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DUAL CONTINUOUS K-FRAMES IN HILBERT SPACES

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Abstract. Frame theory is an active research area in mathematics, computer science and engineering with many exciting applications in a variety of different fields. In this paper we study the notion of dual continuous K-frames in Hilbert spaces. Also we establish some properties.

Keywords: continuous frame; continuous K-frame; dual continuous K-frame.

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1. Introduction

A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are generally called the frame coefficients of the vector.

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Introduced by Duffin and Schaeffer in 1952 [6] to study some deep problems in nonharmonic Fourier series, the theory of frame in Hilbert space has grown rapidly. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frames theory began to be widely used, particularly in the more specialized context of wavelet frames [3] and Gabor frames [8].

A discrete frame in a separable Hilbert space $\mathscr{H}$ is a sequence $\{f_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ called frame bounds such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \ \forall x \in \mathscr{H}.$$

The continuous frames has been defined by Ali, Antoine and Gazeau [1], called frames associated with measurable space. For more details, the reader can refer to [9]. The concept of continuous K-frame in Hilbert space have been introduced in [10].

Many generalizations of the concept of frame have been defined in Hilbert Spaces and Hilbert C^*-modules [11, 12, 13, 14].

In this papers, we characterize the concept of dual continuous K-frames in Hilbert spaces and we give some new properties.

2. Preliminaries

Let X be a Banach space, (Ω, μ) a measure space, and function $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function f has defined Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions.

Let (Ω, μ) be a measure space, let H and K be two separables Hilbert Spaces, we denote $B(H, K)$ the collection of all bounded linear operators from H to K, as well $B(H, H)$ is abbreaviated to $B(H)$.

For $T \in B(H, K)$, we use the notation $\mathscr{R}(T)$ and $\mathscr{N}(T)$ to denote respectively the range and the null space of T.

Definition 2.1. [9] Let H be a complex Hilbert space, and (Ω, μ) be a measure space with positive measure μ.

A map $F : \Omega \longrightarrow H$ is called a continuous frame with respect to (Ω, μ) if :

1 - F is weakly measurable, ie: $\forall f \in H, \quad w \longrightarrow \langle f, F(\omega)\rangle$ is a measurable function on Ω.

2 - There exists two constants $A, B > 0$ such that :

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2 \quad \forall f \in H. \tag{2.1}$$

For continuous frame F, the analysis operator T is defined by :

$$\begin{aligned} T :&H \longrightarrow L^2(\Omega) \\ &f \longrightarrow \{\langle f, F(\omega)\rangle\}_{\omega\in\Omega} \end{aligned}$$

The adjoint operator of T, called synthesis operator, is defined by :

$$\begin{aligned} T^* :&L^2(\Omega) \longrightarrow H \\ &x \longrightarrow \int_\Omega x(\omega)F(\omega)d\mu(\omega) \end{aligned}$$

The frame operator of the continuous frame F is defined by : $S = T^*T$ such that, is bounded and invertible.

Recall that a continuous Bessel sequence G is a dual continuous frame of F if :

$$f = \int_\Omega \langle f, G(\omega)\rangle F(\omega)d\mu(\omega) \qquad \forall f \in H$$

We have :

$$f = \int_\Omega \langle f, S_F^{-1}F(\omega)\rangle F(\omega)d\mu(\omega) \qquad \forall f \in H$$

This show that $S_F^{-1}F$ is a dual continuous frame of F, called the canonical dual continuous frame of F.

Definition 2.2. [10] Let $K \in B(H)$, a map $F : \Omega \longrightarrow H$ is said to be a continuous K-frame, if there exists a constants $0 < A < B < \infty$ such that :

$$A\|K^*f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2 \quad \forall f \in H.$$

The constants A and B are called the lower and upper continuous K-frame bounds.

If $A = B$, F is called a tight continuous K-frame.

If $A = B = 1$, F is called Parseval continuous K-frame.

If (2.1) holds right, F is called continuous K-Bessel sequence.

In the following definition we will recall the definition of dual continuous K-frame.

Definition 2.3. [10] Let $K \in B(H)$, and $F : \Omega \longrightarrow H$ be a continuous Bessel mapping for H, and $G : \Omega \longrightarrow H$ be a continuous Bessel mapping for H, we say that F, G is a continuous $K-$dual pair, if:

$$Kf = \int_{\Omega} \langle f, G(\omega) \rangle F(\omega) d\mu(\omega) \quad \forall f \in H.$$

Definition 2.4. [2] A Bessel mapping F is said to be L^2-independent if $\int_{\Omega} \phi(\omega) F(\omega) d\mu(\omega) = 0$ for $\phi \in L^2(\Omega)$, implies that $\phi = 0$ a. e.

