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AN ITERATIVE ALGORITHM FOR THE GENERALIZED CENTRO-SYMMETRIC SOLUTION OF THE GENERALIZED COUPLED SYLVESTER MATRIX EQUATIONS $AV+BW=EVF+C, \quad MV+NW=GVH+D$

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Abstract: In this paper, an iterative algorithm for solving of the generalized coupled Sylvester matrix equations $AV+BW=EVF+C, \quad MV+NW=GVH+D$ over the generalized centro-symmetric matrices (V,W) is proposed. For any initial generalized centro-symmetric matrices V_0 and W_0, a generalized centro-symmetric solution (V,W) is obtained within a finite number of iterations in the absence of round-off errors. Two numerical examples are presented to support the theoretical results where the efficiency and accuracy of the suggested algorithm are shown.

Keywords: symmetric orthogonal matrix; generalized centro-symmetric matrix; iterative method; the generalized coupled Sylvester matrix equation; accuracy.

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1. INTRODUCTION

In the present paper, we are concerned with the generalized coupled Sylvester matrix equations

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$$\begin{cases} AV + BW = EVF + C, \\ MV + NW = GVH + D. \end{cases} \quad (1)$$

where $A, B, E, C, M, G, N, D, F, H \in R^{nxn}$, $V \in \mathcal{CSR}_p^{nxn}$ and $W \in \mathcal{CSR}_s^{nxn}$.Let R^{nxn} represent the set of all real nxn matrices, $\mathcal{SR}^{nxn}$ be the set of all symmetric matrices in R^{nxn}, $\mathcal{ASR}^{nxn}$ be the set of all anti-symmetric matrices in R^{nxn} and $\mathcal{SOR}^{nxn}$ be the set of all symmetric orthogonal matrices in R^{nxn}. For a matrix$A$, $A^T, tr(A)$, $R(A)$ denote the transpose, the trace and the column space of matrix A respectively.$\langle A, B\rangle = tr(B^T A)$ is defined as the inner product of the two matrices $\mathbf{A}$ and $\mathbf{B}$, which generates the Frobenius norm, i.e. $||A||^2 = \langle A, A\rangle = tr(A^T A)$.Denote by $I_n$the identity matrix with order n.

Definition 1[1]: Generalized centro-symmetric matrix

An $n\ x\ n$ matrix A is called generalized centro-symmetric (generalized central anti-symmetric) with respect to$P$, if $PAP = A$ $(PAP = -A)$, where P be some real symmetric orthogonal nxn matrix, i.e., $P = P^T = P^{-1}$. The set of order n generalized centro-symmetric (generalized central anti-symmetric) matrices with respect to P is denoted by $\mathcal{CSR}_P^{nxn}$ $(\mathcal{CASR}_P^{nxn})$.

Definition 2: Centro-symmetric matrix an n x n matrix $A = (a_{ij})$ is centrosymmetric if its elements satisfy $a_{ij} = a_{n+1-i,n+1-j}$ for all $i, j = 1,2, \ldots, n$.

Some properties

- If $\boldsymbol{A}$ is centro-symmetric matrix over field $\boldsymbol{F}$,then $\boldsymbol{c} \times \boldsymbol{A} (\boldsymbol{c} \in \boldsymbol{F})$ is centro-symmetric.
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are centro-symmetric matrices, then the summation $(\boldsymbol{A} + \boldsymbol{B})$ is centro-symmetric.
- The multiplication of two square centro-symmetric matrices is also centro-symmetric.
- In the case of non-square matrix, if $\boldsymbol{A}^{\boldsymbol{mxn}}$ and $\boldsymbol{B}^{\boldsymbol{nxp}}$ are centro-symmetric matrices, then $(\boldsymbol{A} \times \boldsymbol{B})$ is also centro-symmetric.
- If $\boldsymbol{P}$ is centro-symmetric matrix, then $\boldsymbol{P}^{-1}$ and $\boldsymbol{P}^{\mathbf{T}}$ are also centro-symmetric.

Example:

A square matrix: $A_{3x3} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 6 & 2 \\ 8 & 4 & 3 \end{pmatrix}$ is centro-symmetric.

We find that $A^T = \begin{pmatrix} 3 & 2 & 8 \\ 4 & 6 & 4 \\ 8 & 2 & 3 \end{pmatrix}$, and $A^{-1} = \begin{pmatrix} -0.04 & -0.08 & 0.16 \\ -0.04 & 0.22 & -0.04 \\ 0.16 & -0.08 & -0.04 \end{pmatrix}$

are also centro-symmetric.

The generalized centro-symmetric matrices have wide applications in information theory, linear estimate theory, linear system theory and numerical analysis (see[2,3]).The main concern in this paper is to study the following Problem.

Problem 1

For given matrices ,$\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{C}, \boldsymbol{M}, \boldsymbol{G}, \boldsymbol{N}, \boldsymbol{D}, \boldsymbol{F}, \boldsymbol{H} \in \boldsymbol{R}^{nxn}$, $\boldsymbol{P} \in \boldsymbol{SOR}^{nxn}$ and $\boldsymbol{S} \in \boldsymbol{SOR}^{nxn}$,find the matrices $\boldsymbol{V} \in \boldsymbol{CSR}_{\boldsymbol{P}}^{nxn}$ and $\boldsymbol{W} \in \boldsymbol{CSR}_{\boldsymbol{S}}^{nxn}$ such that

$$\boldsymbol{AV} + \boldsymbol{BW} = \boldsymbol{EVF} + \boldsymbol{C},\ \boldsymbol{MV} + \boldsymbol{NW} = \boldsymbol{GVH} + \boldsymbol{D}.$$

