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SOME FIXED POINT THEOREM ON E-METRIC SPACE USING CYCLIC MAPPINGS

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Abstract. In this manuscript, the notion of cyclic type contractions on E-metric spaces are introduced and subsequently established the fixed point and common fixed point results for this class of mappings in the setting of E-metric spaces. The presented results are extended from some well-known fixed point theorems in the literature.

Keywords: E-metric space; positive cone; cyclic mapping.
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1. Introduction and Preliminaries

In [3] Huang and Zhang introduced the notion of cone metric spaces, replacing the set of real numbers by an ordered Banach space, they have defined the cone metric spaces and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping for cone metric spaces. W.A. Kirk, P.S. Srinivasan and P.

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Veeramani [6] studied cyclic contraction of metric space and obtained some fixed point theorems. Al-Rawashdeh et al. [2] introduced E-metric space. W.A. Kirk [7] introduced fixed points of asymptotic contraction. Achille Basile et al.[1] defined cone with semi-interior points and equilibrium. Some recent works on cone metric space and E-metric space can be found in [4, 5, 8, 9, 10]. In this paper, the cyclic contraction on E-metric space is defined and subsequently establish some fixed point and common fixed point results for this class of mappings in the setting of E-metric spaces. The presented results extend some well known fixed point theorems in the literature.

Definition 1.1. Let E be a real ordered vector space, E^+ be a non-empty closed and convex subset of E, and 0_E be a zero element in E. Then E^+ is called a positive cone if it satisfies

i. for all $x \in E^+$ and $\alpha \geq 0$ imply $\alpha x \in E^+$
ii. for all $x \in E^+$ and $-x \in E^+$ imply $x = 0_E$.

Definition 1.2. An ordered space E is a vector space over the real numbers, with a partial order relation $\preceq$ such that

i. for all x, y, and $z \in E$, $x \preceq y$ implies $x + z \preceq y + z$,
ii. for all $a \in \mathbb{R}^+$ and $x \in E$ with $x \succeq 0_E, ax \succeq 0_E$.

Definition 1.3. The positive cone E^+ of a normed ordered space X is called

(a) *normal*, if there exists a constant $M > 0$, such that $0 \preceq x \preceq y$ implies $||x|| \leq M||y||$;
(b) *solid*, if $intE^+ \neq \phi$;
(c) *reflexive* $\Leftrightarrow E^+ \cap U$ is weakly compact, where U is the unit ball in X;
(d) *strongly reflexive* $\Leftrightarrow E^+ \cap U$ is compact.

Definition 1.4. Let X be a non-empty set and let E be a real ordered vector space. An E-metric function $d_E : X \times X \to E$ such that for all x, y and $z \in X$, we have

(1) $d_E(x,y) \succeq 0_E$ and $d_E(x,y) = 0_E$ if and only if $x = y$;
(2) $d_E(x,y) \overset{e}{=} d_E(y,x)$;
(3) $d_E(x,y) \preceq d_E(x,z) + d_E(y,z)$.

Then the pair (X, d_E) is called an $E-$ metric space.

Let E be an real ordered vector space by the positive cone E^+, we say that $U = \{x \in E : ||x|| \leq 1\}$ be the closed unit ball of E, and by U_+ we mean the positive part of unit ball defined by the set $U_+ = U \cap E^+$.

Definition 1.5. The point $x_0 \in E^+$ is a semi-interior point of E^+, if there exists a real number $\rho > 0$ such that $x_0 - \rho U_+ \subseteq E^+$.

The set of all semi-interior points of E^+ is denoted by $(E^+)^0$, and for $x, y \in E^+$, $x \preceq y$ if and only if $y - x \in (E^+)^0$.

Remark 1.6. Any interior point of E^+ is a semi-interior point.

Definition 1.7. Let E be a real ordered vector space with $E^+ \neq \emptyset$ and (X, d_E) be an $E-$ metric space. Let (x_n) be a sequence in X and $x \in X$.

