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CUBIC PICTURE FUZZY SOFT MATRICES

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Abstract. In this paper, we define the concept of Cubic Picture Fuzzy Soft Matrices (CPFSMs). Cubic Picture Fuzzy Soft Matrices is a combination of Cubic Soft Matrices (CSMs) and Picture Fuzzy Soft Matrices (PFSMs). Furtherly, we develop the P-order and R-order of Union and Intersection of Cubic Picture Fuzzy Soft Matrices with relevant algebraic properties are investigated.

Keywords: cubic set; cubic soft set; cubic soft matrices; picture soft set; picture fuzzy soft matrices; interval valued picture fuzzy sets.

AMS Subject Classification: 03E72.

1. Introduction

Fuzzy set (FS) theory was introduced by Zadeh[9] in 1965 and it is an extention of the classical crisp logic a multivariate form. Intuitionistic Fuzzy Set (IFS) defined by Atanassov[2] in 1983, which is also an extension of FS. Molodtsov[7] developed the concept of Soft Set (SS) theory. This provided a new methodology for studying uncertainty.

A Cubic Set(CS) is a hybrid structure involving an Interval Valued Fuzzy Set(IVFS) by Jun.et.al.,[6] in 2012 have introduced the concept of Cubic Sets as internal Cubic Set(ICS) and external Cubic Set(ECS). Muhiuddin and Al-rogi[8] proposed the concept of $P-(R)$-Cubic

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Soft Subsets(CSSs) $P-(R)$-Union and Intersection of a Cubic Soft Sets(CSS) were developed.

Chinnadurai and Barkavi[4] was introduced the notion of Cubic Soft Matrices(CSM),also they investigated $P-(R)$- order of $P-(R)$-Union and intersection of CSM.

The notion of Picture Fuzzy Set (PFS) was introduced by Coung[3] in 2015, is a recently developed tool to deal with uncertainty which is a direct extension of Intuitionistic Fuzzy Set(IFS). Interval valued Picture Fuzzy Set(IVPFS) also proposed by Coung.

In 2020 Dogra and Pal[5] established the concept of Picture Fuzzy Matrix(PFM) and studied some of its properties. Ashraf[1] recently developed the concept of Cubic Picture Fuzzy Set(CPFS) in 2018.

In this paper, we present the notion of Cubic Picture Fuzzy Soft Matrices(CPFSMs), is a combination of Cubic Soft Matrices(CSM) with Picture Fuzzy Soft Matrices(PFSM). We define $P-(R)$-order of $P-(R)$- Union, Intersection of Cubic Picture Fuzzy Soft Matrices. Also investigate some of its algebraic properties.

2. PRELIMINARIES

Definition 2.1. [3] Let U be a non empty set, A Picture Fuzzy Set P of U is given by $P = \{u, \alpha_p(u), \beta_p(u), \gamma_p(u)/u \in U\}$
where, $\alpha_p : U \to [0,1], \beta_p : U \to [0,1]$ and $\gamma_p : U \to [0,1]$ are subsets of U which define the degree of Positive, degree of Neutral and degree of Negative of any element $u \in U$, to the PFS P, which satisfying the condition
$0 \leq \alpha_p(u) + \beta_p(u) + \gamma_p(u) \leq 1$ for all $u \in U$.

Definition 2.2. [6] Let U be a non empty set, An IVPFS P of U is given by
$P = \{u, \tilde{\alpha}_p(u), \tilde{\beta}_p(u), \tilde{\gamma}_p(u)/u \in U\}$
where, $\tilde{\alpha}_p : U \to [0,1], \tilde{\beta}_p : U \to [0,1]$ and $\tilde{\gamma}_p : U \to [0,1]$. $C[0,1]$ denotes the set of all closed sub intervals of $[0,1]$. Respectively $\tilde{\alpha}_p(u), \tilde{\beta}_p(u), \tilde{\gamma}_p(u)$ are closed sub intervals of $[0,1]$, representing the degree of positive, degree of Neutral and degree of Negative of any element u to the PFS P. The lower and upper ends of $\tilde{\alpha}_p(u), \tilde{\beta}_p(u)$ and $\tilde{\gamma}_p(u)$ are denoted respectively
$\bar{\alpha}_p(u), \underline{\alpha}_p(u), \bar{\beta}_p(u), \underline{\beta}_p(u)$ and $\bar{\gamma}_p(u), \underline{\gamma}_p(u)$, which satisfying the condition
$0 \leq \bar{\alpha}_p(u) + \bar{\beta}_p(u) + \bar{\gamma}_p(u) \leq 1$, for all $u \in U$.

Definition 2.3. [1] Let U be a non empty set, A Cubic Picture Fuzzy Set P is an object of the form

$P=\{u,<\tilde{\alpha}_p(u),\alpha_p(u)>,<\tilde{\beta}_p(u),\beta_p(u)>,<\tilde{\gamma}_p(u),\gamma_p(u)>/u\in U\}$

in which $\tilde{\alpha}_p(u),\tilde{\beta}_p(u),\tilde{\gamma}_p(u)$ are closed sub intervals of $[0,1]$, representing the degree of positive, degree of Neutral and degree of Negative of any element u to the PFS P and $\tilde{\alpha}_{P(u)},\tilde{\beta}_{P(u)},\tilde{\gamma}_{P(u)}$ a degree of positive. degree of neutral and degree of negative of the element u t0 the set P.

A CPFS P can also be denoted as

$P=\left(<\tilde{\alpha}_p,\alpha_p>,<\tilde{\beta}_p,\beta_p>,<\tilde{\gamma}_p,\gamma_p>\right)$ and P^U denotes the collection of all CPFSs defined on U.

Definition 2.4. [7] Let U be a non empty set, E be a set of parameters and $A\subseteq E$. $P(U)$ denote the power set of U.

A pair (F,A) is called a soft set over U, if $F:A\rightarrow P(U)$.

3. Cubic Picture Fuzzy Soft Sets and Cubic Picture Fuzzy Soft Matrices

Definition 3.1. Let U be a non empty set, E be a set of parameters and $A\subseteq E$. A cubic Picture Fuzzy Soft Set over U is defined as a pair (F,A), where $F:A\rightarrow P^U,(F,A)=\{F(e)/e\in A\}$, where $F(e)=\{<\tilde{\alpha}_{F(e)},\alpha_{F(e)}>,<\tilde{\beta}_{F(e)},\beta_{F(e)}>,<\tilde{\gamma}_{F(e)},\gamma_{F(e)}>\}$,

$\tilde{\alpha}_{F(e)},\tilde{\beta}_{F(e)},\tilde{\gamma}_{F(e)}$ are closed sub intervals of $[0,1]$

Definition 3.2. The complement of a CPFSS (F,A) denoted by $(F,A)^c$ is defined as, $(F,A)^c=$ $\{<x,[\underline{\gamma}_{F(e)}(x),\bar{\gamma}_{F(e)}(x)],\gamma_{F(e)}(x),[\underline{\beta}_{F(e)}(x),\bar{\beta}_{F(e)}(x)],\beta_{F(e)}(x),$
$[\underline{\alpha}_{F(e)}(x),\bar{\alpha}_{F(e)}(x)],\alpha_{F(e)}(x)>\forall x\in U$ and $e\in A\}$.