A large number of papers have presented several methods for solving linear matrix equation such as, Chu $[\boldsymbol{4}]$ and Hue $[\boldsymbol{5}]$. Peng constructed an iteration method for symmetric solution of the matrix equation $\boldsymbol{AXB} = \boldsymbol{C}$ and presented an iteration method to solve the minimum frobenius norm to residual this problem$[\boldsymbol{6}]$. In 2008, Hung [7] presented anew iterative method for solving $\boldsymbol{AXB} = \boldsymbol{C}$ over skew-symmerric matrix. In [8] Wang presented an iterative algorithm for solving linear matrix equation $\boldsymbol{AXB} + \boldsymbol{CX^t}\,\boldsymbol{D} = \boldsymbol{E}$. In 2011, M. Hajarian and M. Dehghan $[\boldsymbol{9}]$ presented an iterative algorithm for solving matrix the equation $\boldsymbol{AXB} + \boldsymbol{CX^t}\,\boldsymbol{D} = \boldsymbol{E}$ over centro-symmetric matrices. Ding and Chen $[\boldsymbol{10}, \boldsymbol{11}]$ introduced the hierarchical least squares-iterative (HLSI) algorithms for several coupled matrix equations. Xie $[\boldsymbol{12}]$ considered the inverse eigenvalue problem of the centro-symmetric matrices. M.L. Liang and C.H. You, et.al $[\boldsymbol{13}]$ presented an iterative algorithm for the generalized centro-symmetric matrices equation $\boldsymbol{A}\,\boldsymbol{X}\,\boldsymbol{B} = \boldsymbol{C}$. In $[\boldsymbol{14}]$ M. Hajarian and M. Dehghan constructed an iterative algorithm for solving a pair of matrix equations $\boldsymbol{AYB} = \boldsymbol{E}$, $\boldsymbol{CYD} = \boldsymbol{F}$ over generalized centro-symmetric matrices, And the generalized solution of matrix equations $(\boldsymbol{1})$ over Bisymmetric matrices has been represented in $[\boldsymbol{15}]$. M.A. Ramadan $[\boldsymbol{16}]$ presented explicit and iterative methods for solving the generalized sylvester matrix equation $\boldsymbol{AV} + \boldsymbol{BW} = \boldsymbol{EVF} + \boldsymbol{C}$. The coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q}\left(\boldsymbol{A}_{ij}\boldsymbol{X}_j\boldsymbol{B}_{ij} + \boldsymbol{C}_{ij}\boldsymbol{X}_j^T\boldsymbol{D}_{ij}\right) = \boldsymbol{F}_i, \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{p}$$

over generalized centro-symmetric matrices have been represented in $[\mathbf{17}]$. A gradient-based iterative algorithm for generalized coupled Sylvester matrix equations

$$\begin{cases} A_1X_1B_1 + C_1X_1D_1 = E_1 \\ A_2X_2B_2 + C_2X_2D_2 = E_2 \end{cases}$$

over generalized centro-symmetric matrices have been represented in $[\mathbf{18}]$. The generalized centro-symmetric solutions of the Sylvester matrix equations $(\mathbf{1})$ have not been studied and we will construct an iterative algorithm for it in this paper.

2. MAIN RESULTS

In this section, we will establish an iterative algorithm to solve problem 1. Some basic properties and lemmas will be presented to analyze the properties of the suggested algorithm.

Algorithm I

Step1: Input matrices

$A, B, E, C, M, G, N, D, F, H \in R^{nxn}, V_1 \in \mathcal{CSR}_p^{nxn}, W_1 \in \mathcal{CSR}_s^{nxn}, P \in \mathcal{SOR}^{nxn}, S \in \mathcal{SOR}^{nxn}$.

Step 2: compute

$$X_1 = C - AV_1 + EV_1F - BW_1;$$

$$Y_1 = D - MV_1 + GV_1H - NW_1;$$

$$R_1 = \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix};$$

$$P_1 = \frac{1}{2}[A^TX_1 + PA^TX_1P - E^TX_1F^T - PE^TX_1F^TP + M^TY_1 + PM^TY_1P - G^TY_1H^T - PG^TY_1H^TP];$$

$$Q_1 = \frac{1}{2}[B^TX_1 + SB^TX_1S + N^TY_1 + SN^TY_1S];$$

$$k := 1;$$

Step 3: If $R_k = 0$,then stop and $[V_k, W_k]$is the solution; else, if $R_k \neq 0$ but $P_k = 0$ and $Q_k = 0$, then stop and the generalized coupled sylvester matrix equations are not consistent over centro-symmetric matrices; else $k := k + 1$;

Step 4: compute

$$V_k = V_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2 + \|Q_{k-1}\|^2} P_{k-1};$$

$$W_k = W_{k-1} + \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2 + \|Q_{k-1}\|^2} Q_{k-1};$$

$$R_k = \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix};$$

$$= R_{k-1} - \frac{\|R_{k-1}\|^2}{\|P_{k-1}\|^2 + \|Q_{k-1}\|^2} \begin{pmatrix} AP_{k-1} + BQ_{k-1} - EP_{k-1}F & 0 \\ 0 & MP_{k-1} + NQ_{k-1} - GP_{k-1}H \end{pmatrix};$$

$$P_k = \frac{1}{2}[A^T X_k + PA^T X_k P - E^T X_k F^T - PE^T X_k F^T P + M^T Y_k + PM^T Y_k P - G^T Y_k H^T$$
$$- PG^T Y_k H^T P] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2} P_{k-1};$$

$$Q_k = \frac{1}{2}[B^T X_k + SB^T X_k S + N^T Y_k + SN^T Y_k S] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2} Q_{k-1};$$

Step 5: Go to Step 3.

Remark: Obviously from Algorithm I, $V_k, P_k \in \mathcal{CSR}_p^{nxn}$ and $W_k, S_k \in \mathcal{CSR}_s^{nxn}$ for $k = 1, 2, \ldots$

for $k = 2$

$$PV_2P = PV_1P + \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} PP_1P$$

$$= V_1 + \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \frac{1}{2}[PA^T X_1 P + P^2 A^T X_1 P^2 - PE^T X_1 F^T P - P^2 E^T X_1 F^T P^2 + PM^T Y_1 P$$
$$+ P^2 M^T Y_1 P^2 - PG^T Y_1 H^T P - P^2 G^T Y_1 H^T P^2]$$

$$= V_1 + \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} \frac{1}{2}[PA^T X_1 P + A^T X_1 - PE^T X_1 F^T P - E^T X_1 F^T + PM^T Y_1 P + M^T Y_1$$
$$- PG^T Y_1 H^T P - G^T Y_1 H^T]$$

$$= V_2 \quad .$$

And

$$SW_2S = SW_1S + \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} SQ_1S$$

$$= W_1 + \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}\frac{1}{2}[SB^TX_1S + S^2B^TX_1S^2 + SN^TY_1S + S^2N^TY_1S^2]$$

$$= W_1 + \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2}\frac{1}{2}[SB^TX_1S + B^TX_1 + SN^TY_1S + N^TY_1]$$

$$= W_2 \quad .$$

Lemma 1:

