/\emptyset_n] = [(f_n * h_n)/(g_n * \emptyset_n)]$ respectively then B_L^1 becomes a convolution algebra.

Note that a function $f \in L^1$ with a Boehmian$[(f * \emptyset_n)/\emptyset_n]$; $(\emptyset_n)$is any delta sequence. Here, $L^1$is a subspace of B_L^1.

Also if $F = [f_n/\emptyset_n]\ then\ F * \emptyset_n = f_n\ \therefore F * \emptyset_n \in L^1, \forall\, n \in \mathbb{N}$.

The two type of convergence $\delta\ and\ \Delta$ of sequences of Boehmians on B_L^1 can be defined as-

A sequence of Boehmians F_n is δ-convergent to F if $\exists$ a delta sequence $(\emptyset_n)$ such that $F_n * \emptyset_k \in L^1\ and\ F * \emptyset_k \in L^1, \forall\, n, k \in \mathbb{N}\ and\ \|(F_n - F) * \emptyset_k\| \to 0$ for each $k \in \mathbb{N}$. We can denote this as $\delta - \lim F_n = F$.

A sequence of Boehmians F_n is Δ-convergent to F if $\exists$ a delta sequence $(\emptyset_n)$ such that $(F_n - F) * \emptyset_n \in L^1\ \ \forall\, n \in \mathbb{N}\ and\ \|(F_n - F) * \emptyset_n\| \to 0$ for each $k \in \mathbb{N}$. we can denote this as $\Delta - \lim F_n = F$. With this Δ-convergence B_L^1 is a quasi-normed space [12].

The relation between these two types of convergence is given in [12] as an equivalence.

$\Delta - \lim F_n = F$ if and only if each sub-sequence of (F_n) contains a δ-convergent subsequence which converges to F.

Now, if $\Delta - \lim F_n = F\ \ and\ \Delta - \lim H_n = H\ \ then\ \ \Delta - \lim(F_n * H_n) = F * H$.

If $(\emptyset_n)$ is a delta sequence, $[\emptyset_n/\emptyset_n]$ represent Integrable Boehmian. As it corresponds to the Dirac delta distribution; denoted by $\emptyset$, all the derivatives of $\emptyset$ are also Integrable Boehmians.

Since there are delta sequences $(\emptyset_n)$ such that all functions $\emptyset_n$ are infinitely differentiable and having bounded support, we can define the m^th^ derivative of $\emptyset$ by

$\emptyset^{(m)} = [\emptyset_n^m/\emptyset_n]$. Here, $\emptyset^{(m)} \in B_L^1$, for any $m \in \mathbb{N}$.

The m^th^ derivative of Boehmian $F \in B_L^1$ is defined as $F^{(m)} = F * \emptyset^{(m)}$.

Using the continuity of the convolution in B_L^1, if $\Delta - \lim F_n = F$ then we have

$$\Delta - \lim F_n^{(m)} = F^{(m)}, \text{ for any } m \in \mathbb{N}.$$

Let $F = [f_n/\emptyset_n] \in B_L^1$ then $f_1 * \emptyset_n = f_n * \emptyset_1$ for each $n \in \mathbb{N}$.

Again as $\int_0^\infty \emptyset_n(x)\, dx = 1, \forall\, n \in \mathbb{N}$ we have,

$$\int_0^\infty f_1(x)\,dx = \int_0^\infty (f_1 * \emptyset_n)(x)\,dx$$

$$= \int_0^\infty (f_n * \emptyset_1)(x)\,dx$$

$$= \int_0^\infty f_n(x)\,dx.$$

Also if $[f_n/g_n] = [h_n/\emptyset_n]$ $then$ $f_n * \emptyset_n = h_n * g_n, \forall\, n \in \mathbb{N}$.

$\Rightarrow \int f_n\,dx = \int h_n\,dx\ (since\ \int g_n = \int \emptyset_n = 1)$.

Hence, we can define the integral of a Boehmian as if

$$F = [f_n/\emptyset_n] \in B_L^1\ then\ \int_0^\infty F(x)\,dx = \int_0^\infty f_1\,(x)dx.$$

The integral is same as Lebesgue integral for the function in L^1. However, continuously differentiable functions in L^1 whose derivatives are not in L^1 are Integrable as Boehmians but not Integrable as functions.

4. THE KAMAL TRANSFORM

The Kamal transform of the function $F(t)$ defined in [1] is

$$K\{F(t)\} = \int_0^\infty e^{-t/v}\,F(t)dt = G(v), t \geq 0, k_1 \leq v \leq k_2,$$

provided the integral on R.H.S. exists and $F(t)$ is sectionally continuous and of exponential order on the set A; where

$$A = \{f: |f(t)| < Me^{|t|/\alpha_j}, t \in (-1)^j \times [0,\infty), M, \alpha > 0\};$$

Here the constant M should be finite number, α may be infinite and v is a variable of transform. Off course these conditions are sufficient for existence of Kamal transform of $F(t)$.