Definition 3.3. Let $U=\{u_1,u_2,...,u_m\}$ be a Universal set and $E=\{e_1,e_2,...,e_n\}$ be a set of parameters and $A\in E$, then the cubic picture fuzzy soft matrix (F,A) is represented in matrix form as,

$$P_c^M = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ . & & & \\ . & & & \\ . & & & \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}$$

where, $[p_{ij}] = (<[\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p]>)$

$= (<[\underline{\alpha}_{ij}^p, \bar{\alpha}_{ij}^p], \alpha_{ij}^p>, <[\underline{\beta}_{ij}^p, \bar{\beta}_{ij}^p], \beta_{ij}^p>, <[\underline{\gamma}_{ij}^p, \bar{\gamma}_{ij}^p], \gamma_{ij}^p>)$

$i = 1,2,...,m,\ j = 1,2,...,n$. Also satisfying the condition, $0<\bar{\alpha}_{ij}^p + \bar{\beta}_{ij}^p + \bar{\gamma}_{ij}^p \leq 1$, then P_c^M is an $(m \times n)$ CPFSM.

Definition 3.4. Consider the CPFSM $P_c^M = [<[\tilde{\alpha}_{ij}^P, \alpha_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\gamma}_{ij}^P, \gamma_{ij}^P]>]_{(m\times n)}$, then the complement of the CPFSM is denoted by,
$(P_c^M)^c = [<[\tilde{\gamma}_{ij}^P, \gamma_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\alpha}_{ij}^P, \alpha_{ij}^P]>]_{(m\times n)}$ for all i, j.

Example 3.4.1

$$P_c^M = \begin{bmatrix} <[0.1,0.3],0.2,[0.2,0.3],0.3,[0.1,0.2],0.3> & <[0.1,0.4],0.3,[0.1,0.3],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.2,0.4],0.3,[0.2,0.3],0.3> & <[0.1,0.3],0.3,[0.2,0.3],0.3,[0.1,0.3],0.3> \\ <[0.1,0.3],0.2,[0.1,0.4],0.2,[0.1,0.2],0.2> & <[0.2,0.3],0.2,[0.1,0.2],0.2,[0.1,0.4],0.3> \end{bmatrix}$$

$$(P_c^M)^c = \begin{bmatrix} <[0.1,0.2],0.3,[0.2,0.3],0.3,[0.1,0.3],0.2> & <[0.1,0.3],0.3,[0.1,0.3],0.2,[0.1,0.4],0.3> \\ <[0.2,0.3],0.3,[0.2,0.4],0.3,[0.1,0.2],0.2> & <[0.1,0.3],0.3,[0.2,0.3],0.3,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.1,0.4],0.2,[0.1,0.3],0.2> & <[0.1,0.4],0.2,[0.1,0.2],0.2,[0.2,0.3],0.3> \end{bmatrix}$$

4. P-UNION, P-INTERSECTION, R-UNION, R-INTERSECTION OF CPFSM

In this section, we define P-Union, P-Intersection, R-Union, R-Intersection of two $CPFSM_{(m\times n)}$ and investigate some properties.

Definition 4.1. Let $P_c^M = [<[\tilde{\alpha}_{ij}^P, \alpha_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\gamma}_{ij}^P, \gamma_{ij}^P]>]$
. $Q_c^M = [<[\tilde{\mu}_{ij}^Q, \mu_{ij}^Q], [\tilde{\eta}_{ij}^Q, \eta_{ij}^Q], [\tilde{\xi}_{ij}^Q, \varsigma_{ij}^Q]>] \in CPFSM_{(m\times n)}$.

Then

(1) P-Union of P_c^M and Q_c^M is denoted by $P_c^M \vee_P Q_c^M$ is defined as
$P_c^M \vee_P Q_c^M = M_c^M$, if $M_c^M = [m_{ij}] = <\tilde{M}_c^M, \lambda_{ij}^M>$,
where $\tilde{M}_c^M = \max\left\{\left(<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>\right)\right\}$ and
$\lambda_{ij}^M = \min\left\{\left(<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>\right)\right\}$ for all i, j.

(2) P-Intersection of P_c^M and Q_c^M is denoted by $P_c^M \wedge_P Q_c^M$ is defined as
$P_c^M \wedge_P Q_c^M = M_c^M$, if $M_c^M = [m_{ij}] = <\tilde{M}_c^M, \lambda_{ij}^M>$,
where $\tilde{M}_c^M = \min\left\{\left(<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>\right)\right\}$ and
$\lambda_{ij}^M = \min\left\{\left(<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>\right)\right\}$ for all i, j.

(3) R-Union of P_c^M and Q_c^M is denoted by $P_c^M \vee_R Q_c^M$ is defined as
$P_c^M \vee_R Q_c^M = M_c^M$, if $M_c^M = [m_{ij}] = <\tilde{M}_c^M, \lambda_{ij}^M>$,
where $\tilde{M}_c^M = \max\left\{\left(<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>\right)\right\}$ and
$\lambda_{ij}^M = \min\left\{\left(<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>\right)\right\}$ for all i, j.

(4) R-Intersection of P_c^M and Q_c^M is denoted by $P_c^M \wedge_R Q_c^M$ is defined as
$P_c^M \wedge_R Q_c^M = M_c^M$, if $M_c^M = [m_{ij}] = <\tilde{M}_c^M, \lambda_{ij}^M>$,
where $\tilde{M}_c^M = \min\left\{\left(<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>\right)\right\}$ and
$\lambda_{ij}^M = \max\left\{\left(<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>\right)\right\}$ for all i, j.