Assume that the sequences $\{R_i\}$, $\{P_i\}$, $\{Q_i\}$ are generated by Algorithm I, then we have

$$trace\left(R_{i+1}^TR_j\right) = tr\left(R_i^TR_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2+\|Q_i\|^2} tr\left(P_i^TP_j + Q_i^TQ_j\right) + \frac{\|R_i\|^2\|R_j\|^2}{(\|P_i\|^2+\|Q_i\|^2)\|R_{j-1}\|^2} tr\left(P_i^TP_{j-1} + Q_i^TQ_{j-1}\right), \quad \text{for } i,j = 1,2,\dots \tag{2}$$

$$trace\left(R_{i+1}^TR_j\right) = tr\left(\left[R_i - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2}\begin{pmatrix} AP_i + BQ_i - EP_iF & 0 \\ 0 & MP_i + NQ_i - GP_iH \end{pmatrix}\right]^T R_j\right)$$

$$= tr\left(R_i^TR_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2} tr\left(\begin{pmatrix} P_i^TA^T + Q_i^TB^T - F^TP_i^TE^T & 0 \\ 0 & P_i^TM^T + Q_i^TN^T - H^TP_i^TG^T \end{pmatrix}\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}\right)$$

$$= tr\left(R_i^TR_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2+\|Q_i\|^2} tr\left(\left(P_i^TA^T + Q_i^TB^T - F^TP_i^TE^T\right)X_j + \left(P_i^TM^T + Q_i^TN^T - H^TP_i^TG^T\right)Y_j\right)$$

$$= tr\left(R_i^TR_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2} tr(P_i^T(A^TX_j - E^TX_jF^T + M^TY_j - G^TY_jH^T) + Q_i^T(B^TX_j + N^TY_j))$$

$$= tr\left(R_i^TR_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2} tr\Bigg(P_i^T\Bigg\{\frac{A^TX_j + PA^TX_jP - E^TX_jF^T - PE^TX_jF^TP}{2} + \frac{M^TY_j + PM^TY_jP - G^TY_jH^T - PG^TY_jH^TP}{2} + \frac{A^TX_j - PA^TX_jP - E^TX_jF^T + PE^TX_jF^TP}{2} + \frac{M^TY_j - PM^TY_jP - G^TY_jH^T + PG^TY_jH^TP}{2}\Bigg\} + Q_i^T\Bigg\{\frac{B^TX_j + SB^TX_jS + N^TY_j + SN^TY_jS}{2} + \frac{B^TX_j - SB^TX_jS + N^TY_j - SN^TY_jS}{2}\Bigg\}\Bigg)$$

$$= tr(R_i^T R_j) - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2} tr\left(P_i^T\left\{P_j \frac{\|R_j\|^2}{\|R_{j-1}\|^2} P_{j-1}\right\} + Q_i^T\left\{Q_j \frac{\|R_j\|^2}{\|R_{j-1}\|^2} Q_{j-1}\right\}\right)$$

$$= tr(R_i^T R_j) - \frac{\|R_i\|^2}{\|P_i\|^2 + \|Q_i\|^2} tr(P_i^T P_j + Q_i^T Q_j) + \frac{\|R_i\|^2\|R_j\|^2}{(\|P_i\|^2 + \|Q_i\|^2)\|R_{j-1}\|^2} tr(P_i^T P_{j-1} + Q_i^T Q_{j-1})$$

We complete the proof of lemma 1.

Lemma 2:

Suppose the sequences $\{R_i\}$, $\{P_i\}$, $\{Q_i\}$ are obtained by Algorithm I, then we have

$tr(R_j^T R_i) = 0$ and $tr(P_j^T P_i + Q_j^T Q_i) = 0$ for $i, j = 1,2, \dots, k, i \neq j$. (3)

Proof: For an arbitrary matrix A, $trace(A) = trace(A^T)$ .we prove $tr(R_j^T R_i) = 0$ and

$tr(P_j^T P_i + Q_j^T Q_i) = 0$ for $1 \leq i < j \leq k$, by use induction and two steps are required.

Step 1: We prove that

$tr(R_{i+1}^T R_i) = 0$ and $tr(P_{i+1}^T P_i + Q_{i+1}^T Q_i) = 0$, for $i, j = 1,2, \dots, k$ (4)

We use induction to prove (4)

for $\mathrm{i} = 1$, using the proof of lemma 1, we have

$$trace(R_2^T R_1) = tr\left(\left[R_1 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2}\begin{pmatrix} AP_1 + BQ_1 - EP_1F & 0 \\ 0 & MP_1 + NQ_1 - GP_1H \end{pmatrix}\right]^T R_1\right)$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2} tr\left((P_1^T A^T + Q_1^T B^T - F^T P_1^T E^T)X_1 + (P_1^T M^T + Q_1^T N^T - H^T P_1^T G^T)Y_1\right)$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2} tr\left(P_1^T(A^T X_1 - E^T X_1 F^T + M^T Y_1 - G^T Y_1 H^T) + Q_1^T(B^T X_1 + N^T Y_1)\right)$$

$$\begin{aligned}= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2} tr\Bigg(P_1^T\Bigg\{&\frac{A^T X_1 + PA^T X_1 P - E^T X_1 F^T - PE^T X_1 F^T P}{2} \\ &+ \frac{M^T Y_1 + PM^T Y_1 P - G^T Y_1 H^T - PG^T Y_1 H^T P}{2} \\ &+ \frac{A^T X_1 - PA^T X_1 P - E^T X_1 F^T + PE^T X_1 F^T P}{2} \\ &+ \frac{M^T Y_1 - PM^T Y_1 P - G^T Y_1 H^T + PG^T Y_1 H^T P}{2}\Bigg\} \\ &+ Q_1^T\left\{\frac{B^T X_1 + SB^T X_1 S + N^T Y_1 + SN^T Y_1 S}{2} + \frac{B^T X_1 - SB^T X_1 S + N^T Y_1 - SN^T Y_1 S}{2}\right\}\Bigg)\end{aligned}$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2} tr(P_1^T P_1 + Q_1^T Q_1).$$

$$= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}\ \left(\|P_1\|^2 + \|Q_1\|^2\right) \ = \ 0 \ . \tag{5}$$