(1) (x_n) is e-converges to x whenever for every $e \gg 0_E$, there exists a positive integer n_0 such that if $n \geq n_0$ then $d_E(x_n, x) \preceq e$. We denote this by $\lim_{n\to\infty} x_n \stackrel{e}{=} x$ or $x_n \xrightarrow{e} x$.

(2) (x_n) is an e-Cauchy sequence whenever for every $e \gg 0_E$, there exists a positive integer n_0 such that if $n, m \geq n_0$ then $d_E(x_n, x_m) \preceq e$.

(3) (X, d_E) is $e-$ complete if every $e-$ Cauchy sequence is $e-$ convergent.

Proposition 1.8. *If E is a complete real ordered vector space with closed cone E^+ and generating then any semi-interior point of E^+ is an interior point of E^+.*

Definition 1.9. Let (X, d_E) be a $e-$complete $E-$ metric space. A mapping $T : X \to X$ is said to be an $e-$ asymptotic contraction if for each integer $n \geq 1$, such that

$$d_E(T^n x, T^n y) \preccurlyeq \eta^n d_E(x, y) \tag{1.1}$$

for all $x, y \in X$ and if $\eta \in [0, 1)$.

If $\eta = 1$, then T is non-expansive mapping such that

$$d_E(T^n x, T^n y) \preccurlyeq d_E(x, y)$$

2. FIXED POINT THEOREMS

Definition 2.1. Let (X,d_E) be a $E-$ metric space and let P and Q be two non-empty subsets of X. A mapping $T: P\cup Q \to P\cup Q$ is called an $e-$ cyclic map if $T(P)\subset Q$ and $T(Q)\subset P$.

Definition 2.2. Let (X,d_E) be a $E-$ metric space and let P and Q be two non-empty subsets of X. An cyclic map $T: P\cup Q \to P\cup Q$ is said to be Zamfirescu contraction such that at least one of the following is true.

For all $x\in P$ and $y\in Q$ and some $\theta_1\in(0,1), \theta_2,\theta_3\in(0,\frac{1}{2})$.

(1) $d_E(Tx,Ty)\preccurlyeq \theta_1 d_E(x,y)$

(2) $d_E(Tx,Ty)\preccurlyeq \theta_2[d_E(Tx,x)+d_E(Ty,y)]$

(3) $d_E(Tx,Ty)\preccurlyeq \theta_3[d_E(Tx,y)+d_E(x,Ty)], \forall\, x\in P$ and $\forall\, y\in Q$

Theorem 2.3. *Let P and Q be non-empty closed subsets of a complete $E-$metric space (X,d_E) and $T: P\cup Q\to P\cup Q$ be a Zamfirescu contraction. Then T has a unique fixed point in $P\cap Q$.*

Proof. According to Zamfirescu contraction, we want to divide the proof into three parts

Part 1:

Fix $x\in P$ and $y=Tx\in Q$, we have

$$d_E(T^2x,Tx)\preccurlyeq \theta_1 d_E(Tx,x) \tag{2.1}$$

In general, $d_E(T^{n+1}x,T^nx)\preccurlyeq \theta_1^n(d_E(Tx,x))$.

Now for $n>m$, consider

$$\begin{aligned} d_E(T^mx,T^nx) &\preccurlyeq d_E(T^mx,T^{m+1}x)+d_E(T^{m+1}x,T^{m+2}x)+\cdots d_E(T^{n-1}x,T^nx)\\ &\preccurlyeq (\theta_1^m+\theta_1^{m+1}+\cdots+\theta_1^{n-1})d_E(Tx,x)\\ &\preccurlyeq \theta_1^m\left(\frac{1-\theta_1^{n-m}}{1-\theta_1}\right)d_E(Tx,x). \end{aligned}$$