Example 4.1.1

$$P_c^M = \begin{bmatrix} <[0.1,0.3],0.2,[0.2,0.3],0.3,[0.1,0.2],0.3> & <[0.1,0.4],0.3,[0.1,0.3],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.2,0.4],0.3,[0.2,0.3],0.3> & <[0.1,0.3],0.3,[0.2,0.3],0.3,[0.1,0.3],0.3> \\ <[0.1,0.3],0.2,[0.1,0.4],0.2,[0.1,0.2],0.2> & <[0.2,0.3],0.2,[0.1,0.2],0.2,[0.1,0.4],0.3> \end{bmatrix}$$

$$Q_c^M = \begin{bmatrix} <[0.1,0.3],0.2,[0.2,0.3],0.3,[0.2,0.3],0.3> & <[0.1,0.4],0.3,[0.1,0.3],0.2,[0.2,0.3],0.3> \\ <[0.2,0.3],0.3,[0.1,0.4],0.2,[0.1,0.2],0.2> & <[0.1,0.4],0.3,[0.1,0.2],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.2,0.3],0.3,[0.1,0.3],0.2> & <[0.1,0.3],0.2,[0.1,0.3],0.2,[0.2,0.3],0.3> \end{bmatrix}$$

1. (P-Union) $P_c^M \vee_P Q_c^M = M_c^M$, is defined by,

$$M_c^M = \begin{bmatrix} <[0.1,0.3],0.2,[0.2,0.3],0.3,[0.2,0.3],0.3> & <[0.1,0.4],0.3,[0.1,0.3],0.2,[0.1,0.4],0.3> \\ <[0.2,0.3],0.3,[0.2,0.4],0.3,[0.2,0.3],0.3> & <[0.1,0.4],0.3,[0.2,0.3],0.3,[0.1,0.3],0.3> \\ <[0.1,0.3],0.2,[0.2,0.4],0.3,[0.1,0.3],0.2> & <[0.2,0.3],0.2,[0.1,0.3],0.2,[0.2,0.4],0.3> \end{bmatrix}$$

2. (P-Intersection) $P_c^M \wedge_P Q_c^M = M_c^M$, is defined by,

$$M_c^M = \begin{bmatrix} <[0.1,0.2],0.2,[0.1,0.3],0.3,[0.1,0.2],0.3> & <[0.1,0.3],0.2,[0.1,0.2],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.1,0.4],0.2,[0.1,0.2],0.2> & <[0.1,0.3],0.3,[0.1,0.2],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.1,0.3],0.2,[0.1,0.2],0.2> & <[0.1,0.2],0.2,[0.1,0.2],0.2,[0.1,0.3],0.3> \end{bmatrix}$$

3. (R-Union) $P_c^M \vee_R Q_c^M = M_c^M$, is defined by,

$$M_c^M = \begin{bmatrix} <[0.1,0.3],0.2,[0.2,0.3],0.3,[0.2,0.3],0.3> & <[0.1,0.4],0.2,[0.1,0.3],0.2,[0.2,0.3],0.3> \\ <[0.2,0.3],0.2,[0.2,0.4],0.2,[0.2,0.3],0.2> & <[0.1,0.4],0.3,[0.2,0.3],0.2,[0.1,0.3],0.3> \\ <[0.1,0.3],0.2,[0.2,0.4],0.2,[0.1,0.3],0.2> & <[0.2,0.3],0.2,[0.1,0.3],0.2,[0.2,0.4],0.3> \end{bmatrix}$$

4. (R-Intersection) $P_c^M \wedge_R Q_c^M = M_c^M$, is defined by,

$$M_c^M = \begin{bmatrix} <[0.1,0.2],0.2,[0.1,0.3],0.3,[0.1,0.2],0.3> & <[0.1,0.3],0.3,[0.1,0.2],0.2,[0.1,0.3],0.3> \\ <[0.1,0.2],0.3,[0.1,0.4],0.3,[0.1,0.2],0.3> & <[0.1,0.3],0.3,[0.1,0.2],0.3,[0.1,0.3],0.3> \\ <[0.1,0.2],0.2,[0.1,0.3],0.3,[0.1,0.2],0.2> & <[0.1,0.2],0.2,[0.1,0.2],0.2,[0.1,0.3],0.3> \end{bmatrix}$$

Property 4.2. Let $P_c^M, Q_c^M, R_c^M, S_c^M, T_c^M \in CPFSM_{(m\times n)}$, then the following conditions are holds,

(1) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then $P_c^M \subseteq_P (Q_c^M \wedge_P R_c^M) \wedge_P (R_c^M \wedge_P S_c^M) \wedge_P (S_c^M \wedge_P T_c^M)$.

(2) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then $P_c^M \subseteq_P (Q_c^M \vee_P R_c^M) \vee_P (R_c^M \vee_P S_c^M) \vee_P (S_c^M \vee_P T_c^M)$.

(3) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then $(P_c^M \vee_P Q_c^M \vee_P R_c^M \vee_P S_c^M) \subseteq T_c^M$.

(4) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then $(P_c^M \wedge_P Q_c^M \wedge_P R_c^M \wedge_P S_c^M) \subseteq_P T_c^M$.

(5) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then
(a)$(P_c^M \vee_P Q_c^M) \subseteq_P (R_c^M \vee_P S_c^M) \subseteq_P T_c^M$
(b)$(P_c^M \wedge_P Q_c^M) \subseteq_P (R_c^M \wedge_P S_c^M) \subseteq_P T_c^M$.

(6) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_P R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_P T_c^M$, then

(a)$(P_c^M \vee_P Q_c^M) \vee_P (R_c^M \vee_P S_c^M) \subseteq_P T_c^M$

(b)$(P_c^M \wedge_P Q_c^M) \vee_P (R_c^M \vee_P S_c^M) \subseteq_P T_c^M$

(c)$(P_c^M \vee_P Q_c^M) \vee_P (R_c^M \wedge_P S_c^M) \subseteq_P T_c^M$

(d)$(P_c^M \wedge_P Q_c^M) \wedge_P (R_c^M \wedge_P S_c^M) \subseteq_P T_c^M$

(e)$(P_c^M \vee_P Q_c^M) \wedge_P (R_c^M \wedge_P S_c^M) \subseteq_P T_c^M$

(f)$(P_c^M \wedge_P Q_c^M) \wedge_P (R_c^M \vee_P S_c^M) \subseteq_P T_c^M$.

Property 4.3. Let $P_c^M, Q_c^M, R_c^M, S_c^M, T_c^M \in CPFSM_{(m\times n)}$, then the following conditions are holds,

(1) If, $P_c^M \subseteq_R Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_P S_c^M$ and $S_c^M \subseteq_R T_c^M$, then
$P_c^M \subseteq_R (Q_c^M \wedge_R R_c^M) \wedge_R (R_c^M \wedge_R S_c^M) \wedge_R (S_c^M \wedge_R T_c^M)$.