The following lemmas will be used to prove our mains results.

Lemma 2.5. [4] *Let $\Lambda \in B(H,K)$ has a closed range, then there exists a unique operator $\Lambda^\dagger \in B(K,H)$, called the pseudo-inverse of Λ, satisfying :*

$$\Lambda\Lambda^\dagger\Lambda = \Lambda \qquad (\Lambda\Lambda^\dagger)^* = \Lambda\Lambda^\dagger$$

$$\Lambda^\dagger\Lambda\Lambda^\dagger = \Lambda^\dagger \qquad (\Lambda^\dagger\Lambda)^* = \Lambda^\dagger\Lambda \qquad (\Lambda^*)^\dagger = (\Lambda^\dagger)^*$$

$$\mathscr{N}(\Lambda^\dagger) = (\mathscr{R}(\Lambda))^\perp \qquad \mathscr{R}(\Lambda^\dagger) = (\mathscr{N}(\Lambda))^\perp$$

Lemma 2.6. [5] *Let H, H_1 and H_2 be three Hilbert Spaces, also let $S \in B(H_1, H)$ and $T \in B(H_2, H)$. The following statements are equivalent:*

1 - $\mathscr{R}(S) \subset \mathscr{R}(T)$.

2 - *There exist $\lambda > 0$ such that $SS^* \leq \lambda TT^*$.*

3 - *There exists $\theta \in B(H_1, H_2)$ such that $S = T\theta$.*

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator θ such that :

a - $\|\theta\|^2 = inf\{\mu : SS^* \leq \mu TT^*\}$.

b - $\mathscr{N}(S) = \mathscr{N}(\theta)$.

c - $\mathscr{R}(\theta) \subset \overline{\mathscr{R}(T^*)}$.

Lemma 2.7. [15] *Let (Ω, μ) be a measure space, X and Y are two Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to X$ measurable function; then,*

$$\lambda\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} (\lambda f) d\mu.$$

3. MAIN RESULT

Before giving our main results, we will first demonstrate the following lemmas.

Lemma 3.1. *Let $K \in B(H)$ and F be a continuous Bessel sequence of H with analysis operator T. Then F is a continuous K-frame of H if and only if:*
$\mathscr{R}(K) \subset \mathscr{R}(T^*)$.

Proof. It is an immediate consequence of Lemma 2.6. □

Lemma 3.2. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H, then $K^\dagger F$ is a dual continuous K-Bessel sequence of F.*

Proof. F is a parseval continuous K-frame of H, then :

$$\|K^* f\|^2 = \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \qquad \forall f \in H. \tag{3.1}$$

Let $g \in \mathscr{R}(K^*)$, we have : $g = K^*(K^*)^\dagger g = K^*(K^\dagger)^* g$. Replace f by $(K^*)^\dagger g$ in (3.1), then :

$$\|K^*(K^*)^\dagger g\|^2 = \int_\Omega |\langle (K^*)^\dagger g, F(\omega)\rangle|^2 d\mu(\omega)$$

so,

$$\|g\|^2 = \int_\Omega |\langle g, K^\dagger F(\omega)\rangle|^2 d\mu(\omega).$$

Hence, $K^\dagger F$ is a continuous Bessel sequence.

Since F is a parseval continuous K-frame, one has

$$KK^* f = \int_\Omega \langle f, F(\omega)\rangle F(\omega) d\mu(\omega).$$

Then we have : $g = K^*(K^*)^\dagger g = K^*(K^\dagger)^* g$,

$$\begin{aligned} Kg &= KK^*(K^\dagger)^* g \\ &= \int_\Omega \langle (K^\dagger)^* g, F(\omega)\rangle F(\omega) d\mu(\omega) \\ &= \int_\Omega \langle g, K^\dagger F(\omega)\rangle F(\omega) d\mu(\omega). \end{aligned}$$

If $h \in (\mathscr{R}(K^*))^{\perp} = \mathscr{N}(K)$

lemma 2.5 $\Longrightarrow h \in (\mathscr{N}((K^*)^{\dagger}) = \mathscr{N}((K^{\dagger})^*)$,

then :

$$\int_{\Omega} \langle h, K^{\dagger} F(\omega) \rangle F(\omega) d\mu(\omega) = \int_{\Omega} \langle (K^{\dagger})^* h, F(\omega) \rangle F(\omega) d\mu(\omega) = 0 = Kh.$$