Also we can write

$$trace(P_2^T P_1 + Q_2^T Q_1) = tr\ \left(\left[\frac{A^T X_2 + PA^T X_2 P - E^T X_2 F^T - PE^T X_2 F^T P + M^T Y_2 + PM^T Y_2 P - G^T Y_2 H^T - PG^T Y_2 H^T P}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} P_1\right]^T P_1 + \left[\frac{B^T X_2 + SB^T X_2 S + N^T Y_2 + SN^T Y_2 S}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} Q_1\right]^T Q_1\right) = \frac{\|R_2\|^2}{\|R_1\|^2} tr(\|P_1\|^2 + \|Q_1\|^2) +$$

$$tr\left(X_2^T(AP_1 + BQ_1 - EP_1F) + Y_2^T(MP_1 + NQ_1 - GP_1H)\right) = \frac{\|R_2\|^2}{\|R_1\|^2} tr(\|P_1\|^2 + \|Q_1\|^2) +$$

$$tr\left(R_2^T \frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2}(R_1 - R_2)\right) = \frac{\|R_2\|^2}{\|R_1\|^2} tr(\|P_1\|^2 + \|Q_1\|^2) + \frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2} tr(R_2^T R_1 - R_2^T R_2) =$$

$$\frac{\|R_2\|^2}{\|R_1\|^2} tr(\|P_1\|^2 + \|Q_1\|^2) + \frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2} tr(R_2^T R_1) - \frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2}\|R_2\|^2 \quad = 0\ . \tag{6}$$

Now, assume (4) holds for $i = v - 1$.

for $i = v$, we have

$$trace(R_{v+1}^T R_v) = tr\left(\left[R_v - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2}\begin{pmatrix} AP_v + BQ_v - EP_vF & 0 \\ 0 & MP_v + NQ_v - GP_vH\end{pmatrix}\right]^T R_v\right)$$

$$= \|R_v\|^2 - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2} tr\left(P_v^T(A^T X_v - E^T X_v F^T + M^T Y_v - G^T Y_v H^T) + Q_v^T(B^T X_v + N^T Y_v)\right)$$

$$= \|R_v\|^2 - \frac{\|R_v\|^2}{\|P_v\|^2+\|Q_v\|^2} tr\left[P_v^T\left(P_v - \frac{\|R_v\|^2}{\|R_{v-1}\|^2} P_{v-1}\right) + Q_v^T\left(Q_v \frac{\|R_v\|^2}{\|R_{v-1}\|^2} Q_{v-1}\right)\right]$$

$$= \|R_v\|^2 - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2}\left(\|P_v\|^2 + \|Q_v\|^2\right) + \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2}\frac{\|R_v\|^2}{\|R_{v-1}\|^2} tr(P_v^T P_{v-1} + Q_v^T Q_{v-1})$$

$$= 0. \tag{7}$$

And

$$trace(P_{v+1}^T P_v + Q_{v+1}^T Q_v)$$

$$= tr\ \left(\left[\frac{A^T X_{v+1} + PA^T X_{v+1} P - E^T X_{v+1} F^T - PE^T X_{v+1} F^T P}{2}\right.\right.$$

$$\left. + \frac{M^T Y_{v+1} + PM^T Y_{v+1} P - G^T Y_{v+1} H^T - PG^T Y_{v+1} H^T P}{2} + \frac{\|R_{v+1}\|^2}{\|R_v\|^2} P_v\right]^T P_v$$

$$\left. + \left[\frac{B^T X_{v+1} + SB^T X_{v+1} S + N^T Y_{v+1} + SN^T Y_{v+1} S}{2} + \frac{\|R_{v+1}\|^2}{\|R_v\|^2} Q_v\right]^T Q_v\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(\|P_v\|^2 + \|Q_v\|^2\right) + tr\left(x_{v+1}^T(AP_v + BQ_v - EP_vF) + Y_{v+1}^T(MP_v + NQ_v - GP_vH)\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(\|P_v\|^2 + \|Q_v\|^2\right) + tr\left(R_{v+1}^T \frac{\|P_v\|^2 + \|Q_v\|^2}{\|R_v\|^2}(R_v - R_{v+1})\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(\|P_v\|^2 + \|Q_v\|^2\right) + \frac{\|P_v\|^2 + \|Q_v\|^2}{\|R_v\|^2} tr(R_{v+1}^T R_v - R_{v+1}^T R_{v+1})$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(\|P_v\|^2 + \|Q_v\|^2\right) + \frac{\|P_v\|^2 + \|Q_v\|^2}{\|R_v\|^2} tr(R_{v+1}^T R_v) - \frac{\|P_v\|^2 + \|Q_v\|^2}{\|R_v\|^2}\|R_{v+1}\|^2$$

$$= 0 \quad . \tag{8}$$

Therefore, (4) holds for i = v .Then, (4) holds by the principle of induction.

Step 2: In this step, we show that

$trace\left(R_{v+1}^T R_j\right) = 0$ and $tr\left(P_{v+1}^T P_j + Q_{v+1}^T Q_j\right) = 0$, for $i, j = 1,2, \ldots, v-1$

Suppose that $tr\left(R_v^T R_j\right) = 0$ and $tr\left(P_v^T P_j + Q_v^T Q_j\right) = 0$ for $i, j = 1,2, \ldots, v$

By using the induction assumption and previous lemma, we have

$$trace\left(R_{v+1}^T R_j\right) = tr\left(\left[R_v - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2}\begin{pmatrix} AP_v + BQ_v - EP_vF & 0 \\ 0 & MP_v + NQ_v - GP_vH \end{pmatrix}\right]^T R_j\right)$$

$$= tr\left(R_v^T R_j\right) - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2} tr\left[\begin{pmatrix} AP_v + BQ_v - EP_vF & 0 \\ 0 & MP_v + NQ_v - GP_vH \end{pmatrix}\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}\right]$$

$$= tr\left(R_v^T R_j\right) - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2} tr\left[P_v^T\left(A^T X_j - E^T X_j F^T + M^T Y_j - G^T Y_j H^T\right) + Q_v^T\left(B^T X_j + N^T Y_j\right)\right]$$