(2) If, $P_c^M \subseteq_R Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_R S_c^M$ and $S_c^M \subseteq_R T_c^M$, then
$P_c^M \subseteq_R (Q_c^M \vee_R R_c^M) \vee_R (R_c^M \vee_R S_c^M) \vee_R (S_c^M \vee_R T_c^M)$.

(3) If, $P_c^M \subseteq_R Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_R S_c^M$ and $S_c^M \subseteq_R T_c^M$, then
$(P_c^M \vee_R Q_c^M \vee_R R_c^M \vee_R S_c^M) \subseteq T_c^M$.

(4) If, $P_c^M \subseteq_R Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_R S_c^M$ and $S_c^M \subseteq_R T_c^M$, then
$(P_c^M \wedge_R Q_c^M \wedge_R R_c^M \wedge_R S_c^M) \subseteq_R T_c^M$.

(5) If, $P_c^M \subseteq_P Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_R S_c^M$ and $S_c^M \subseteq_R T_c^M$, then

(a)$(P_c^M \vee_R Q_c^M) \subseteq_R (R_c^M \vee_R S_c^M) \subseteq_R T_c^M$

(b)$(P_c^M \wedge_R Q_c^M) \subseteq_R (R_c^M \wedge_R S_c^M) \subseteq_R T_c^M$.

(6) If, $P_c^M \subseteq_R Q_c^M, Q_c^M \subseteq_R R_c^M, R_c^M \subseteq_R S_c^M$ and $S_c^M \subseteq_R T_c^M$, then

(a)$(P_c^M \vee_R Q_c^M) \vee_R (R_c^M \vee_R S_c^M) \subseteq_R T_c^M$

(b)$(P_c^M \wedge_R Q_c^M) \vee_R (R_c^M \vee_R S_c^M) \subseteq_R T_c^M$

(c)$(P_c^M \vee_R Q_c^M) \vee_R (R_c^M \wedge_R S_c^M) \subseteq_R T_c^M$

(d)$(P_c^M \wedge_R Q_c^M) \wedge_R (R_c^M \wedge_R S_c^M) \subseteq_R T_c^M$

(e)$(P_c^M \vee_R Q_c^M) \wedge_R (R_c^M \wedge_R S_c^M) \subseteq_R T_c^M$

(f)$(P_c^M \wedge_R Q_c^M) \wedge_R (R_c^M \vee_R S_c^M) \subseteq_R T_c^M$.

Property 4.4. Let P_c^M be a $CPFSM_{(m\times n)}$, then

(i) $(P_c^M)^c \vee_P (P_c^M)^c = (P_c^M)^c$

(ii) $(P_c^M)^c \wedge_P (P_c^M)^c = (P_c^M)^c$

(iii) $(P_c^M)^c \vee_R (P_c^M)^c = (P_c^M)^c$

(iv) $(P_c^M)^c \wedge_R (P_c^M)^c = (P_c^M)^c$

Proof:

$P_c^M = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >]$

$(P_c^M)^c = [< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$

(i) $(P_c^M)^c \vee_P (P_c^M)^c = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >] \vee_P$

$[< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{F(e)}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$ for all i, j.

$\Rightarrow \max\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >\right\}, \max\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$\Rightarrow [< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$

$= (P_c^M)^c$.

(ii) $(P_c^M)^c \wedge_P (P_c^M)^c = [< [\tilde{\alpha}_{ij}^p], \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p], \gamma_{ij}^p] >] \wedge_P$

$[< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{F(e)}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$ for all i, j.

$\Rightarrow \min\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >\right\}, \min\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$\Rightarrow [< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$

$= (P_c^M)^c$.

(iii) $(P_c^M)^c \vee_R (P_c^M)^c = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >] \vee_P$

$[< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{F(e)}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$ for all i, j.

$\Rightarrow \max\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >\right\}, \max\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$\Rightarrow [< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$

$= (P_c^M)^c$.

(iv) $(P_c^M)^c \wedge_R (P_c^M)^c = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >] \wedge_P$

$[< [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{F(e)}^p], [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p] >]$ for all i, j.

$\Rightarrow \min\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >\right\}, \min\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$\Rightarrow [<[\tilde{\gamma}^{p}_{ij},\gamma^{p}_{ij}],[\tilde{\beta}^{p}_{ij},\beta^{p}_{ij}],[\tilde{\alpha}^{p}_{ij},\alpha^{p}_{ij}]>]$
$=(P^M_c)^c.$

Property 4.5. Let P^M_c be a $CPFSM_{(m\times n)}$, then
(i) $(P^M_c)^c\vee_P(Q^M_c)^c=(Q^M_c)^c\vee_P(P^M_c)^c$
(ii) $(P^M_c)^c\wedge_P(Q^M_c)^c=(Q^M_c)^c\wedge_P(P^M_c)^c$
(iii) $(P^M_c)^c\vee_R(Q^M_c)^c=(Q^M_c)^c\vee_R(P^M_c)^c$
(iv) $(P^M_c)^c\wedge_R(Q^M_c)^c=(Q^M_c)^c\wedge_R(P^M_c)^c$

Proof:
$P^M_c=[<[\tilde{\alpha}^P_{ij},\alpha^P_{ij}],[\tilde{\beta}^P_{ij},\beta^P_{ij}],[\tilde{\gamma}^P_{ij},\gamma^P_{ij}]>]$
$Q^M_c=[<[\tilde{\mu}^Q_{ij},\mu^Q_{ij}],[\tilde{\eta}^Q_{ij},\eta^Q_{ij}],[\tilde{\varsigma}^Q_{ij},\varsigma^Q_{ij}]>]$
$(P^M_c)^c=[<[\tilde{\gamma}^P_{ij}],\gamma^P_{ij}],[\tilde{\beta}^P_{ij},\beta^P_{ij}],[\tilde{\alpha}^P_{ij}],\alpha^P_{ij}]>]$
$(Q^M_c)^c=[<[\tilde{\varsigma}^Q_{ij}],\varsigma^Q_{ij}],[\tilde{\eta}^Q_{ij},\eta^Q_{ij}],[\tilde{\mu}^Q_{ij}],\mu^Q_{ij}]>]$