The inverse Kamal transform: If $K\{F(t)\} = G(v)$ then $F(t)$ is called inverse Kamal transform of $G(v)$.

$$F(t) = K^{-1}\{G(v)\};\ \text{Where}\ K^{-1}\ \text{is inverse Kamal operator.}$$

Lemma 4.1. If $[f_n/\emptyset_n] \in B_L^1$ then the sequence

$$\widetilde{f_n}(x) = \int_0^\infty f_n\,(t)e^{it/x}dt;$$

Where the Kernel $k(x,t) = e^{it/x}$, converges uniformly on each compact set in$(0,\infty)$.

Proof: If $(\emptyset_n)$ is a delta sequence then $(\widetilde{\emptyset_n})$ converges uniformly on each compact set to constant function 1. Hence for each compact set K, $\widetilde{\emptyset_k} > 0$ on K for all most all $k \in$ K and

$$\widetilde{f_n} = \widetilde{f_n}.\frac{\widetilde{\emptyset_k}}{\widetilde{\emptyset_k}} = \frac{(\widetilde{f_n * \emptyset_k})}{\widetilde{\emptyset_k}} = \frac{(\widetilde{f_k * \emptyset_n})}{\widetilde{\emptyset_k}} = \frac{\widetilde{f_k}}{\widetilde{\emptyset_k}}.\widetilde{\emptyset_n} \text{ on K.}$$

Here, the Kamal transform of Integrable Boehmians $F = [f_n/\emptyset_n]$ is defined as the $\lim(\widetilde{f_n})$ in the space of continuous functions on$(0, \infty)$. Therefore Kamal transform of Integrable Boehmians is a continuous function.

Theorem 4.2. Suppose $F_1, F_2 \in B_L^1$ then

(i) $(\widetilde{\alpha F}) = \alpha\tilde{F}$, for any complex number$\alpha$.

(ii) $(\widetilde{F_1 + F_2}) = \widetilde{F_1} + \widetilde{F_2}$

(iii) $(\widetilde{F_1 * F_2}) = \widetilde{F_1}\widetilde{F_2}$

(iv) If $\tilde{F} = 0 \; then \; F = 0$.

(v) If $\Delta - \lim F_n = F \;\; then \; \widetilde{F_n} \to \tilde{F}$ uniformly on each compact set.

Proof: To prove (v) it is sufficient if we show that $\delta - \lim F_n = F$ which implies that $\widetilde{F_n} \to \tilde{F}$, converges uniformly on each compact set.

Let $(\emptyset_n)$ be a delta sequence such that $F_n * \emptyset_k, F * \emptyset_k \in L^1$, for all $n, k \in \mathbb{N} \; and$

$$\|(F_n - F) * \emptyset_k\| \to 0 \text{ for each } k \in \mathbb{N}.$$

Let S be a compact set in$(0, \infty)$. Then $\widetilde{\emptyset_k} > 0$ on S, for some $k \in \mathbb{N}$. Since $\widetilde{\emptyset_k}$ is a continuous function, Consider

$$\widetilde{F_n}.\widetilde{\emptyset_k} - \tilde{F}.\widetilde{\emptyset_k} = [(F_n \widetilde{- F) *} \emptyset_k] \; and \; \|(F_n - F) * \emptyset_k\| \to 0 \; as \; n \to \infty.$$

$$i.e. \widetilde{F_n}.\widetilde{\emptyset_k} \to \tilde{F}.\widetilde{\emptyset_k} \text{ , converges uniformly on S.}$$

The case (i) to (iv) can be proved easily.

Lemma 4.3. Let $f \in L^1 \; and \; f_n(x) = \frac{1}{2\pi}\int_{-r}^{r} \tilde{f}(t)\, e^{it/x} dt$. Then (f_n) converges to f in L^1.

Theorem 4.4. Let $F \in B_L^1 \; and \; f_n(x) = \frac{1}{2\pi}\int_{-r}^{r} \tilde{f}(t)\, e^{it/x} dt$. Then $\delta - \lim f_n = F$.

Hence we can also have $\Delta - \lim f_n = F$.

Proof: Let $F = [g_n/\emptyset_n]$ and $k \in \mathbb{N}$. Then

$$(f_n * \emptyset_k)(x) = \int_0^\infty f_n(x-u)\,\emptyset_k(u)du$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \tilde{F}(t)\, e^{it\left(\frac{1}{x}-\frac{1}{u}\right)} \emptyset_k(u)dtdu$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \tilde{F}(t)\, e^{\frac{it}{x}}dt \int_0^\infty \emptyset_k(u)e^{-\frac{it}{u}}\,du$$

$$= \frac{1}{2\pi}\int_{-r}^{r} \tilde{F}(t)\widetilde{\emptyset_k}(t)\, e^{it/x}dt.$$

Therefore by lemma (4.1), $\|f_n * \emptyset_k - F * \emptyset_k\| \to 0$ as $n \to \infty$. Since $k \in \mathbb{Z}^+$is arbitrary we have, $\delta - \lim f_n = F$.

From (iv) and (v) in Theorem (4.2), there are separate points for the family of continuous functional onB_L^1.

Hence, we can have the following characterization-

Theorem 4.5. If a function Γ(t) is defined on [0,1] with values inB_L^1 such that the derivative Γ'(t) exists and equal to 0 at each point then Γ(t) is continuous function.

5. CONCLUSION

The space B_L^1 contains some elements which are not Schwartz distributions. The Kamal transform for Integrable Boehmians is obtained with some basic properties. An inversion theorem for Kamal transform is also discussed.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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