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STOCHASTIC DECOMPOSITION OF THE $M/M/1$ QUEUE WITH ENVIRONMENT DEPENDENT WORKING VACATION

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Abstract. We consider an M/M/1 queue with n types of working vacation. After a non zero busy period if the server finds the system empty, it opts for one of the n types of working vacation depending on the environment. On vacation completion epoch finding an empty system, it remains in the respective vacation. We demonstrate the stochastic decomposition structure of the queue length and waiting time of this M/M/1 queue and obtain the distribution of the additional queue length and additional delay.

Keywords: working vacation; environment; stochastic decomposition.

2010 AMS Subject Classification: 60K25, 60K30.

1. Introduction

If a queue is empty the server remains idle. The idle time of the server can be utilized for supplementary jobs. This gives rise to extensive research work in the field of vacation queueing models. The details regarding the research on queueing models can be found in the survey of Doshi [1], the monograph of Takagi [2] and Tian and Zhang [3]. If the number of customers in the queue is less, the functioning of the server at a slow rate will reduce the operating cost, energy consumption, and start-up cost. These advantages are pointing towards

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working vacation. A working vacation is an extension of regular vacation. In working vacation, instead of completely stopping the service, the server provides service at a slow rate. Working vacation reduces the chance of reneging of the customers compared to normal vacation. In this era of high demand for commodities and services which are available in a short spell, the concept of working vacation is very useful. This may be the main reason for the extensive research work going on in working vacation queueing models.

Servi and Finn [4] introduced the idea of working vacation in queueing models and applied the results to the performance analysis of gateway routers in fiber communication networks. Baba [5] studied GI/M/1 queue with working vacation.[6] gives a comprehensive overview of the research results and analysis methods of vacation queue, including its applications in the communication networks. Stochastic decomposition results for the number of customers in the system in the case of an exhaustive service were first obtained by Fuhrmann [7] and then confirmed by Doshi [1]. Fuhrmann and Cooper [8] and Shanthikumar [9] established stochastic decomposition structures for a classical M/G/1 queue with general vacations. [10] demonstrates stochastic decomposition in an M/M/1 queue with working vacation. In [11] the authors discuss an M/M/1 queue with n types of vacations where the server opts out one among these n vacations depending on the environment.

In this model, we consider a single server queueing system with working vacation. On completion of service, if the server finds the system empty, he goes for a working vacation. There are n types of working vacations. After a busy period, depending on the environment, the server opts for i^{th} type of vacation with probability $p_i, 1 \leq i \leq n$. During vacation, if customers arrive, the server provides service at a lower rate. On completion of service during vacation, if there is no customer in the system the server continues to be on vacation. Otherwise, the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service at the normal rate. On completion of vacation, if the server finds the system empty, he remains on the corresponding vacation. We demonstrate stochastic decomposition of the queue length and waiting time using the method of induction and Little's formula[12].

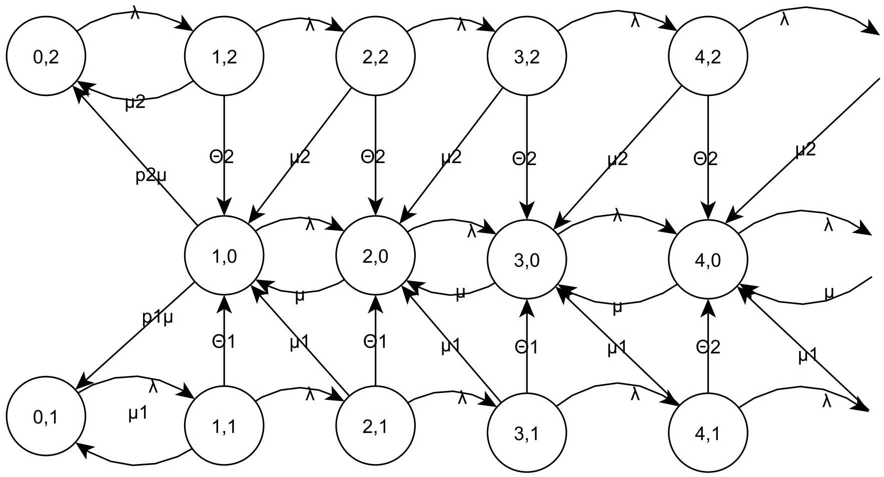

FIGURE 1. Model description

The rest of the paper is organized as follows. In section 2 the model is described in detail. Section 3 discusses the stochastic decomposition structures of the number of customers in the system and waiting time and obtains the distributions of additional queue length and additional delay.