So, for all $f \in H$, we have :

$$Kf = \int_{\Omega} \langle g, K^{\dagger} F(\omega) \rangle F(\omega) d\mu(\omega).$$

□

Lemma 3.3. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H with analysis operator T, then G is a dual continuous K-Bessel sequence of F if and only if there exists $\varphi \in B(H, L^2(\Omega))$ such that : $T^*\varphi = 0$ and $(\varphi f)_{\omega} = \langle f, G(\omega) - K^{\dagger} F(\omega) \rangle \quad \forall f \in H, \quad \forall \omega \in \Omega$.*

Proof. Let G be a dual continuous K-Bessel sequence of F,

$$\varphi : H \longrightarrow L^2(\Omega)$$

$$f \longrightarrow \varphi f$$

wich is defined by: $(\varphi f)_{\omega} = \langle f, G(\omega) - K^{\dagger} F(\omega) \rangle$. One has

$$\begin{aligned} T^*(\varphi f) &= \int_{\Omega} \langle f, G(\omega) - K^{\dagger} F(\omega) \rangle F(\omega) d\mu(\omega) \\ &= \int_{\Omega} \langle f, G(\omega) \rangle F(\omega) d\mu(\omega) - \int_{\Omega} \langle f, K^{\dagger} F(\omega) \rangle F(\omega) d\mu(\omega) \\ &= Kf - Kf = 0. \end{aligned}$$

Conversely, suppose that exist $\varphi \in B(H, L^2(\Omega))$ such that : $T^*\varphi = 0$ and

$$(\varphi f)_{\omega} = \langle f, G(\omega) - K^{\dagger} F(\omega) \rangle \quad \forall f \in H, \quad \forall \omega \in \Omega.$$

We have :

$$\begin{aligned} \int_{\Omega} \langle f, G(\omega) \rangle F(\omega) d\mu(\omega) &= \int_{\Omega} \langle f, K^{\dagger} F(\omega) \rangle F(\omega) d\mu(\omega) \\ &= Kf. \end{aligned}$$

□

Suppose that $K \in B(H)$ and F is a continuous K-frame. The dual continuous K-Bessel sequence of F such that the square of the norm of its analysis operator equals to the optimal lower continuous K-frame bound of F is called the canonical dual continuous K-Bessel sequence of F.

Theorem 3.4. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H with analysis operator T_F, then $K^\dagger F$ is the canonical dual continuous K-Bessel sequence of F.*

Proof. From lemma 3.2, we know that $K^\dagger F$ is a dual continuous K-Bessel sequence of F. To complete the proof, by definition of the canonical dual continuous K-Bessel sequence of F it only needs to prove: $\|T_{F'}\| \leq \|T_G\|$ for any dual continuous K-Bessel sequence G of F, where $T_{F'}$ is the analysis operator of $K^\dagger F$.
By lemma 3.3, there exist $\varphi \in B(H, L^2(\Omega))$ such that $T_F^*\varphi = 0$ and

$$(\varphi f)_\omega = \langle f, G(\omega) - K^\dagger F(\omega)\rangle \quad \forall f \in H, \quad \forall \omega \in \Omega.$$

On the other hand : $T_G^* = T_{F'}^* + \varphi$,

$$\begin{aligned}
\|T_G^* f\|^2 &= \langle T_G^* f, T_G^* f\rangle \\
&= \langle T_{F'}^* f + \varphi f, T_{F'}^* f + \varphi f\rangle \\
&= \|T_{F'}^* f\|^2 + \langle T_{F'}^* f, \varphi f\rangle + \langle \varphi f, T_{F'}^* f\rangle + \|\varphi f\|^2 \\
&= \|T_{F'}^* f\|^2 + \|\varphi f\|^2 \geq \|T_{F'}^* f\|^2
\end{aligned}$$

Hence, $\|T_{F'}\| \leq \|T_G\|$ as desired. □

Lemma 3.5. 1 - *The canonical continuous dual K-Bessel sequence of a parseval continuous K-frame F, wich will be denoted by $\tilde{F}$ later, is actually a parseval continuous frame on $(\mathcal{N}(K))^\perp$.*

2 - *The canonical dual continuous K-Bessel sequence of parseval continuous K-frame F is precisely a parseval continuous $K^\dagger K$-frame. But in general it is not a parseval continuous K-frame. It can naturally generate a new one in the form $K\tilde{F}$.*