$$\begin{aligned} = tr\left(R_v^T R_j\right) - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2} tr\Bigg(P_v^T \Bigg\{ & \frac{A^T X_j + PA^T X_j P - E^T X_j F^T - PE^T X_j F^T P}{2} \\ & + \frac{M^T Y_j + PM^T Y_j P - G^T Y_j H^T - PG^T Y_j H^T P}{2} \\ & + \frac{A^T X_j - PA^T X_j P - E^T X_j F^T + PE^T X_j F^T P}{2} \\ & + \frac{M^T Y_j - PM^T Y_j P - G^T Y_j H^T + PG^T Y_j H^T P}{2}\Bigg\} \\ & + Q_v^T\left\{\frac{B^T X_j + SB^T X_j S + N^T Y_j + SN^T Y_j S}{2} + \frac{B^T X_j - SB^T X_j S + N^T Y_j - SN^T Y_j S}{2}\right\}\Bigg) \end{aligned}$$

$$= tr\left(R_v^T R_j\right) - \frac{\|R_v\|^2}{\|P_v\|^2 + \|Q_v\|^2} tr\left(P_v^T P_j + Q_v^T Q_j\right)$$

$$= 0. \tag{9}$$

And

$$trace\left(P_{v+1}^{T}P_j + Q_{v+1}^{T}Q_j\right)$$

$$= tr\left(\left[\frac{A^TX_{v+1} + PA^TX_{v+1}P - E^TX_{v+1}F^T - PE^TX_{v+1}F^TP}{2}\right.\right.$$

$$\left. + \frac{M^TY_{v+1} + PM^TY_{v+1}P - G^TY_{v+1}H^T - PG^TY_{v+1}H^TP}{2} + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}P_v\right]^T P_j$$

$$\left. + \left[\frac{B^TX_{v+1} + SB^TX_{v+1}S + N^TY_{v+1} + SN^TY_{v+1}S}{2} + \frac{\|R_{v+1}\|^2}{\|R_v\|^2}Q_v\right]^T Q_j\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}tr\left(P_v^TP_j + Q_v^TQ_j\right) + tr\left(x_{v+1}^T\left(AP_j + BQ_j - EP_jF\right) + Y_{v+1}^T\left(MP_j + NQ_j - GP_jH\right)\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(P_v^TP_j + Q_v^TQ_j\right) + tr\left(R_{v+1}^T\frac{\|P_j\|^2 + \|Q_j\|^2}{\|R_j\|^2}\left(R_j - R_{j+1}\right)\right)$$

$$= \frac{\|R_{v+1}\|^2}{\|R_v\|^2}\left(P_v^TP_j + Q_v^TQ_j\right) + \frac{\|P_j\|^2 + \|Q_j\|^2}{\|R_j\|^2}tr\left(R_{v+1}^TR_j\right) - \frac{\|P_j\|^2 + \|Q_j\|^2}{\|R_j\|^2}tr\left(R_{v+1}^TR_{j+1}\right)$$

$$= 0 \ . \tag{10}$$

From Steps 1 and step 2, the conclusion$(\mathbf{3})$ holds by the principle of induction.

Lemma 3:

Assume that problem 1 is consistent over centro-symmetric matrices, and $[V^*, W^*]$ is an arbitrary solution pair of problem1 , then for any initial generalized centro-symmetric matrix generated by Algorithm I , we have

$$trace((V^* - V_i)^TP_i + (W^* - W_i)^TQ_i) = \|R_i\|^2 \qquad \text{for} \qquad i, j = 1,2, \dots \ . \tag{11}$$

where the sequences $\{R_i\}$, $\{P_i\}$, $\{Q_i\}$, $\{V_i\}$ and $\{W_i\}$ are generated by Algorithm I .

Proof

We prove conclusion (11) by induction. for $i = 1$, we have

$$trace((V^* - V_1)^TP_1 + (W^* - W_1)^TQ_1)$$

$$= tr\left((V^* - V_1)^T\left(\frac{A^TX_1 + PA^TX_1P - E^TX_1F^T - PE^TX_1F^TP}{2} + \frac{M^TY_1 + PM^TY_1P - G^TY_1H^T - PG^TY_1H^TP}{2}\right) + \right.$$

$$\left.(W^* - W_1)^T\left(\frac{B^TX_1 + SB^TX_1S + N^TY_1 + SN^TY_1S}{2}\right)\right)$$

$$= tr\big((V^* - V_1)^T (A^T X_1 - E^T X_1 F^T + M^T Y_1 - G^T Y_1 H^T) + (W^* - W_1)^T (B^T X_1 + N^T Y_1)\big)$$

$$= tr\Big(X_1^T\big(A(V^* - V_1) - E(V^* - V_1)F + B(W^* - W_1)\big) + Y_1^T\big(M(V^* - V_1) - G(V^* - V_1)H + N(W^* - W_1)\big)\Big)$$

$$= tr\left(\begin{pmatrix} X_1^T & 0 \\ 0 & Y_1^T \end{pmatrix}\begin{pmatrix} C - AV_1 + EV_1F - BW_1 & 0 \\ 0 & D - MV_1 + GV_1H - NW_1 \end{pmatrix}\right)$$

$$= tr(R_1^T R_1)$$

$$= \|R_1\|^2 \quad . \tag{12}$$

Suppose the conclusion (11) holds for $i = z$.

For $i = z + 1$, we have

$$\boldsymbol{trace}((V^* - V_{z+1})^T P_{z+1} + (W^* - W_{z+1})^T Q_{z+1})$$

$$= tr\left((V^* - V_{z+1})^T\left(\frac{A^T X_{z+1} + PA^T X_{z+1} P - E^T X_{z+1} F^T - PE^T X_{z+1} F^T P}{2} + \frac{M^T Y_{z+1} + PM^T Y_{z+1} P - G^T Y_{z+1} H^T - PG^T Y_{z+1} H^T P}{2} + \frac{\|R_{z+1}\|^2}{\|R_z\|^2} P_z\right) + \left(\frac{B^T X_{z+1} + SB^T X_{z+1} S + N^T Y_{z+1} + SN^T Y_{z+1} S}{2} + \frac{\|R_{z+1}\|^2}{\|R_z\|^2} Q_z\right)\right)$$

$$= tr\big((V^* - V_{z+1})^T (A^T X_{z+1} - E^T X_{z+1} F^T + M^T Y_{z+1} - G^T Y_{z+1} H^T) + (W^* - W_{z+1})^T (B^T X_{z+1} + N^T Y_{z+1})\big) + \frac{\|R_{z+1}\|^2}{\|R_z\|^2} tr((V^* - V_{z+1})^T P_z + (W^* - W_{z+1})^T Q_z)$$