2. MODEL DESCRIPTION

Consider a single server queueing system with working vacation in which arrival occurs according to a Poisson process with parameter λ. The service time is exponentially distributed with parameter μ. On completion of a service if the server finds the system empty it goes for a working vacation. There are n types of working vacations. Depending on the environment, after a busy period, the server goes for i^{th} type of vacation with probability $p_i, 1 \leq i \leq n$. The duration of i^{th} type of vacation is exponentially distributed with parameter $\gamma_i, 1 \leq i \leq n$. During vacation, if customers arrive, the server provides service at a lower rate μ_i, provided the server is in i^{th} type of vacation, $1 \leq i \leq n$. On completion of service during vacation, if there is no customer in the system the server continues to stay on vacation. Otherwise, the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service at the normal rate μ. On completion of vacation, if the server finds the system empty, it remains on the corresponding vacations. Figure 1 is a diagrammatic representation of the model.

3. Mathematical Description

We establish the stochastic decomposition of the state space by induction on the number of environmental factors.

Case.1 First we consider the case of $n = 2$.

Let $N(t)$ be the number of customers in the system and $S(t)$ be the status of the server at time t:

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ 1, & \text{if server is in the type I working vacation;} \\ 2, & \text{if server is in the type II working vacation;} \end{cases}$$

Then $X = \{X(t), t \geq 0\}$ where $X(t) = (N(t), S(t))$ is a continuous time Markov chain with state space $\{0,1\} \cup \{0,2\} \cup \{(j,k),\ j = 1,2,\ldots; k = 0,1,2\}$. The infinitesimal generator associated with the Markov chain is

$$Q_1 = \begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \text{ where } -B_0 = A_0 = \lambda I,$$

$$B_1 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 \\ \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, A_2 = \begin{bmatrix} \mu & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda - \mu & 0 & 0 \\ \theta_1 & -\lambda - \mu_1 - \theta_1 & 0 \\ \theta_2 & 0 & -\lambda - \mu_2 - \theta_2 \end{bmatrix}$$

Stability analysis. We have $A = A_0 + A_1 + A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \theta_1 + \mu_1 & -\mu_1 - \theta_1 & 0 \\ \theta_2 + \mu_2 & 0 & -\mu_2 - \theta_2 \end{bmatrix}$

Then A is the infinitesimal generator of a Markov chain with state space $\{0,1,2\}$ which represents the status of the server. Let $\mathbf{y} = (y_0, y_1, y_2)$ be the invariant probability vector of A. Then

$\mathbf{y}A = 0$ and $\mathbf{y}e = 1$. The left drift rate of the original Markov chain is $\mathbf{y}A_2e$ and that for right drift is $\mathbf{y}A_0e$. Left drift indicates a service completion and right drift represents arrival of customer. Thus the system is stable if and only if $\mathbf{y}A_0e < \mathbf{y}A_2e$. Here $\mathbf{y}A_0e = \lambda$ and $\mathbf{y}A_2e = \mu$.

Hence we have

Theorem: The system is stable if and only if $\lambda < \mu$.

3.1. Steady State Analysis.

For the analysis of the model it is necessary to solve for the minimal non-negative solution R_1 of the matrix quadratic equation

(1) $$R_1^2A_2 + R_1A_1 + A_0 = 0.$$

Since the Matrices A_2, A_1, A_0 are lower triangular R_1 is also lower triangular. Solving (1) we obtain R_1 as $R_1 = \begin{bmatrix} r_0 & 0 & 0 \\ r_1 & \overline{r}_1 & 0 \\ r_2 & 0 & \overline{r}_2 \end{bmatrix}$ where $r_0 = \rho$, $r_1 = \frac{\rho(\lambda+\theta_1)}{(\lambda+\mu_1+\theta_1)}$, $\overline{r}_1 = \frac{\lambda}{(\lambda+\mu_1+\theta_1)}$, $r_2 = \frac{\rho(\lambda+\theta_2)}{(\lambda+\mu_2+\theta_2)}$ and $\overline{r}_2 = \frac{\lambda}{(\lambda+\mu_2+\theta_2)}$.