(i) $(P^M_c)^c\vee_P(Q^M_c)^c=[<[\tilde{\alpha}^p_{ij},\alpha^p_{ij}],[\tilde{\beta}^p_{ij},\beta^p_{ij}],[\tilde{\gamma}^p_{ij},\gamma^p_{ij}]>]\vee_P$
$[<[\tilde{\varsigma}^Q_{ij},\varsigma^Q_{ij}],[\tilde{\eta}^Q_{ij},\eta^Q_{ij}],[\tilde{\mu}^Q_{ij},\mu^Q_{ij}]>]$ for all i,j.
$\Rightarrow\ \max\left\{<\tilde{\gamma}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\alpha}^P_{ij}>,<\tilde{\varsigma}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\mu}^Q_{ij}>\right\},\max\left\{<\gamma^P_{ij},\beta^P_{ij},\alpha^P_{ij}>,<\varsigma^Q_{ij},\eta^Q_{ij},\mu^Q_{ij}>\right\}$ for all i,j
$\Rightarrow\ \max\left\{<\tilde{\varsigma}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\mu}^Q_{ij}>,<\tilde{\gamma}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\alpha}^P_{ij}>,\right\},\max\left\{<\varsigma^Q_{ij},\eta^Q_{ij},\mu^Q_{ij}>,<\gamma^P_{ij},\beta^P_{ij},\alpha^P_{ij}>\right\}$ for all i,j
$=(Q^M_c)^c\vee_P(P^M_c)^c$

(ii) $(P^M_c)^c\wedge_P(Q^M_c)^c=[<[\tilde{\alpha}^p_{ij},\alpha^p_{ij}],[\tilde{\beta}^p_{ij},\beta^p_{ij}],[\tilde{\gamma}^p_{ij},\gamma^p_{ij}]>]\wedge_P$
$[<[\tilde{\varsigma}^Q_{ij},\varsigma^Q_{ij}],[\tilde{\eta}^Q_{ij},\eta^Q_{ij}],[\tilde{\mu}^Q_{ij},\mu^Q_{ij}]>]$ for all i,j.
$\Rightarrow\min\left\{<\tilde{\gamma}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\alpha}^P_{ij}>,<\tilde{\varsigma}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\mu}^Q_{ij}>\right\},\min\left\{<\gamma^P_{ij},\beta^P_{ij},\alpha^P_{ij}>,<\varsigma^Q_{ij},\eta^Q_{ij},\mu^Q_{ij}>\right\}$ for all i,j
$\Rightarrow\ \min\left\{<\tilde{\varsigma}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\mu}^Q_{ij}>,<\tilde{\gamma}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\alpha}^P_{ij}>,\right\},\min\left\{<\varsigma^Q_{ij},\eta^Q_{ij},\mu^Q_{ij}>,<\gamma^P_{ij},\beta^P_{ij},\alpha^P_{ij}>\right\}$ for all i,j
$=(Q^M_c)^c\wedge_P(P^M_c)^c$

(iii) $(P_c^M)^c \vee_R (Q_c^M)^c = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >] \vee_R$

$[< [\tilde{\zeta}_{ij}^Q, \varsigma_{ij}^Q], [\tilde{\eta}_{ij}^Q, \eta_{ij}^Q], [\tilde{\mu}_{ij}^Q, \mu_{ij}^Q] >]$ for all i, j.

$\Rightarrow \max\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\zeta}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\mu}_{ij}^Q >\right\}, \min\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \varsigma_{ij}^Q, \eta_{ij}^Q, \mu_{ij}^Q >\right\}$ for all i, j

$\Rightarrow \max\left\{< \tilde{\zeta}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\mu}_{ij}^Q >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, \right\}, \min\left\{< \varsigma_{ij}^Q, \eta_{ij}^Q, \mu_{ij}^Q >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$= (Q_c^M)^c \vee_R (P_c^M)^c$

(iv) $(P_c^M)^c \wedge_R (Q_c^M)^c = [< [\tilde{\alpha}_{ij}^p, \alpha_{ij}^p], [\tilde{\beta}_{ij}^p, \beta_{ij}^p], [\tilde{\gamma}_{ij}^p, \gamma_{ij}^p] >] \wedge_R$

$[< [\tilde{\zeta}_{ij}^Q, \varsigma_{ij}^Q], [\tilde{\eta}_{ij}^Q, \eta_{ij}^Q], [\tilde{\mu}_{ij}^Q, \mu_{ij}^Q] >]$ for all i, j.

$\Rightarrow \min\left\{< \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, < \tilde{\zeta}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\mu}_{ij}^Q >\right\}, \max\left\{< \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >, < \varsigma_{ij}^Q, \eta_{ij}^Q, \mu_{ij}^Q >\right\}$ for all i, j

$\Rightarrow \min\left\{< \tilde{\zeta}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\mu}_{ij}^Q >, < \tilde{\gamma}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\alpha}_{ij}^P >, \right\}, \max\left\{< \varsigma_{ij}^Q, \eta_{ij}^Q, \mu_{ij}^Q >, < \gamma_{ij}^P, \beta_{ij}^P, \alpha_{ij}^P >\right\}$ for all i, j

$= (Q_c^M)^c \wedge_R (P_c^M)^c$.

5. Some Operations on $CPFSM_{(m\times n)}$

In this section, we discuss the P-Union, P-Intersection, R-Uion, R-Intersection of Cubic Picture Fuzzy Soft Matrices are defined and their relevant properties are investigated.

Theorem 5.1. *Let*
$P_c^M, Q_c^M, R_c^M, S_c^M$ *be a* $CPFSM_{(m\times n)}$, *then*
$(P_c^M \vee_P Q_c^M) \wedge_R (R_c^M \vee_P S_c^M) =$
$\left[(P_c^M \vee_P R_c^M) \wedge_R (P_c^M \vee_P S_c^M)\right] \wedge_R \left[(Q_c^M \vee_P R_c^M) \wedge_R (Q_c^M \vee_P S_c^M)\right]$

Proof:

Let

$P_c^M = [< [\tilde{\alpha}_{ij}^P, \alpha_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\gamma}_{ij}^P, \gamma_{ij}^P] >]$

$Q_c^M = [< [\tilde{\mu}_{ij}^Q, \mu_{ij}^Q], [\tilde{\eta}_{ij}^Q, \eta_{ij}^Q], [\tilde{\zeta}_{ij}^Q, \varsigma_{ij}^Q] >]$

$R_c^M = [< [\tilde{\rho}_{ij}^R, \rho_{ij}^R], [\tilde{\sigma}_{ij}^R, \sigma_{ij}^R], [\tilde{\tau}_{ij}^R, \tau_{ij}^R] >]$

$S_c^M = [< [\tilde{\chi}_{ij}^S, \chi_{ij}^S], [\tilde{\psi}_{ij}^S, \psi_{ij}^S], [\tilde{\omega}_{ij}^S, \omega_{ij}^S] >]$ *all* $\in CPFSM_{(m\times n)}$.