Let $e\succeq 0_E$ be given, choose $\rho>0$ such that $e-\rho U_+\subseteq E^+$ and there is a positive integer n_0 such that $\theta_1^m(\frac{1-\theta_1^{n-m}}{1-\theta_1})d_E(Tx,x)\in\frac{\rho}{2}U_+$, for every $m,n\geq n_0$, therefore $e-\theta_1^m(\frac{1-\theta_1^{n-m}}{1-\theta_1})d_E(Tx,x)-\frac{\rho}{2}U_+\subseteq e-\rho U_+\subseteq E^+$, hence

$$d_E(T^mx,T^nx)\preccurlyeq \theta_1^m\left(\frac{1-\theta_1^{n-m}}{1-\theta_1}\right)d_E(Tx,x)\preceq e\,\forall n,m\geq n_0.$$

Therefore, $\{T^n x\}$ is Cauchy sequence. Then, there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore $P \cap Q$ is non empty.

Now

$$\begin{aligned} d_E(Tz,z) &= \lim_{n\to\infty}(d_E(Tz,T^{2n}z)) \\ &\preccurlyeq \theta_1 \lim_{n\to\infty} d_E(z,T^{2n-1}x) \\ &= \theta_1 d_E(z,z) \\ \Rightarrow d_E(Tz,z) &\stackrel{e}{=} 0_E. \end{aligned}$$

Therefore z is a fixed point of T in $P \cap Q$.

Uniqueness

Suppose w and z are two fixed points of T in $P \cap Q$ and T is a cyclic, we get $w, z \in P \cap Q$

$$\begin{aligned} d_E(z,w) &= d_E(Tz,Tw) \\ &\preccurlyeq \theta_1(d_E(z,w)) \\ \Rightarrow \ d_E(z,w) &\stackrel{e}{=} 0_E \end{aligned}$$

Hence z is an unique fixed point of T.

Part 2:

Fix $x \in P$ and $y = Tx \in Q$, we have

$$\begin{aligned} d_E(T^{n+1}x,T^n x) &\preccurlyeq \frac{\theta_2}{1-\theta_2} d_E(T^n x,T^{n-1}x) \\ &= \eta d_E(T^n x,T^{n-1}x). \end{aligned}$$

Where $\eta = \frac{\theta_2}{1-\theta_2}$ and clearly $\eta \in (0,1)$ since $\theta_2 \in (0,1/2)$, we have

$$d_E(T^{n+1}x,T^n x) \preccurlyeq \eta^n d_E(Tx,x)$$

Now for $n > m$, consider

$$\begin{aligned} d_E(T^m x,T^n x) &\preccurlyeq d_E(T^m x,T^{m+1}x) + d_E(T^{m+1}x,T^{m+2}x) + \cdots d_E(T^{n-1}x,T^n x) \\ &\preccurlyeq (\eta^m + \eta^{m+1} + \cdots + \eta^{n-1}) d_E(Tx,x) \\ &\preccurlyeq \eta^m \left(\frac{1-\eta^{n-m}}{1-\eta}\right) d_E(Tx,x). \end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_1 such that $\eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \in \frac{\rho}{2}U_+$, for every $m,n \geq n_1$, therefore $e - \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^m x, T^n x) \preccurlyeq \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \preceq e \forall n,m \geq n_1.$$

Therefore, $\{T^n x\}$ is Cauchy sequence. Then, there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore $P \cap Q$ is non empty.

Now

$$\begin{aligned} d_E(Tz,z) &= \lim_{n\to\infty} d_E(Tz, T^{2n}x) \\ &\preccurlyeq \theta_2 \lim_{n\to\infty}[d_E(Tz,z) + d_E(T^{2n}x, T^{2n-1}x)] \\ &\quad \theta_2[d_E(Tz,z) + d_E(z,z)] \\ &\stackrel{\text{e}}{=} d_E(Tz,z) \\ &\preccurlyeq \theta_2 d_E(Tz,z) \end{aligned}$$

$\Rightarrow \ d_E(Tz,z) \preceq 0_E$, z is a fixed point of T in $P \cap Q$.