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$ be the steady state probability vector associated with the Markov process X. Here $\mathbf{x}_0 = (x_{01}, x_{02})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}), i = 1, 2, \ldots, \infty$. Assume that $\mathbf{x}_i = \mathbf{x}_1R_1^{i-1}, i = 2, 3, \ldots$, then $\mathbf{x}$ can be obtained by solving $\mathbf{x}Q = 0$ using the boundary condition

(2) $$\mathbf{x}_0e + \mathbf{x}_1(I - R_1)^{-1}e = 1.$$

From $\mathbf{x}Q = 0$ we get

(3) $$\mathbf{x}_0B_0 + \mathbf{x}_1B_2 = 0.$$

(4) $$\mathbf{x}_0B_1 + \mathbf{x}_1(A_1 + R_1A_2) = 0.$$

From (3) and (4) we will get

(5) $$\mu p_1x_{10} + \mu_1x_{11} = (\lambda)x_{01}.$$

(6) $$\mu p_2x_{10} + \mu_2x_{12} = (\lambda)x_{02}.$$

(7) $$\mu x_{10} = (\lambda + \theta_1)x_{11} + (\lambda + \theta_2)x_{12}.$$

(8) $$\lambda x_{01} = (\lambda + \mu_1 + \theta_1)x_{11}.$$

(9) $$\lambda x_{02} = (\lambda + \mu_2 + \theta_2)x_{12}.$$

Assume $x_{01} = k_1$ and $x_{02} = k_2$, then from (8) and (9), $x_{11} = \bar{r}_1 k_1,\ \ x_{12} = \bar{r}_2 k_2$. Substituting the values of x_{11} and x_{01} in (5) we will get $x_{10} = \frac{k_1 \bar{r}_1}{p_1}$. Also

$$k_2 = \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} k_1$$

To find the value of k_1 we use the normalizing condition

$\mathbf{x}_0 e + \mathbf{x}_1 (I - R_1)^{-1} e = 1.$

Let $r_0^{'} = 1 - r_0, \bar{r}_1^{'} = 1 - \bar{r}_1, \bar{r}_2^{'} = 1 - \bar{r}_2$; then $(I - R_1)^{-1} = \begin{bmatrix} 1/r_0^{'} & 0 & 0 \\ -r_1/\bar{r}_0^{'} r_1^{'} & 1/\bar{r}_1^{'} & 0 \\ -r_2/r_0^{'} \bar{r}_2^{'} & 0 & 1/\bar{r}_2^{'} \end{bmatrix}$

Using (2)

(10) $$k_1 \left[1 + \frac{r_1}{p_1 r_0^{'}} + \frac{\bar{r}_1}{\bar{r}_1^{'}} - \frac{r_1 \bar{r}_1}{r_0^{'} \bar{r}_1^{'}} \right] + k_2 \left[1 + \frac{\bar{r}_2}{\bar{r}_2^{'}} - \frac{r_2 \bar{r}_2}{r_0^{'} \bar{r}_2^{'}} \right] = 1.$$

Substituting k_2 in (10)

(11) $$k_1 \left[1 + \frac{r_1}{p_1 r_0^{'}} + \frac{\bar{r}_1}{\bar{r}_1^{'}} - \frac{r_1 \bar{r}_1}{r_0^{'} \bar{r}_1^{'}} + \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} \left[1 + \frac{\bar{r}_2}{\bar{r}_2^{'}} - \frac{r_2 \bar{r}_2}{r_0^{'} \bar{r}_2^{'}} \right] \right] = 1.$$

From (11) $k_1 = \frac{1}{\left[1 + \frac{r_1}{p_1 r_0^{'}} + \frac{\bar{r}_1}{\bar{r}_1^{'}} - \frac{r_1 \bar{r}_1}{r_0^{'} \bar{r}_1^{'}} + \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} \left[1 + \frac{\bar{r}_2}{\bar{r}_2^{'}} - \frac{r_2 \bar{r}_2}{r_0^{'} \bar{r}_2^{'}} \right] \right]}$.

Now $R_1^{k-1} = \begin{bmatrix} r_0^{(k-1)} & 0 & 0 \\ r_1 \frac{(r_0^{k-1} - \bar{r}_1^{k-1})}{(r_0 - \bar{r}_1)} & \bar{r}_1^{(k-1)} & 0 \\ r_2 \frac{(r_0^{k-1} - \bar{r}_2^{k-1})}{(r_0 - \bar{r}_2)} & 0 & \bar{r}_2^{(k-1)} \end{bmatrix}$ and

$\mathbf{x}_k \mathbf{e} = x_{10} r_0^{k-1} + x_{11} \left[\bar{r}_1^{(k-1)} + r_1 \frac{(r_0^{k-1} - \bar{r}_1^{k-1})}{(r_0 - \bar{r}_1)} \right] + x_{12} \left[\bar{r}_2^{(k-1)} + r_2 \frac{(r_0^{k-1} - \bar{r}_2^{k-1})}{(r_0 - \bar{r}_2)} \right]$ for $k > 1$.