$$= tr\Big(X_{z+1}^T\big(A(V^* - V_{z+1}) - E(V^* - V_{z+1})F + B(W^* - W_{z+1})\big) + Y_{z+1}^T\big(M(V^* - V_{z+1}) - G(V^* - V_{z+1})H + N(W^* - W_{z+1})\big)\Big) + \frac{\|R_{z+1}\|^2}{\|R_z\|^2} tr((V^* - V_{z+1})^T P_z + (W^* - W_{z+1})^T Q_z)$$

$$= tr\left(\begin{pmatrix} X_{z+1}^T & 0 \\ 0 & Y_{z+1}^T \end{pmatrix}\begin{pmatrix} C - AV_{z+1} + EV_{z+1}F - BW_{z+1} & 0 \\ 0 & D - MV_{z+1} + GV_{z+1}H - NW_{z+1} \end{pmatrix}\right)$$

$$+ \frac{\|R_{z+1}\|^2}{\|R_z\|^2} tr((V^* - V_z)^T P_z + (W^* - W_z)^T Q_z) - \frac{\|R_{z+1}\|^2}{\|R_z\|^2} \frac{\|R_z\|^2}{\|P_z\|^2 + \|Q_z\|^2} tr\big({P_z}^T P_z + {Q_z}^T Q_z\big)$$

$$= tr(R_{z+1}^T R_{z+1}) + \frac{\|R_{z+1}\|^2}{\|R_z\|^2} \|R_z\|^2 - \|R_{z+1}\|^2$$

$$= \|R_{z+1}\|^2 \tag{13}$$

By the principle of induction, conclusion(11) holds for all $i = 1,2, \ldots$.

Theorem:

When problem 1 is consistent over centro-symmetric matrices, then for any arbitrary initial generalized centro-symmetric matrices $V_1 \in \mathcal{CSR}_P^{nxn}$ and $W_1 \in \mathcal{CSR}_S^{nxn}$,a generalized centro-symmetric solution of problem 1 can be obtained with finite iterative steps in the absence of round-off errors.

Proof:

Assume that $R_i \neq 0$ for $i = 1,2, \ldots\ldots 2n^2$. from lemma 3 ,we have $P_i \neq 0$ or $Q_i \neq 0$ for $i = 1,2, \ldots\ldots 2n^2$. then R_{2n^2+1} and $[V_{2n^2+1}, W_{2n^2+1}]$ can be calculated by Algorithm I. Also, we can write from lemma 2, $tr\left(R_{2n^2+1}^T R_i\right) = 0$ for $i = 1,2, \ldots\ldots 2n^2$ and $tr\left(R_i^T R_j\right) = 0$ for $i, j = 1,2, \ldots . 2n^2$, $(i \neq j)$. Hence, the sequence $\{R_i\}$ consists of an orthogonal basis of matrix space

$$S = \left\{ N | N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, where\ N_1 \in R^{nxn}, N_2 \in R^{nxn} \right\}.$$

Then, $R_{2n^2+1} = 0$ and $[V_{2n^2+1}, W_{2n^2+1}]$ is the generalized centro-symmetric solution of our stated problem.

Hence, the proof is completed.

3. Numerical Examples

In this section, we will give some numerical examples to illustrate our results. We implemented Algorithm I in MATLAB.

Example 1 Consider the pair of matrix equations

$$\begin{cases} AV + BW = EVF + C \\ MV + NW = GVH + D \end{cases}$$

where

$$A = \begin{pmatrix} 2 & 1 & 4 & 1 \\ 3 & 3 & 4 & 6 \\ 5 & 4 & 2 & 2 \\ 5 & 3 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 9 & -5 & -2 \\ -6 & 9 & -2 & 10 \\ 0 & -9 & 1 & -4 \\ -3 & 5 & 9 & 4 \end{pmatrix}, \quad E = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 4 & 0 & 4 & 2 \\ 5 & 8 & 7 & 3 \\ 5 & 0 & 5 & 8 \end{pmatrix}$$

$$F=\begin{pmatrix}-3&-2&4&-1\\8&1&-2&-1\\5&2&1&4\\1&-4&4&-1\end{pmatrix},\ M=\begin{pmatrix}2&6&-3&2\\-2&6&1&7\\4&5&3&6\\3&-3&5&9\end{pmatrix},\ N=\begin{pmatrix}2&5&-3&1\\-1&0&-2&7\\-4&-4&1&2\\3&-2&2&8\end{pmatrix}$$

$$G=\begin{pmatrix}6&1&1&2\\10&6&0&8\\1&7&6&6\\6&-2&8&-2\end{pmatrix},\ H=\begin{pmatrix}4&2&1&-1\\1&2&5&5\\7&2&2&2\\-2&5&-2&-1\end{pmatrix}$$

and

$$C=\begin{pmatrix}-22&45&-65&-37\\-96&118&-86&1\\-10&-69&-165&-144\\-331&-13&-41&-23\end{pmatrix},\ D=\begin{pmatrix}-301&-79&-118&-50\\-490&-118&-280&-167\\-207&-72&-125&-106\\-432&-95&-47&17\end{pmatrix}$$

Then, we can verify that this system is consistent over the generalized centro-symmetric matrices and has the generalized centro-symmetric solutions V^*, W^* as:

$$V^*=\begin{pmatrix}4&0&4&0\\0&-2&0&0\\3&0&2&0\\0&5&0&-1\end{pmatrix}\in \mathcal{CSR}_P^{4x4},\quad \text{where } P=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

and

$$W^*=\begin{pmatrix}1&0&0&0\\0&6&0&3\\0&0&-1&0\\0&5&0&4\end{pmatrix}\in \mathcal{CSR}_S^{4x4},\quad \text{where } S=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

Let the arbitrary initial matrices $V_1=W_1=0$. Apply Algorithm I, we get the generalized centro-symmetric solutions as follows:

$$V_{21}=\begin{pmatrix}4.0000&0.0000&4.0000&0.0000\\-0.0000&-2.0000&-0.0000&0.0000\\3.0000&0.0000&2.0000&0.0000\\0.0000&5.0000&-0.0000&-1.0000\end{pmatrix}\in \mathcal{CSR}_P^{4x4}$$

and

$$W_{21}=\begin{pmatrix}1.0000&0.0000&-0.0000&-0.0000\\-0.0000&6.0000&-0.0000&3.0000\\0.0000&-0.0000&-1.0000&0.0000\\-0.0000&5.0000&0.0000&4.0000\end{pmatrix}\in \mathcal{CSR}_S^{4x4}$$

with

$\|R_{21}\| = \left\|\begin{pmatrix} X_{21} & 0 \\ 0 & Y_{21} \end{pmatrix}\right\| = 3.1 \times 10^{-3}$.