Proof. 1 -

$$\int_\Omega |\langle f,\tilde{F}(\omega)\rangle|^2 d\mu(\omega) = \|K^*(K^\dagger)^* f\|^2 = \|(K^\dagger K)^* f\|^2 = \|K^\dagger K f\|^2 = \|f\|^2.$$

2 -

$$\begin{aligned}\int_\Omega |\langle f,\tilde{F}(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle f,K^\dagger F(\omega)\rangle|^2 d\mu(\omega)\\ &= \|K^*(K^\dagger)^* f\|^2\\ &= \|(K^\dagger K)^* f\|^2 \qquad \forall f\in H,\end{aligned}$$

$$\begin{aligned}\int_\Omega |\langle f,K\tilde{F}(\omega)\rangle|^2 d\mu(\omega) &= \int_\Omega |\langle f,KK^\dagger F(\omega)\rangle|^2 d\mu(\omega)\\ &= \int_\Omega |\langle (KK^\dagger)^* f,F(\omega)\rangle|^2 d\mu(\omega)\\ &= \|K^*(KK^\dagger)^* f\|^2 = \|(KK^\dagger K)^* f\|^2\\ &= \|K^* f\|^2 \qquad \forall f\in H.\end{aligned}$$

□

Theorem 3.6. *Suppose that $K\in B(H)$ has closed range and F is a parseval continuous K-frame of H with a dual continuous K-Bessel sequence G. Then G is the canonical dual continuous K-Bessel sequence of F if and only if $T_G^*T_G = T_G^*T_H$ for any dual continuous K-Bessel sequence H of F, where T_G and T_H denote the analysis operators of G and H respectively.*

Proof. Let us first assume that $G=\tilde{F}$.

If we denote by T_F the analysis operator of F then a direct calculation can show that $T_G = T_F(K^\dagger)^*$.

From this fact and taking into account the fact that :

$$T_F^*(T_G f - T_H f) = \int_\Omega \langle f,\tilde{F}(\omega)\rangle F(\omega)d\mu(\omega) - \int_\Omega \langle f,H(\omega)\rangle F(\omega)d\mu(\omega) = 0.$$

We obtain for any $f,g\in H$:

$$\langle (T_G - T_H)f, T_G g\rangle = \langle (T_G - T_H)f, T_F(K^\dagger)^* g\rangle = \langle K^\dagger T_F^*(T_G - T_H)f, g\rangle = 0.$$

Thus $T_G^*(T_G f - T_H f) = 0$ then $T_G^* T_G = T_G^* T_H$.

For the converse, suppose that $T_G^* T_G = T_G^* T_H$, for any dual continuous K-Bessel sequence H of F. Then :

$$\|T_G\|^2 = \|T_G^* T_G\| = \|T_G^* T_H\| \leq \|T_G\| \|T_H\|.$$

So, $\|T_G\| \leq \|T_H\|$ implying that G is the canonical continous K-Bessel sequence of F. □

Now it is legitimate to pose the following question: Under what condition will a parseval continuous K-frame admit a unique dual continuous K-Bessel sequence?

Theorem 3.7. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H with analysis operator T_F. Then F has a unique dual continuous K-Bessel sequence if and only if $\mathscr{R}(T_F) = L^2(\Omega)$.*

Proof. Suppose that $\mathscr{R}(T_F) = L^2(\Omega)$, then T_F^* is injective. Let G and Q be two dual continuous K-Bessel sequences of F. Then: $\{\langle f, G(\omega) - Q(\omega)\rangle\}_{\omega\in\Omega} \in L^2(\Omega)$ and that :

$$\begin{aligned}
0 = Kf - Kf &= \int_\Omega \langle f, G(\omega)\rangle F(\omega) d\mu(\omega) - \int_\Omega \langle f, Q(\omega)\rangle F(\omega) d\mu(\omega) \\
&= \int_\Omega \langle f, G(\omega) - Q(\omega)\rangle F(\omega) d\mu(\omega) \\
&= T_F^*(\{\langle f, G(\omega) - Q(\omega)\rangle\}_{\omega\in\Omega}).
\end{aligned}$$

Since, T_F^* is injective, we have

$$\langle f, G(\omega) - Q(\omega)\rangle = 0 \quad \forall \omega \in \Omega \quad and \quad \forall f \in H,$$

hence

$$G(\omega) = Q(\omega) \quad \forall \omega \in \Omega,$$

so $G = Q$.