Let $Q_v(z)$ be the PGF associated with the number of customers in the system. Then

$Q_v(z) = \sum_{n=0}^{\infty} \mathbf{x}_n \mathbf{e} z^n$

$= x_{01} + x_{02} + \frac{x_{10} z}{1 - r_0 z} + \frac{x_{11} z}{1 - \bar{r}_1 z} + \frac{x_{12} z}{1 - \bar{r}_2 z} + \frac{x_{11} r_1 z}{r_0 - \bar{r}_1} \left[\frac{1}{1 - r_0 z} - \frac{1}{1 - \bar{r}_1 z} \right] + \frac{x_{12} r_2 z}{r_0 - \bar{r}_2} \left[\frac{1}{1 - r_0 z} - \frac{1}{1 - \bar{r}_2 z} \right]$

$= \frac{1 - r_0}{1 - r_0 z} \left[x_{01} \frac{(1 - r_0 z)}{(1 - r_0)} + x_{02} \frac{(1 - r_0 z)}{(1 - r_0)} + \frac{x_{10} z}{1 - r_0} + \frac{x_{11} z}{1 - \bar{r}_1 z} \frac{(1 - r_0 z)}{(1 - r_0)} + \frac{x_{12} z}{1 - \bar{r}_2 z} \frac{(1 - r_0 z)}{(1 - r_0)} + \right.$

$\left. \frac{x_{11} r_1 z}{r_0 - \bar{r}_1} \frac{(1 - r_0 z)}{(1 - r_0)} \left[\frac{1}{1 - r_0 z} - \frac{1}{1 - \bar{r}_1 z} \right] + \frac{x_{12} r_2 z}{r_0 - \bar{r}_2} \frac{(1 - r_0 z)}{(1 - r_0)} \left[\frac{1}{1 - r_0 z} - \frac{1}{1 - \bar{r}_2 z} \right] \right].$

$Q_v^{'}(z) = \frac{r_0}{1 - r_0 z} \left[x_{01} \frac{(1 - r_0 z)}{(1 - r_0)} + x_{02} \frac{(1 - r_0 z)}{(1 - r_0)} + \frac{x_{10} z}{1 - r_0} \frac{x_{11} z}{1 - \bar{r}_1 z} \frac{(1 - r_0 z)}{(1 - r_0)} + \frac{x_{12} z}{1 - \bar{r}_2 z} \frac{(1 - r_0 z)}{(1 - r_0)} + \right.$

$$\frac{x_{11}r_1z}{r_0-\bar{r}_1}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\bar{r}_1z}\right]+\frac{x_{12}r_2z}{r_0-\bar{r}_2}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\bar{r}_2z}\right]\Bigg]+\left(\frac{1-r_0}{1-r_0z}\right)\left(\frac{1}{1-r_0}\right)$$
$$\Bigg[-r_0x_{01}-r_0x_{02}+x_{10}+\frac{x_{11}r_1}{(r_0-\bar{r}_1)}+\frac{x_{12}r_2}{(r_0-\bar{r}_2)}+x_{11}\left(\frac{r_0-r_1-\bar{r}_1}{r_0-\bar{r}_1}\right)\left(\frac{1-2r_0z+r_0\bar{r}_1z^2}{(1-\bar{r}_1z)^2}\right)+$$
$$x_{12}\left(\frac{r_0-r_2-\bar{r}_2}{r_0-\bar{r}_2}\right)\left(\frac{1-2r_0z+r_0\bar{r}_2z^2}{(1-\bar{r}_2z)^2}\right)\Bigg]$$