Consider,

$(P_c^M \vee_P Q_c^M) \wedge_R (R_c^M \vee_P S_c^M) =$
$\Big(\max\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>\Big\},\max\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>\Big\}\Big) \wedge_R$
$\Big(\max\Big\{<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},\max\Big\{<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}\Big)$ *for all i,j*
$\Rightarrow \Big(\max\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\max\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\}\Big)$
$\wedge_R \Big(\max\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},\max\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}\Big)$
$\wedge_R \Big(\max\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\max\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\}\Big)$
$\wedge_R \Big(\max\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},\max\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}\Big)$ *for all i,j*
$\Rightarrow \min\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},$
$\max\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\},\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}$
$\wedge_R \min\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},$
$\max\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\},\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}$ *for all i,j.*
$\Rightarrow \min\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\Big\{<\tilde{\alpha}_{ij}^P,\tilde{\beta}_{ij}^P,\tilde{\gamma}_{ij}^P>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},$
$\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\tilde{\varsigma}_{ij}^Q>,<\tilde{\rho}_{ij}^R,\tilde{\sigma}_{ij}^R,\tilde{\tau}_{ij}^R>\Big\},\Big\{<\tilde{\mu}_{ij}^Q,\tilde{\eta}_{ij}^Q,\varsigma_{ij}^Q>,<\tilde{\chi}_{ij}^S,\tilde{\psi}_{ij}^S,\tilde{\omega}_{ij}^S>\Big\},$
$\max\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\},\Big\{<\alpha_{ij}^P,\beta_{ij}^P,\gamma_{ij}^P>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\},$
$\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\rho_{ij}^R,\sigma_{ij}^R,\tau_{ij}^R>\Big\},\Big\{<\mu_{ij}^Q,\eta_{ij}^Q,\varsigma_{ij}^Q>,<\chi_{ij}^S,\psi_{ij}^S,\omega_{ij}^S>\Big\}$, *for all i,j.*
$= \big[(P_c^M \vee_P R_c^M) \wedge_R (P_c^M \vee_P S_c^M)\big] \wedge_R \big[(Q_c^M \vee_P R_c^M) \wedge_R (Q_c^M \vee_P S_c^M)\big].$

Theorem 5.2. *Let $P_c^M, Q_c^M, R_c^M, S_c^M$ be a $CPFSM_{(m\times n)}$, then*

$$(P_c^M \vee_R Q_c^M) \wedge_P (R_c^M \vee_R S_c^M) = \big[(P_c^M \vee_R R_c^M) \wedge_P (P_c^M \vee_R S_c^M)\big] \wedge_P \big[(Q_c^M \vee_R R_c^M) \wedge_P (Q_c^M \vee_R S_c^M)\big]$$

Proof:

Let

$P_c^M = [<[\tilde{\alpha}_{ij}^P,\alpha_{ij}^P],[\tilde{\beta}_{ij}^P,\beta_{ij}^P],[\tilde{\gamma}_{ij}^P,\gamma_{ij}^P]>]$

$Q_c^M = [<[\tilde{\mu}_{ij}^Q,\mu_{ij}^Q],[\tilde{\eta}_{ij}^Q,\eta_{ij}^Q],[\tilde{\varsigma}_{ij}^Q,\varsigma_{ij}^Q]>]$

$R_c^M = [<[\tilde{\rho}_{ij}^R,\rho_{ij}^R],[\tilde{\sigma}_{ij}^R,\sigma_{ij}^R],[\tilde{\tau}_{ij}^R,\tau_{ij}^R]>]$

$S_c^M = [<[\tilde{\chi}_{ij}^S,\chi_{ij}^S],[\tilde{\psi}_{ij}^S,\psi_{ij}^S],[\tilde{\omega}_{ij}^S,\omega_{ij}^S]>]$ *all* $\in CPFSM_{(m\times n)}$.

Consider,

$(P_c^M \vee_R Q_c^M) \wedge_P (R_c^M \vee_R S_c^M) =$
$\left(\max\left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>\right\}, \min\left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>\right\}\right) \wedge_P$
$\left(\max\left\{<\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\}, \min\left\{<\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}\right)$ *for all i, j*
$\Rightarrow \left(\max\left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \min\left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}\right)$
$\wedge_P \left(\max\left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\}, \min\left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}\right)$
$\wedge_P \left(\max\left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \min\left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}\right)$
$\wedge_P \left(\max\left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\}, \min\left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}\right)$ *for all i, j*
$\Rightarrow \min\left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\},$
$\min\left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}, \left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}$
$\wedge_P \min\left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\},$
$\min\left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}, \left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}$ *for all i, j*
$\Rightarrow \min\left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \left\{<\tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\},$
$\left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R>\right\}, \left\{<\tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q>, <\tilde{\chi}_{ij}^S, \tilde{\psi}_{ij}^S, \tilde{\omega}_{ij}^S>\right\},$
$\min\left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}, \left\{<\alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\},$
$\left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R>\right\}, \left\{<\mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q>, <\chi_{ij}^S, \psi_{ij}^S, \omega_{ij}^S>\right\}$, *for all i, j.*
$= \left[(P_c^M \vee_R R_c^M) \wedge_P (P_c^M \vee_R S_c^M)\right] \wedge_P \left[(Q_c^M \vee_R R_c^M) \wedge_P (Q_c^M \vee_R S_c^M)\right].$

Theorem 5.3. *Let* $P_c^M, Q_c^M, R_c^M, S_c^M$ *be a* $CPFSM_{(m\times n)}$*, then*

$$(P_c^M \wedge_P Q_c^M) \vee_R (R_c^M \wedge_P S_c^M) = \left[(P_c^M \wedge_P R_c^M) \vee_R (P_c^M \wedge_P S_c^M)\right] \vee_R \left[(Q_c^M \wedge_P R_c^M) \vee_R (Q_c^M \wedge_P S_c^M)\right]$$

Proof:

Let

$P_c^M = [<[\tilde{\alpha}_{ij}^P, \alpha_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\gamma}_{ij}^P, \gamma_{ij}^P]>]$

$Q_c^M = [<[\tilde{\mu}_{ij}^Q, \mu_{ij}^Q], [\tilde{\eta}_{ij}^Q, \eta_{ij}^Q], [\tilde{\varsigma}_{ij}^Q, \varsigma_{ij}^Q]>]$

$R_c^M = [<[\tilde{\rho}_{ij}^R, \rho_{ij}^R], [\tilde{\sigma}_{ij}^R, \sigma_{ij}^R], [\tilde{\tau}_{ij}^R, \tau_{ij}^R]>]$

$S_c^M = [<[\tilde{\chi}_{ij}^S, \chi_{ij}^S], [\tilde{\psi}_{ij}^S, \psi_{ij}^S], [\tilde{\omega}_{ij}^S, \omega_{ij}^S]>]$ *all* $\in CPFSM_{(m\times n)}$.