Uniqueness

Suppose w is an another fixed point of T in $P \cap Q$ T is a cyclic, we get $w \in P \cap Q$

$$\begin{aligned} d_E(z,w) &= d_E(Tz,Tw) \\ &\preccurlyeq \theta_2[d_E(Tz,z) + d_E(Tw,w)] \\ &= \theta_2[d_E(z,z) + d_E(w,w)] \\ d_E(z,w) &\preccurlyeq 0_E \\ \Rightarrow \ d_E(z,w) &\stackrel{\text{e}}{=} 0_E \ \text{ since } \theta_2 \in (0,1/2) \\ \Rightarrow \quad z &= w \end{aligned}$$

Hence z is an unique fixed point of T.

Part 3:

Fix $x \in P$ and $y = Tx \in Q$, we have

$$\begin{aligned} d_E(T^{n+1}x, T^n x) &\preccurlyeq \frac{\theta_3}{1-\theta_3} d_E(T^n x, T^{n-1}x) \\ &= \eta d_E(T^n x, T^{n-1}x) \end{aligned}$$

Where $\eta = \frac{\theta_3}{1-\theta_3}$ and clearly $\eta \in (0,1)$ since $\theta_3 \in (0, 1/2)$, we have

$$d_E(T^{n+1}x, T^n x) \preccurlyeq \eta^n d_E(Tx, x)$$

Now for $n > m$, consider

$$\begin{aligned} d_E(T^m x, T^n x) &\preccurlyeq d_E(T^m x, T^{m+1}x) + d_E(T^{m+1}x, T^{m+2}x) + \cdots d_E(T^{n-1}x, T^n x) \\ &\preccurlyeq (\eta^m + \eta^{m+1} + \cdots + \eta^{n-1}) d_E(Tx, x) \\ &\preccurlyeq \eta^m (\frac{1-\eta^{n-m}}{1-\eta}) d_E(Tx, x). \end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_2 such that $\eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \in \frac{\rho}{2}U_+$, for every $m,n \geq n_2$, therefore $e - \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^m x, T^n x) \preccurlyeq \eta^m (\frac{1-\eta^{n-m}}{1-\eta}) d_E(Tx, x) \preceq e \forall n, m \geq n_2.$$

Hence $\{T^n x\}$ is Cauchy sequence. Then, there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed and $z \in P \cap Q$. Therefore $P \cap Q$ is non empty.

Now

$$\begin{aligned} d_E(Tz, z) &= \lim_{n\to\infty} d_E(Tz, T^{2n}z) \\ &\preccurlyeq \theta_3 \lim_{n\to\infty} [d_E(Tz, T^{2n-1}x) + d_E(z, T^{2n}z)] \\ &\stackrel{\text{e}}{=} \theta_3 [d_E(Tz, z) + d_E(z, z)] \\ d_E(Tz, z) &\preccurlyeq \theta_3 d_E(Tz, z) \end{aligned}$$

z is a fixed point of T in $P \cap Q$

Uniqueness

Suppose w is an another fixed point of T in $P\cap Q$ and T is a cyclic, we get $w\in P\cap Q$.

$$\begin{aligned}
d_E(z,w) &= d_E(Tz,Tw)\\
&\preccurlyeq \theta_3[d_E(Tz,w)+d_E(z,Tw)\\
&= \theta_3[d_E(z,w)+d_E(z,w)]\\
d_E(z,w) &\preccurlyeq 2\theta_3 d_E(z,w)\\
d_E(z,w) &\stackrel{e}{=} 0_E
\end{aligned}$$

Hence $z=w$

□

Theorem 2.4. *Let P and Q be non-empty closed subsets of $e-$complete $E-$ metric space (X,d_E). Suppose that a map $T:P\cup Q\to P\cup Q$ is satisfying $T(P)\subset Q$ and $T(Q)\subset P$ and there exists $\theta_4\in(0,\frac{1}{3})$ such that*

$$d_E(Tx,Ty)\preccurlyeq \theta_4[d_E(x,y)+d_E(Tx,y)+d_E(x,Ty)] \tag{2.2}$$

for all $x\in P$ and $y\in Q$. Then T has an unique fixed point in $P\cap Q$.