Expected queue length $E(\overline{L})=Q_v'(1)=\frac{r_0}{1-r_0}+\left(\frac{1}{1-r_0}\right)[-r_0x_{01}-r_0x_{02}+x_{10}$

$$+\frac{x_{11}r_1}{(r_0-\bar{r}_1)}+\frac{x_{12}r_2}{(r_0-\bar{r}_2)}+x_{11}\left(\frac{r_0-r_1-\bar{r}_1}{r_0-\bar{r}_1}\right)\left(\frac{1-2r_0+r_0\bar{r}_1}{(1-\bar{r}_1)^2}\right)+x_{12}\left(\frac{r_0-r_2-\bar{r}_2}{r_0-\bar{r}_2}\right)\left(\frac{1-2r_0+r_0\bar{r}_2}{(1-\bar{r}_2)^2}\right)\Bigg]$$
$$=\frac{r_0}{1-r_0}+\left(\frac{1}{1-r_0}\right)\Bigg[-r_0k_1-r_0k_2+\frac{k_1r_1}{p_1}+\frac{r_1k_1\bar{r}_1}{(1-\bar{r}_1)(r_0-\bar{r}_1)}+\frac{k_2\bar{r}_2r_2}{(r_0-\bar{r}_2)}+k_1\bar{r}_1\left(\frac{r_0-r_1-\bar{r}_1}{r_0-\bar{r}_1}\right)$$
$$\left(\frac{1-2r_0+r_0\bar{r}_1}{(1-\bar{r}_1)^2}\right)+k_2\bar{r}_2\left(\frac{r_0-r_2-\bar{r}_2}{r_0-\bar{r}_2}\right)\left(\frac{1-2r_0+r_0\bar{r}_2}{(1-\bar{r}_2)^2}\right)\Bigg]$$
$$=\frac{r_0}{1-r_0}+\left(\frac{k_1}{1-r_0}\right)\Bigg[-r_0+\frac{r_1}{p_1}+\frac{r_1\bar{r}_1}{(1-\bar{r}_1)(r_0-\bar{r}_1)}+\bar{r}_1\left(\frac{r_0-r_1-\bar{r}_1}{r_0-\bar{r}_1}\right)\left(\frac{1-2r_0+r_0\bar{r}_1}{(1-\bar{r}_1)^2}\right)\Bigg]+\left(\frac{k_2}{1-r_0}\right)$$
$$\Bigg[-r_0+\frac{\bar{r}_2r_2}{(r_0-\bar{r}_2)}+\bar{r}_2\left(\frac{r_0-r_2-\bar{r}_2}{r_1-\bar{r}_2}\right)\left(\frac{1-2r_0+r_1\bar{r}_2}{(1-\bar{r}_2)^2}\right)\Bigg]$$

Case.2 Now consider the case of $n=3$. Then $S(t)$ has four states.

$$S(t)=\begin{cases}0, & \text{if the server is serving in normal mode;}\\ 1, & \text{if server is in the type I working vacation;}\\ 2, & \text{if server is in the type II working vacation;}\\ 3, & \text{if server is in the type III working vacation;}\end{cases}$$

The state space of X is $\{(0,k)|k=1,2,3\}\cup\{(j,k),\ j=1,2,\ldots;k=0,1,2,3\}$. The infinitesimal generator associated with the Markov chain is $Q_2=\begin{bmatrix}B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0\\ & & & \ddots & \ddots & \ddots\\ & & & & & \end{bmatrix}$

where $-B_0=A_0=\lambda I_3,\ B_1=\begin{bmatrix}0 & \lambda & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \lambda\end{bmatrix}, B_2=\begin{bmatrix}\mu p_1 & \mu p_2 & \mu p_3\\ \mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_3\end{bmatrix},$

$$A_2 = \begin{bmatrix} \mu & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda-\mu & 0 & 0 & 0 \\ \theta_1 & -\lambda-\mu_1-\theta_1 & 0 & 0 \\ \theta_2 & 0 & -\lambda-\mu_2-\theta_2 & 0 \\ \theta_3 & 0 & 0 & -\lambda-\mu_3-\theta_3 \end{bmatrix}.$$

$$A = A_0+A_1+A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \theta_1+\mu_1 & -\mu_1-\theta_1 & 0 & 0 \\ \theta_2+\mu_2 & 0 & -\mu_2-\theta_2 & 0 \\ \theta_3+\mu_3 & 0 & 0 & -\mu_3-\theta_3 \end{bmatrix}$$

We get $\lambda < \mu$ as the condition for stability.

$$R_2 = \begin{bmatrix} r_0 & 0 & 0 & 0 \\ r_1 & \overline{r}_1 & 0 & 0 \\ r_2 & 0 & \overline{r}_2 & 0 \\ r_3 & 0 & 0 & \overline{r}_3 \end{bmatrix}$$ where $r_0 = \rho$, $r_1 = \frac{\rho(\lambda+\theta_1)}{(\lambda+\mu_1+\theta_1)}$, $\overline{r}_1 = \frac{\lambda}{(\lambda+\mu_1+\theta_1)}$, $r_2 = \frac{\rho(\lambda+\theta_2)}{(\lambda+\mu_2+\theta_2)}$, $\overline{r}_2 = \frac{\lambda}{(\lambda+\mu_2+\theta_2)}$, $r_3 = \frac{\rho(\lambda+\theta_3)}{(\lambda+\mu_3+\theta_3)}$ and $\overline{r}_3 = \frac{\lambda}{(\lambda+\mu_3+\theta_3)}$