Consider,

$(P_c^M \wedge_P Q_c^M) \vee_R (R_c^M \wedge_P S_c^M) =$

$\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\xi}_{ij}^{Q}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\}\right)\vee_{R}$
$\left(\min\left\{<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},\min\left\{<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$
$\vee_{R}\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}\right)$
$\vee_{R}\left(\min\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$
$\vee_{R}\left(\min\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},\min\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}\right)$ *for all i, j*
$\Rightarrow\max\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},$
$\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\},\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}$
$\vee_{R}\max\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},$
$\min\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\},\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}$ *for all i, j*
$\Rightarrow\max\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},$
$\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\chi}_{ij}^{S},\tilde{\psi}_{ij}^{S},\tilde{\omega}_{ij}^{S}>\right\},$
$\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\},\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\},$
$\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\},\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\chi_{ij}^{S},\psi_{ij}^{S},\omega_{ij}^{S}>\right\}$, *for all i, j.*
$=\left[(P_c^M\wedge_P R_c^M)\vee_R(P_c^M\wedge_P S_c^M)\right]\vee_R\left[(Q_c^M\wedge_P R_c^M)\vee_R(Q_c^M\wedge_P S_c^M)\right]$

Theorem 5.4. *Let $P_c^M, Q_c^M, R_c^M, S_c^M$ be a $CPFSM_{(m\times n)}$, then*

$$(P_c^M\wedge_R Q_c^M)\vee_P(R_c^M\wedge_R S_c^M)=$$
$$\left[(P_c^M\wedge_R R_c^M)\vee_P(P_c^M\wedge_R S_c^M)\right]\vee_P\left[(Q_c^M\wedge_R R_c^M)\vee_P(Q_c^M\wedge_R S_c^M)\right]$$

Proof:

Let

$P_c^M=[<[\tilde{\alpha}_{ij}^{P},\alpha_{ij}^{P}],[\tilde{\beta}_{ij}^{P},\beta_{ij}^{P}],[\tilde{\gamma}_{ij}^{P},\gamma_{ij}^{P}]>]$

$Q_c^M=[<[\tilde{\mu}_{ij}^{Q},\mu_{ij}^{Q}],[\tilde{\eta}_{ij}^{Q},\eta_{ij}^{Q}],[\tilde{\xi}_{ij}^{Q},\varsigma_{ij}^{Q}]>]$

$R_c^M=[<[\tilde{\rho}_{ij}^{R},\rho_{ij}^{R}],[\tilde{\sigma}_{ij}^{R},\sigma_{ij}^{R}],[\tilde{\tau}_{ij}^{R},\tau_{ij}^{R}]>]$

$S_c^M=[<[\tilde{\chi}_{ij}^{S},\chi_{ij}^{S}],[\tilde{\psi}_{ij}^{S},\psi_{ij}^{S}],[\tilde{\omega}_{ij}^{S},\omega_{ij}^{S}]>]$ *all* $\in CPFSM_{(m\times n)}$.

Consider,

$(P_c^M\wedge_R Q_c^M)\vee_P(R_c^M\wedge_R S_c^M)=$
$\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\xi}_{ij}^{Q}>\right\},\max\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\}\right)\vee_{P}$

$\left(\min\left\{<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},\max\left\{<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}\right)$ *for all i, j*

$\Rightarrow\left(\min\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\max\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\}\right)$

$\vee_P\left(\min\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},\max\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}\right)$

$\vee_P\left(\min\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\max\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\}\right)$

$\vee_P\left(\min\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},\max\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}\right)$ *for all i, j*

$\Rightarrow\max\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},$

$\max\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\},\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}$

$\vee_P\max\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},$

$\max\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\},\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}$ *for all i, j*

$\Rightarrow\max\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},$

$\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\chi}^S_{ij},\tilde{\psi}^S_{ij},\tilde{\omega}^S_{ij}>\right\},$

$\max\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\},\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\},$

$\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\},\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\chi^S_{ij},\psi^S_{ij},\omega^S_{ij}>\right\}$, *for all i, j.*

$=\left[(P^M_c\wedge_R R^M_c)\vee_P(P^M_c\wedge_R S^M_c)\right]\vee_P\left[(Q^M_c\wedge_R R^M_c)\vee_P(Q^M_c\wedge_R S^M_c)\right].$

Theorem 5.5. *Let* P^M_c,Q^M_c,R^M_c *be a* $CPFSM_{(m\times n)}$*, then*

(i) $P^M_c\vee_P(Q^M_c\wedge_R R^M_c)=(P^M_c\vee_P Q^M_c)\wedge_R(P^M_c\vee_P R^M_c)$

(ii) $P^M_c\wedge_P(Q^M_c\vee_R R^M_c)=(P^M_c\wedge_P Q^M_c)\vee_R(P^M_c\wedge_P R^M_c)$

Proof:

Let

$P^M_c=[<[\tilde{\alpha}^P_{ij},\alpha^P_{ij}],[\tilde{\beta}^P_{ij},\beta^P_{ij}],[\tilde{\gamma}^P_{ij},\gamma^P_{ij}]>]$

$Q^M_c=[<[\tilde{\mu}^Q_{ij},\mu^Q_{ij}],[\tilde{\eta}^Q_{ij},\eta^Q_{ij}],[\tilde{\varsigma}^Q_{ij},\varsigma^Q_{ij}]>]$

$R^M_c=[<[\tilde{\rho}^R_{ij},\rho^R_{ij}],[\tilde{\sigma}^R_{ij},\sigma^R_{ij}],[\tilde{\tau}^R_{ij},\tau^R_{ij}]>]$ *all* $\in CPFSM_{(m\times n)}$.