Table 1: The number of iteration and corresponding residual for the generalized centro-symmetric solutions.

Number of iterations K	Norm of residual $\|R\|$
20	3.11×10^{-2}
21	3.1×10^{-3}
22	2.4×10^{-3}
23	7.6678×10^{-4}
24	8.757×10^{-7}

Example 2

Consider the pair of matrix equations (1) and suppose that the given matrices:

$$A = \begin{pmatrix} 20 & 7 & 7 & 6 & 6 & 18 & 2 \\ 6 & 14 & 4 & 6 & 20 & 4 & 20 \\ 12 & 8 & 18 & 7 & 10 & 2 & 14 \\ 18 & 3 & 6 & 12 & 9 & 7 & 13 \\ 5 & 20 & 11 & 8 & 12 & 13 & 3 \\ 15 & 11 & 12 & 5 & 13 & 8 & 1 \\ 9 & 11 & 20 & 14 & 17 & 19 & 12 \end{pmatrix},$$

$$B = \begin{pmatrix} -10 & -10 & 8 & -9 & -7 & -9 & -10 \\ 4 & -4 & 8 & -6 & 7 & 10 & -7 \\ -6 & 10 & -9 & 6 & 1 & 3 & -3 \\ -5 & -5 & 5 & -2 & 10 & -2 & 8 \\ 10 & -8 & 8 & 7 & -9 & 2 & -1 \\ -7 & 2 & 0 & -7 & 9 & -3 & 3 \\ 6 & 3 & -9 & 5 & -3 & -6 & -2 \end{pmatrix},$$

$$E = \begin{pmatrix} 7 & 5 & 4 & 6 & 0 & 5 & 0 \\ 6 & 5 & 2 & 3 & -1 & -1 & 6 \\ 0 & 8 & 8 & 2 & 0 & 5 & 3 \\ -1 & 5 & 8 & 1 & 4 & 4 & -1 \\ 5 & 3 & 7 & 7 & -1 & -1 & 5 \\ 2 & 4 & 6 & 8 & -1 & 1 & 7 \\ 0 & 8 & 5 & 8 & -1 & 2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 7 & 6 & -3 & 3 & 0 & 5 & 7 \\ -1 & 2 & 6 & 0 & 0 & -4 & 9 \\ -5 & 2 & 1 & 8 & 7 & 6 & 1 \\ 3 & 10 & -3 & -5 & 4 & 4 & -3 \\ 9 & 4 & 5 & -2 & 8 & 7 & 1 \\ 7 & 7 & 2 & 9 & -4 & -3 & 0 \\ 1 & 6 & -4 & -2 & -2 & 6 & 2 \end{pmatrix},$$

$$M=\begin{pmatrix}-10 & -5 & 9 & -7 & -7 & -6 & 9\\ -7 & -3 & -7 & 3 & -8 & -9 & -6\\ -2 & -10 & -1 & 0 & -10 & 14 & 11\\ 13 & 11 & -10 & 5 & -6 & -4 & -8\\ 5 & 9 & -10 & -3 & 3 & 6 & 15\\ -10 & 8 & -10 & 5 & 11 & -5 & 6\\ -2 & 11 & -6 & 5 & -3 & -4 & -3\end{pmatrix},$$

$$N=\begin{pmatrix}17 & 14 & -2 & 17 & 3 & 18 & 2\\ -5 & -5 & 12 & 7 & 1 & 19 & 18\\ 14 & 0 & -4 & 15 & 0 & 8 & 18\\ 18 & -3 & -2 & 2 & 14 & -5 & 15\\ 11 & -1 & -1 & 20 & 4 & 5 & 18\\ 13 & 19 & 11 & 20 & -1 & -2 & -5\\ 7 & 17 & 17 & -3 & -1 & 19 & 19\end{pmatrix},$$

$$G=\begin{pmatrix}-2 & 0 & -2 & 1 & 3 & 1 & 1\\ 1 & 1 & -2 & 9 & 3 & 5 & -1\\ 4 & 3 & 3 & 8 & 7 & 0 & 0\\ 10 & 3 & 10 & 3 & -1 & 10 & 10\\ 5 & 3 & 5 & 9 & 7 & -2 & 1\\ 1 & 5 & 9 & 3 & -1 & 0 & 1\\ -1 & 3 & 2 & 4 & 4 & 6 & 6\end{pmatrix}, \quad H=\begin{pmatrix}-8 & 1 & -3 & 2 & -7 & -5 & 6\\ -7 & 1 & -7 & -4 & 1 & 6 & 1\\ -3 & 1 & -2 & 0 & 4 & -1 & -5\\ -5 & 6 & -5 & -3 & 6 & -5 & 1\\ 3 & -1 & 2 & 6 & 4 & 4 & 6\\ 3 & -2 & 5 & 4 & 7 & 7 & -5\\ 6 & 2 & -1 & -6 & -6 & 5 & 4\end{pmatrix}$$

and

$$C=\begin{pmatrix}-1632 & -1401 & 108 & 324 & 65 & -1105 & -76\\ -1400 & -2457 & 939 & -381 & -560 & -2486 & -481\\ -1620 & -2527 & 710 & -898 & 157 & -1524 & -712\\ -1201 & -1218 & 338 & -380 & 529 & -593 & -320\\ -1836 & -2625 & 1072 & 6 & -761 & -2604 & -871\\ -1668 & -2649 & 987 & -387 & -582 & -2852 & -1031\\ -689 & -1406 & 904 & 23 & 161 & -651 & 304\end{pmatrix},$$

$$D=\begin{pmatrix}528 & -224 & 281 & -263 & -628 & 82 & 571\\ -305 & -293 & -33 & -110 & -456 & 83 & 55\\ 348 & -295 & 373 & -679 & -181 & 429 & -691\\ 650 & -686 & 990 & -197 & 2191 & -527 & -3214\\ 405 & -547 & 613 & -641 & 436 & 840 & -1259\\ 391 & -552 & 452 & -260 & 930 & 519 & -1379\\ 585 & -539 & 540 & -168 & 349 & -12 & -591\end{pmatrix}.$$