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$ be the steady state probability vector associated with the Markov process X. Here $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}, x_{i3}), i = 1, 2, \ldots$. Then assuming $x_{01} = k_1$ $x_{02} = k_2$ and $x_{03} = k_3$, we get $x_{11} = \overline{r}_1 k_1$, $x_{12} = \overline{r}_2 k_2$ $x_{13} = \overline{r}_3 k_3$. $x_{10} = \frac{k_1 r_1}{p_1}$. Also $k_2 = \frac{\mu p_2 r_1}{p_1(\lambda-\mu_2 r_2)} k_1$, $k_3 = \frac{\mu p_3 r_1}{p_1(\lambda-\mu_3 r_3)} k_1$.

Let $r_0^{'} = 1-r_0$, $\overline{r}_1^{'} = 1-\overline{r}_1$, $\overline{r}_2^{'} = 1-\overline{r}_2$, $\overline{r}_3^{'} = 1-\overline{r}_3$ then

$$(I-R_2)^{-1} = \begin{bmatrix} 1/r_0^{'} & 0 & 0 & 0 \\ -r_1/\overline{r}_1^{'} r_0^{'} & 1/\overline{r}_1^{'} & 0 & 0 \\ -r_2/\overline{r}_2^{'} r_0^{'} & 0 & 1/\overline{r}_2^{'} & 0 \\ -r_3/\overline{r}_3^{'} r_0^{'} & 0 & 0 & 1/\overline{r}_3^{'} \end{bmatrix}$$

Using the normalizing condition $\mathbf{x}_0 e + \mathbf{x}_1 (I-R_2)^{-1} e = 1$, we get

$$k_1 = \frac{1}{\left[1+\frac{\bar{r}_1}{\bar{r}'_1}-\frac{r_1\bar{r}_1}{r'_0\bar{r}'_1}+\frac{r_1}{p_1r'_0}+\sum_{j=2}^{3}\frac{\mu p_j r_1}{p_1(\lambda-\mu_j r_j)}\left(1+\frac{\bar{r}_j}{\bar{r}'_j}-\frac{r_j\bar{r}_j}{r'_0\bar{r}'_j}\right)\right]}$$

Now $R_2^{k-1} = \begin{bmatrix} r_1^{(k-1)} & 0 & 0 & 0 \\ r_2\frac{(r_1^{k-1}-r_3^{k-1})}{(r_1-r_3)} & r_3^{(k-1)} & 0 & 0 \\ r_4\frac{(r_1^{k-1}-r_5^{k-1})}{(r_1-r_5)} & 0 & r_5^{(k-1)} & 0 \\ r_6\frac{(r_1^{k-1}-r_7^{k-1})}{(r_1-r_7)} & 0 & 0 & r_7^{(k-1)} \end{bmatrix}$ and

$\mathbf{x}_k\mathbf{e} = x_{10}r_1^{k-1} + x_{11}\left[r_3^{(k-1)} + r_2\frac{(r_1^{k-1}-r_3^{k-1})}{(r_1-r_3)}\right] + x_{12}\left[r_5^{(k-1)} + r_4\frac{(r_1^{k-1}-r_5^{k-1})}{(r_1-r_5)}\right] +$
$x_{13}\left[r_7^{(k-1)} + r_6\frac{(r_1^{k-1}-r_7^{k-1})}{(r_1-r_7)}\right]$ for $k > 1$.

$Q_v(z) = \sum_{n=0}^{\infty}\mathbf{x}_n\mathbf{e}z^n$
$= x_{01} + x_{02} + x_{03} + \frac{x_{10}z}{1-r_1z} + \frac{x_{11}z}{1-r_3z} + \frac{x_{12}z}{1-r_5z} + \frac{x_{13}z}{1-r_7z} + \frac{x_{11}r_2z}{r_1-r_3}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_3z}\right] +$
$\frac{x_{12}r_4z}{r_1-r_5}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_5z}\right] + \frac{x_{13}r_6z}{r_1-r_7}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_7z}\right]$