Consider,

(i) $P^M_c\vee_P(Q^M_c\wedge_R R^M_c)=<[\tilde{\alpha}^P_{ij},\alpha^P_{ij}],[\tilde{\beta}^P_{ij},\beta^P_{ij}],[\tilde{\gamma}^P_{ij},\gamma^P_{ij}]>\vee_P$

$\left(\min\left\{<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>,<\tilde{\rho}^R_{ij},\tilde{\sigma}^R_{ij},\tilde{\tau}^R_{ij}>\right\},\max\left\{<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>,<\rho^R_{ij},\sigma^R_{ij},\tau^R_{ij}>\right\}\right)$ *for all i, j*

$\Rightarrow\left(\max\left\{<\tilde{\alpha}^P_{ij},\tilde{\beta}^P_{ij},\tilde{\gamma}^P_{ij}>,<\tilde{\mu}^Q_{ij},\tilde{\eta}^Q_{ij},\tilde{\varsigma}^Q_{ij}>\right\},\max\left\{<\alpha^P_{ij},\beta^P_{ij},\gamma^P_{ij}>,<\mu^Q_{ij},\eta^Q_{ij},\varsigma^Q_{ij}>\right\}\right)\wedge_R$

$\left(\max\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\max\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>\right\},\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\}\right)$
$\left(\max\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\},\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$=(P_c^M\vee_P Q_c^M)\wedge_R(P_c^M\vee_P R_c^M).$

(ii) $P_c^M\wedge_P(Q_c^M\vee_R R_c^M)=<[\tilde{\alpha}_{ij}^{P},\alpha_{ij}^{P}],[\tilde{\beta}_{ij}^{P},\beta_{ij}^{P}],[\tilde{\gamma}_{ij}^{P},\gamma_{ij}^{P}]>\wedge_P$
$\left(\max\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\}\right)\vee_R$
$\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\beta_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\max\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>\right\},\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\}\right)$
$\left(\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\},\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$=(P_c^M\wedge_P Q_c^M)\vee_R(P_c^M\wedge_P R_c^M).$

Theorem 5.6. *Let* P_c^M, Q_c^M, R_c^M *be a* $CPFSM_{(m\times n)}$*, then*
(i) $P_c^M\vee_R(Q_c^M\wedge_P R_c^M)=(P_c^M\vee_R Q_c^M)\wedge_P(P_c^M\vee_R R_c^M)$
(ii) $P_c^M\wedge_R(Q_c^M\vee_P R_c^M)=(P_c^M\wedge_R Q_c^M)\vee_P(P_c^M\wedge_R R_c^M)$

Proof:

Let
$P_c^M=[<[\tilde{\alpha}_{ij}^{P},\alpha_{ij}^{P}],[\tilde{\beta}_{ij}^{P},\beta_{ij}^{P}],[\tilde{\gamma}_{ij}^{P},\gamma_{ij}^{P}]>]$
$Q_c^M=[<[\tilde{\mu}_{ij}^{Q},\mu_{ij}^{Q}],[\tilde{\eta}_{ij}^{Q},\eta_{ij}^{Q}],[\tilde{\varsigma}_{ij}^{Q},\varsigma_{ij}^{Q}]>]$
$R_c^M=[<[\tilde{\rho}_{ij}^{R},\rho_{ij}^{R}],[\tilde{\sigma}_{ij}^{R},\sigma_{ij}^{R}],[\tilde{\tau}_{ij}^{R},\tau_{ij}^{R}]>]$ *all* $\in CPFSM_{(m\times n)}$.
Consider,

(i) $P_c^M\vee_R(Q_c^M\wedge_P R_c^M)=<[\tilde{\alpha}_{ij}^{P},\alpha_{ij}^{P}],[\tilde{\beta}_{ij}^{P},\beta_{ij}^{P}],[\tilde{\gamma}_{ij}^{P},\gamma_{ij}^{P}]>\vee_R$
$\left(\min\left\{<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\max\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\}\right)\wedge_P$
$\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\},\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*
$\Rightarrow\left(\min\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\mu}_{ij}^{Q},\tilde{\eta}_{ij}^{Q},\tilde{\varsigma}_{ij}^{Q}>\right\},\left\{<\tilde{\alpha}_{ij}^{P},\tilde{\beta}_{ij}^{P},\tilde{\gamma}_{ij}^{P}>,<\tilde{\rho}_{ij}^{R},\tilde{\sigma}_{ij}^{R},\tilde{\tau}_{ij}^{R}>\right\}\right)$
$\left(\min\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\mu_{ij}^{Q},\eta_{ij}^{Q},\varsigma_{ij}^{Q}>\right\},\left\{<\alpha_{ij}^{P},\beta_{ij}^{P},\gamma_{ij}^{P}>,<\rho_{ij}^{R},\sigma_{ij}^{R},\tau_{ij}^{R}>\right\}\right)$ *for all i, j*

$= (P_c^M \vee_R Q_c^M) \wedge_P (P_c^M \vee_R R_c^M)$.

(ii) $P_c^M \wedge_R (Q_c^M \vee_P R_c^M) = < [\tilde{\alpha}_{ij}^P, \alpha_{ij}^P], [\tilde{\beta}_{ij}^P, \beta_{ij}^P], [\tilde{\gamma}_{ij}^P, \gamma_{ij}^P] > \wedge_R$
$\left(\max\left\{< \tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\varsigma}_{ij}^Q >, < \tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R >\right\}, \max\left\{< \mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q >, < \rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R >\right\}\right)$ *for all* i, j
$\Rightarrow \left(\min\left\{< \tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P >, < \tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\xi}_{ij}^Q >\right\}, \max\left\{< \alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P >, < \mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q >\right\}\right) \vee_P$
$\left(\min\left\{< \tilde{\alpha}_{ij}^P, \beta_{ij}^P, \tilde{\gamma}_{ij}^P >, < \tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R >\right\}, \max\left\{< \alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P >, < \rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R >\right\}\right)$ *for all i, j*
$\Rightarrow \left(\max\left\{< \tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P >, < \tilde{\mu}_{ij}^Q, \tilde{\eta}_{ij}^Q, \tilde{\xi}_{ij}^Q >\right\}, \left\{< \tilde{\alpha}_{ij}^P, \tilde{\beta}_{ij}^P, \tilde{\gamma}_{ij}^P >, < \tilde{\rho}_{ij}^R, \tilde{\sigma}_{ij}^R, \tilde{\tau}_{ij}^R >\right\}\right)$
$\left(\max\left\{< \alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P >, < \mu_{ij}^Q, \eta_{ij}^Q, \varsigma_{ij}^Q >\right\}, \left\{< \alpha_{ij}^P, \beta_{ij}^P, \gamma_{ij}^P >, < \rho_{ij}^R, \sigma_{ij}^R, \tau_{ij}^R >\right\}\right)$ *for all i, j*
$= (P_c^M \wedge_R Q_c^M) \vee_P (P_c^M \wedge_R R_c^M)$.

6. CONCLUSION

In this paper, we have introduced the concept of Cubic Picture Fuzzy Soft Matrices(CPFSMs). Also, we discussed some of its algebraic properties with P-(R)-order of Union, Intersection of Cubic Picture Fuzzy Soft Matrices. In future, we extend this concept to Internal and External Cubic Picture Fuzzy Soft Matrices.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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