Conversely, assume contrarity that $\mathscr{R}(T_F) \neq L^2(\Omega)$.

Since F is a parseval continuous K-frame, it is easely seen that $KK^* = T_F^* T_F$.

Hence, $\mathscr{R}(T_F^*) = \mathscr{R}(K)$, by lemma 2.6, and T_F^* has closed range as a consequence.

Let $S \in B(H, L^2(\Omega))$ be an invertible operator and $0 \neq \alpha \in (\mathscr{R}(T_F))^{\perp}$.

Taking $h = S^{-1}(\alpha)$ and $G(\omega) = \alpha(\bar{\omega})h$, for each $\omega \in \Omega$, for every $f \in H$, we have:

$$\begin{aligned}\int_{\Omega} |\langle f, G(\omega)\rangle|^2 d\mu(\omega) &= \int_{\Omega} |\langle f, \alpha(\bar{\omega})h\rangle|^2 d\mu(\omega)\\ &= \int_{\Omega} |\langle f, h\rangle|^2 |\alpha(\omega)|^2 d\mu(\omega)\\ &= |\langle f, h\rangle|^2 \|\alpha\|_2^2\\ &\leq \|\alpha\|_2^2 \|h\|^2 \|f\|^2.\end{aligned}$$

Hence, G is a continuous K-Bessel sequence for H.

Now, let $Q(\omega) = \tilde{F}(\omega) + G(\omega)$ for every $\omega \in \Omega$, then it is easely seen that Q is a continuous K-Bessel sequence for H.

Since α is orthogonal to $\mathscr{R}(T_F)$,

$$\begin{aligned}\langle \int_{\Omega} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega), e\rangle &= \int_{\Omega} \langle f, \alpha(\bar{\omega})h\rangle \langle F(\omega), e\rangle d\mu(\omega)\\ &= \int_{\Omega} \langle f, h\rangle \alpha(\omega) \langle F(\omega), e\rangle d\mu(\omega)\\ &= \langle f, h\rangle \langle \alpha, \{\langle e, F(\omega)\rangle\}_{\omega\in\Omega}\rangle\\ &= \langle f, h\rangle \langle \alpha, T_F e\rangle = 0 \quad \forall e, f \in H.\end{aligned}$$

Then $\int_{\Omega} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega) = 0 \quad \forall f \in H$.

Which give:

$$\begin{aligned}\int_{\Omega} \langle f, Q(\omega)\rangle F(\omega) d\mu(\omega) &= \int_{\Omega} \langle f, \tilde{F}(\omega)\rangle F(\omega) d\mu(\omega) + \int_{\Omega} \langle f, G(\omega)\rangle F(\omega) d\mu(\omega)\\ &= \int_{\Omega} \langle f, \tilde{F}(\omega)\rangle F(\omega) d\mu(\omega) = Kf.\end{aligned}$$

Since $\alpha \neq 0$, there exists $w_0 \in \Omega$ such that $\alpha(w_0) \neq 0$, and thus $G(w_0) \neq 0$. A simple calculation gives $(\frac{\alpha(w_0)}{|\alpha(w_0)|^2}) S(G(w_0)) = \alpha$.

Hence Q is a dual continuous K-Bessel sequence of F and its different from $\tilde{F}$ wich is a contradiction. □

Theorem 3.8. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H then the following results hold:*

1 - *F is continuous L^2-independent if and only if $\tilde{F}$ is continuous L^2-independent.*

2 - *If F admits a unique dual continuous K- Bessel sequence then $\tilde{F}$ admits a unique dual continuous K^*-Bessel sequence.*

Proof. (1) One has

$$\begin{aligned}\int_\Omega \langle f, F(\omega)\rangle F(\omega)d\mu(\omega) &= T_F^* T_F f = KK^* f\\ &= \int_\Omega \langle K^* f, \tilde{F}(\omega)\rangle F(\omega)d\mu(\omega)\\ &= \int_\Omega \langle f, K\tilde{F}(\omega)\rangle F(\omega)d\mu(\omega) \qquad \forall f\in H.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= \int_\Omega \langle f, F(\omega)\rangle F(\omega)d\mu(\omega) - \int_\Omega \langle f, K\tilde{F}(\omega)\rangle F(\omega)d\mu(\omega)\\ &= \int_\Omega \langle f, F(\omega) - K\tilde{F}(\omega)\rangle F(\omega)d\mu(\omega).\end{aligned}$$

Since, F is continuous L^2-independent, it is follows that:

$$\langle f, F(\omega) - K\tilde{F}(\omega)\rangle = 0 \qquad \forall f\in H, \quad \forall \omega\in\Omega.$$

So $F = K\tilde{F}$.