Expected queue length $E(\overline{L}) = Q'_v(1)$
$= \frac{r_1}{1-r_1} + \left(\frac{k_1}{1-r_1}\right)\left[-r_1 + \frac{r_2}{p_1} + \frac{r_2r_3}{(1-r_3)(r_1-r_3)} + r_3\left(\frac{r_1-r_2-r_3}{r_1-r_3}\right)\left(\frac{1-2r_1+r_1r_3}{(1-r_3)^2}\right)\right] +$
$\left(\frac{k_2}{1-r_1}\right)\left[-r_1 + \frac{r_5r_4}{(r_1-r_5)} + r_5\left(\frac{r_1-r_4-r_5}{r_1-r_5}\right)\left(\frac{1-2r_1+r_1r_5}{(1-r_5)^2}\right)\right] + \left(\frac{k_3}{1-r_1}\right)\left[-r_1 + \frac{r_7r_6}{(r_1-r_7)} +\right.$
$\left. r_7\left(\frac{r_1-r_6-r_7}{r_1-r_7}\right)\left(\frac{1-2r_1+r_1r_7}{(1-r_7)^2}\right)\right]$

Case.3

Now we consider the case where there are $n \geq 4$ distinct type of vacations. Then

$S(t)$ has $n+1$ distinct values.

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ i, & \text{if server is in the } i^{th} \text{ type working vacation, } 1 \leq i \leq n; \end{cases}$$

The state space of X is $\{(0,k)/k = 1,2,\ldots,n\} \cup \{(j,k)/j = 0,1,2,\ldots;k = 1,2,\ldots,n\}$ The infinitesimal generator associated with the Markov chain is

$$Q_n = \begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ where } B_1 = \begin{bmatrix} 0 & \lambda & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \\ & & & & \lambda \end{bmatrix}_{n\times(n+1)},$$

$$B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 & \dots & \mu p_n \\ \mu_1 & & & \\ & \mu_2 & & \\ & & & \\ & & & \mu_n \end{bmatrix}_{(n+1)\times n} \quad A_2 = \begin{bmatrix} \mu & & \\ \mu_1 & & \\ \vdots & & \\ \mu_n & & \end{bmatrix}_{(n+1)\times(n+1)}, -B_0 = A_0 = \lambda I_n$$

$$A_1 = \begin{bmatrix} -\lambda-\mu & & & & & \\ \theta_1 & -\lambda-\mu_1-\theta_1 & & & & \\ \theta_2 & & -\lambda-\mu_2-\theta_2 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ \theta_n & & & & & -\lambda-\mu_n-\theta_n \end{bmatrix}$$

As in the earlier sections

$$A_0+A_1+A_2 = \begin{bmatrix} -\lambda-\mu & & & & & \\ \theta_1+\mu_1 & -\mu_1-\theta_1 & & & & \\ \theta_2+\mu_2 & & -\mu_2-\theta_2 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ \theta_n+\mu_n & & & & & -\mu_n-\theta_n \end{bmatrix}$$

Let $\mathbf{y} = (y_0, y_1, y_2, \dots, y_n)$ be the invariant probability vector of A satisfying $\mathbf{y}A = 0$ and $\mathbf{y}e = 1$. The system is stable if and only if $\mathbf{y}A_0e < \mathbf{y}A_2e$. Here $\mathbf{y}A_0e = \lambda$ and $\mathbf{y}A_2e = \mu$.

Theorem: The system is stable if and only if $\lambda < \mu$

$$R_n = \begin{bmatrix} r_0 & & & & & \\ r_1 & \bar{r}_1 & & & & \\ r_2 & 0 & \bar{r}_2 & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ r_n & & & & 0 & \bar{r}_n \end{bmatrix} \text{ where } r_0 = \rho, r_i = \frac{\rho(\lambda+\theta_i)}{(\lambda+\mu_i+\theta_i)},$$

$\bar{r}_i = \frac{\lambda}{(\lambda+\mu_i+\theta_i)}$.