We observe that these matrix equations are consistent over generalized centro-symmetric matrices and have the solutions V with respect to P and W with respect to S.

$$V^* = \begin{pmatrix} 2 & 0 & -3 & 0 & 9 & 0 & 6 \\ 0 & -4 & 0 & 4 & 0 & 9 & 0 \\ 14 & 0 & -3 & 0 & 3 & 0 & 6 \\ 0 & -3 & 0 & 4 & 0 & -5 & 0 \\ 0 & 0 & 4 & 0 & 5 & 0 & -5 \\ 0 & -4 & 0 & -2 & 0 & 3 & 0 \\ 14 & 0 & 14 & 0 & 5 & 0 & 11 \end{pmatrix} \in \mathcal{CSR}_P^{7\times 7}, \quad \text{where}$$

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

And

$$W^* = \begin{pmatrix} 7 & 0 & 8 & 0 & -9 & 0 & 2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 9 & 0 & 2 \\ 0 & -6 & 0 & -7 & 0 & 8 & 0 \\ 2 & 0 & 1 & 0 & 5 & 0 & -2 \\ 0 & -7 & 0 & -1 & 0 & -9 & 0 \\ -2 & 0 & 5 & 0 & 5 & 0 & -1 \end{pmatrix} \in \mathcal{CSR}_S^{7\times 7}, \quad \text{where}$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let the initial matrices $V_1 = W_1 = 0$, Apply Algorithm I and we obtain that the sequences $\{V_k\}$ and $\{W_k\}$ after 71 steps will be as follows:

$$V_{71} = \begin{pmatrix} 2.0000 & -0.0000 & -3.0000 & 0.0000 & 9.0000 & 0.0000 & 6.0000 \\ 0.0000 & -4.0000 & 0.0000 & 4.0000 & 0.0000 & 9.0000 & 0.0000 \\ 14.0000 & 0.0000 & -3.0000 & 0.0000 & 3.0000 & 0.0000 & 6.0000 \\ 0.0000 & -3.0000 & -0.0000 & 4.0000 & 0.0000 & -5.0000 & -0.0000 \\ 0.0000 & -0.0000 & 4.0000 & -0.0000 & 5.0000 & 0.0000 & -5.0000 \\ 0.0000 & 4.0000 & -0.0000 & -2.0000 & -0.0000 & 3.0000 & -0.0000 \\ 14.0000 & 0.0000 & 14.0000 & -0.0000 & 5.0000 & -0.0000 & 11.0000 \end{pmatrix} \in \mathcal{CSR}_P^{7\times 7}$$

and

$$W_{71} = \begin{pmatrix} 7.0000 & 0.0000 & 8.0000 & -0.0000 & -9.0000 & 0.0000 & 2.0000 \\ 0.0000 & -2.0000 & 0.0000 & 0.0000 & -0.0000 & 1.0000 & -0.0000 \\ -1.0000 & -0.0000 & -1.0000 & -0.0000 & 9.0000 & -0.0000 & 2.0000 \\ 0.0000 & -6.0000 & 0.0000 & -7.0000 & 0.0000 & 8.0000 & -0.0000 \\ 2.0000 & -0.0000 & 1.0000 & 0.0000 & 5.0000 & -0.0000 & -2.0000 \\ -0.0000 & -7.0000 & -0.0000 & -1.0000 & -0.0000 & -9.0000 & -0.0000 \\ -2.0000 & -0.0000 & 5.0000 & -0.0000 & 5.0000 & -0.0000 & -1.0000 \end{pmatrix} \in \mathcal{CSR}_S^{7\times 7}$$

with

$$\|R_{71}\| = \left\|\begin{pmatrix} X_{71} & 0 \\ 0 & Y_{71} \end{pmatrix}\right\| = 1.42 \times 10^{-2} \ .$$

Also, we can verify that $V_{71}$is generalized centro-symmetric matrix

$PV_{71}P - V_{71} =$

$$10^{-14}\begin{pmatrix} 0 & 0.0038 & 0 & -0.1172 & 0 & -0.2779 & 0 \\ -0.0029 & 0 & -0.8135 & -0 & -0.3370 & 0 & -0.0703 \\ 0 & -0.8135 & 0 & -0.0920 & 0 & -0.7404 & 0 \\ -0.4243 & 0 & 0.3035 & 0 & -0.5437 & 0 & 0.8405 \\ 0 & 0.5644 & 0 & 0.5217 & 0 & -0.5518 & 0 \\ -0.3646 & 0 & 0.01 & 0 & 0.1246 & 0 & 0.1834 \\ -2.0000 & -0.4123 & 0 & 0.1983 & 0 & 0.7233 & 0 \end{pmatrix}$$

and we have $\|PV_{71}P - V_{71}\| = 2.2966 \times 10^{-14} < 10^{-10}$. Moreover, It can be verified that $\|SW_{71}S - W_{71}\| = 2.748 \times 10^{-15} < 10^{-10}$ .So, V_{71} and W_{71} are centro-symmetric matrices.

Table 2: The number of iteration (K) and corresponding residual $\|R\|$ for the generalized centro- symmetric solutions of example 2:

Number of iterations K	Norm of residual $\|R\|$
70	0.94×10^{-2}
71	1.42×10^{-2}
72	5.9×10^{-3}
73	3.9×10^{-3}
74	3.1×10^{-3}

4. Conclusion

In this paper, an iterative Algorithm I is introduced to solve the generalized coupled Sylvester matrix equations $AV + BW = EVF + C$, $MV + NW = GVH + D$ over generalized centro-symmetric matrices V, W. By applying Algorithm I, we can determine the solvability of problem 1 automatically. When problem 1 is consistent over centro-symmetric matrices, then for any arbitrary initial generalized centro-symmetric matrices V_0 and W_0, a generalized centro-symmetric solution of problem 1 can be obtained with finite iterative steps. And finally, some numerical examples are presented to illustrate the efficiency of the presented method.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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