Suppose now that $\int_\Omega c(\omega)\tilde{F}(\omega)d\mu(\omega) = 0 \quad$ for some $c\in L^2(\Omega)$, then by lemma 2.7:

$$0 = K\int_\Omega c(\omega)\tilde{F}(\omega)d\mu(\omega) = \int_\Omega c(\omega)K\tilde{F}(\omega)d\mu(\omega) = \int_\Omega c(\omega)F(\omega)d\mu(\omega).$$

So, $c(\omega) = 0 \qquad \forall\omega\in\Omega$, because F is continuous $L^2(\Omega)$ independent.

For the converse, let $\int_\Omega c(\omega)F(\omega)d\mu(\omega) = 0 \quad$ for $c\in L^2(\Omega)$, then by lemma 2.7:

$$0 = K^\dagger\int_\Omega c(\omega)F(\omega)d\mu(\omega) = \int_\Omega c(\omega)K^\dagger F(\omega)d\mu(\omega) = \int_\Omega c(\omega)\tilde{F}(\omega)d\mu(\omega)$$

Then $c(\omega) = 0 \qquad \forall\omega\in\Omega$.

(2) Since, F has a unique dual continuous K-Bessel, by theorem 3.7 we know that its analysis operator T_F is surjective and thus T_F^* is injective, which implies that F is continuous L^2-independent.

Hence, by (1), $\tilde{F}$ is also continuous L^2-independent, from wich we conclude that $\tilde{F}$ has a unique dual continuous K^*-Bessel sequence. ☐

Theorem 3.9. *Suppose that $K \in B(H)$ has closed range and F is a parseval continuous K-frame of H. Then for any $c \in L^2(\Omega)$ satisfying the equation:*
$Kf = \int_\Omega c(\omega)F(\omega)d\mu(\omega)$, we have:

$$\int_\Omega |c(\omega)|^2 d\mu(\omega) = \int_\Omega |c(\omega) - \langle f, \tilde{F}(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega |\langle f, \tilde{F}(\omega)\rangle|^2 d\mu(\omega).$$

Proof. We have :

$$\begin{aligned}
\int_\Omega (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\langle \tilde{F}(\omega), f\rangle d\mu(\omega) &= \int_\Omega \langle (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\tilde{F}(\omega), f\rangle d\mu(\omega) \\
&= \langle \int_\Omega (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\tilde{F}(\omega), f\rangle d\mu(\omega) \\
&= \langle K^\dagger \int_\Omega (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)F(\omega), f\rangle d\mu(\omega) \\
&= \langle K^\dagger (Kf - Kf), f\rangle = 0 \qquad \forall f \in H.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega |c(\omega)|^2 d\mu(\omega) \\
&= \int_\Omega c(\omega)\overline{c(\omega)} d\mu(\omega) \\
&= \int_\Omega [(c(\omega) - \langle f, \tilde{F}(\omega)\rangle) + \langle f, \tilde{F}(\omega)\rangle]\overline{[(c(\omega) - \langle f, \tilde{F}(\omega)\rangle) + \langle f, \tilde{F}(\omega)\rangle]} d\mu(\omega) \\
&= \int_\Omega (((c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\overline{(c(\omega) - \langle f, \tilde{F}(\omega)\rangle)} + (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\langle \tilde{F}(\omega), f\rangle \\
&+ \langle f, \tilde{F}(\omega)\rangle\overline{(c(\omega) - \langle f, \tilde{F}(\omega)\rangle)} + \langle f, \tilde{F}(\omega)\rangle\langle \tilde{F}(\omega), f\rangle)) d\mu(\omega) \\
&= \int_\Omega (c(\omega) - \langle f, \tilde{F}(\omega)\rangle)\overline{(c(\omega) - \langle f, \tilde{F}(\omega)\rangle)} d\mu(\omega) + \int_\Omega \langle f, \tilde{F}(\omega)\rangle\langle \tilde{F}(\omega), f\rangle d\mu(\omega) \\
&= \int_\Omega |c(\omega) - \langle f, \tilde{F}(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega |\langle f, \tilde{F}(\omega)\rangle|^2 d\mu(\omega).
\end{aligned}$$

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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