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots,)$ be the steady state probability vector associated with the Markov

chain X. Here $\mathbf{x}_0 = (x_{01}, x_{02}, \ldots, x_{0n})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}, \ldots, x_{in}), i = 1, 2, \ldots$. Then assuming $x_{0j} = k_j, 1 \le i \le n$, we get $x_{1j} = \overline{r}_j k_j$, $x_{10} = \frac{k_1 \overline{r}_1}{p_1}$.
Also $k_j = \frac{\mu p_j \overline{r}_1}{p_1(\lambda - \mu_j \overline{r}_j)} k_1$. Let $\overline{r}_i^{'} = 1 - \overline{r}_i, 1 \le i \le n, r_0^{'} = 1 - r_0$,
$\ell_i = 1/\overline{r}_i^{'}, 0 \le i \le n, \chi_i = -r_i/(\overline{r}_i^{'} \overline{r}_0^{'}), 1 \le i \le n$ then,

$$(I - R_n)^{-1} = \begin{bmatrix} \ell_0 & & & \\ \chi_1 & \ell_2 & & \\ \chi_2 & & \ell_2 & \\ & & & \\ \chi_n & & & \ell_n \end{bmatrix}$$

$$k_1 = \frac{1}{\left[1 + \frac{\overline{r}_1}{\overline{r}_1^{'}} - \frac{r_1 \overline{r}_1}{r_0^{'} \overline{r}_1^{'}} + \frac{r_1}{p_1 r_0^{'}} + \sum_{j=2}^{n} \frac{\mu p_j r_1}{p_1(\lambda - \mu_j r_j)} \left(1 + \frac{\overline{r}_j}{\overline{r}_j^{'}} - \frac{r_j \overline{r}_j}{r_0^{'} \overline{r}_j^{'}} \right) \right]}$$

$$\text{Now } R_n^{k-1} = \begin{bmatrix} r_0^{(k-1)} & & & & & \\ r_1 \frac{(r_0^{k-1} - \overline{r}_1^{k-1})}{(r_0 - \overline{r}_1)} & \overline{r}_1^{(k-1)} & & & & \\ r_2 \frac{(r_0^{k-1} - \overline{r}_2^{k-1})}{(r_0 - \overline{r}_2)} & 0 & \overline{r}_2^{(k-1)} & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ r_n \frac{(r_0^{k-1} - \overline{r}_n^{k-1})}{(r_0 - \overline{r}_n)} & & & & & \overline{r}_n^{(k-1)} \end{bmatrix} \text{ and}$$

$$\mathbf{x}_k \mathbf{e} = x_{10} r_0^{k-1} + \sum_{i=1}^{n} x_{1i} \left[\overline{r}_i^{(k-1)} + r_i \frac{(r_0^{k-1} - \overline{r}_i^{k-1})}{(r_0 - \overline{r}_i)} \right] \text{ for } k > 1.$$

Then $Q_v(z) = \sum_{n=0}^{\infty} \mathbf{x}_n z^n$

$$= \sum_{j=1}^{n} x_{0j} + \frac{x_{10} z}{1 - r_0 z} + \sum_{j=1}^{n} \frac{x_{1j} z}{1 - \overline{r}_j z} + \sum_{j=1}^{n} \frac{x_{1j} r_j z}{r_0 - \overline{r}_j} \left[\frac{1}{1 - r_0 z} - \frac{1}{1 - \overline{r}_j z} \right]$$

Expected queue length $E(\overline{L}) = Q_v^{'}(1)$

$$= \frac{r_0}{1 - r_0} + \sum_{j=1}^{n} \left(\frac{k_j}{1 - r_0} \right) \left[-r_0 + \frac{r_j}{p_1} + \frac{r_j \overline{r}_j}{(1 - \overline{r}_j)(r_0 - \overline{r}_j)} + \overline{r}_j \left(\frac{r_0 - r_j - \overline{r}_j}{r_0 - \overline{r}_j} \right) \left(\frac{1 - 2r_0 + r_0 \overline{r}_j}{(1 - \overline{r}_j)^2} \right) \right].$$

The above discussions lead to

***Theorem*(Stochastic decomposition):** The expected queue length $E(\overline{L})$ can be decomposed into the sum of the expectations of $n+1$ independent random variables as:

$E(\overline{L}) = E(L) + \sum_{i=1}^{n} E(L_{V_i})$ where $E(L)$ is the queue length of classical $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{V_i})$ is the additional queue length due to n types of vacations.

3.2. Stationary waiting time.

Using Little's formula the expected waiting time $E(\overline{W}) = \frac{E(L)}{\lambda}$.

$$E(\overline{W}) = (\frac{1}{\mu-\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} E(L_{V_i})) \tag{12}$$

From (12) it is clear that the expected waiting time can be decomposed into the sum of $n+1$ independent random variables: $E(\overline{W}) = E(W) + \sum_{i=1}^{n} E(W_{V_i})$. where $E(W)$ is the expected waiting time of a customer in the $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{W_i})$ is the additional waiting time due to n types of vacations.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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