Proof. Let us take $x\in P$ and $y=Tx\in Q$. consider the above equation (2.2), we have

$$\begin{aligned}
d_E(T^{n+1}x,T^nx) &\preccurlyeq \frac{2\theta_4}{1-\theta_4}d_E(T^nx,T^{n-1}x)\\
&= \eta d_E(T^nx,T^{n-1}x)
\end{aligned}$$

where $\eta=\frac{2\theta_4}{1-\theta_4}$ and clearly $\eta\in(0,1)$ since $\theta_4\in(0,1/3)$, we have

$$d_E(T^{n+1}x,T^nx)\preccurlyeq \eta^n d_E(Tx,x)$$

Now for $n>m$, consider

$$\begin{aligned}
d_E(T^mx,T^nx) &\preccurlyeq d_E(T^mx,T^{m+1}x)+d_E(T^{m+1}x,T^{m+2}x)+\cdots d_E(T^{n-1}x,T^nx)\\
&\preccurlyeq (\eta^m+\eta^{m+1}+\cdots+\eta^{n-1})d_E(Tx,x)\\
&\preccurlyeq \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x).
\end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_3 such that $\eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \in \frac{\rho}{2}U_+$, for every $m,n \geq n_3$, therefore $e - \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^m x, T^n x) \preccurlyeq \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \preceq e \,\forall n,m \geq n_3.$$

$\{T^n x\}$ is a Cauchy sequence. Then there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore $P \cap Q \neq \emptyset$

Now

$$\begin{aligned} d_E(Tz,z) &= \lim_{n\to\infty} d_E(Tz, T^{2n}z) \\ &\preccurlyeq \theta_4 \lim_{n\to\infty}[d_E(z,T^{2n-1}x) + d_E(Tz,T^{2n-1}x) + d_E(z,T^{2n}x)] \\ &= \theta_4[d_E(z,z) + d_E(Tz,z) + d_E(z,z)] \\ d_E(Tz,z) &\preccurlyeq \theta_4 d_E(Tz,z) \end{aligned}$$

This is not possible, but $\theta_4 \in (0,1/3)$

Therefore, $d_E(Tz,z) \preceq 0_E$, Hence, z is a fixed point of T in $P \cap Q$

Uniqueness

Suppose w is an another fixed point of T in $P \cap Q$ and T is a cyclic, we get $w \in P \cap Q$.

$$\begin{aligned} d_E(z,w) &= d_E(Tz,Tw) \\ &\preccurlyeq \theta_4[d_E(z,w) + d_E(Tz,w) + d_E(z,Tw)] \\ &= [d_E(z,w) + d_E(z,w) + d_E(z,w)] \\ d_E(z,w) &\preccurlyeq 3\theta_4(d_E(z,w)) \\ (1-3\theta_4)d_E(z,w) &\preccurlyeq 0_E \\ \Rightarrow\ d_E(z,w) &\overset{\mathrm{e}}{=} 0_E \ \text{ since } \theta_4 \in (0,1/3) \\ \Rightarrow\quad z &= w \end{aligned}$$

Hence z is an unique fixed point of T. $\square$

Definition 2.5. Let P and Q be two non-empty subsets of $E-$metric space (X,d_E). A cyclic map $T : P\cup Q \to P\cup Q$ is said to be Hardy and Rogers contraction if there exists $\theta_5 \in (0,\frac{1}{5})$ such that

$$d_E(Tx,Ty) \preccurlyeq \theta_5[d_E(x,y)+d_E(Tx,x)+d_E(Ty,y)+d_E(Tx,y)+d_E(x,Ty)]$$

for all $x\in P$ and $y\in Q$

Example 2.6. Consider the $E-$metric space $X=\mathbb{R}$.

Suppose $P=Q=[0,1]$ defined $T : P\cup Q \to P\cup Q$ by

$$Tx=\begin{cases} 4/7 & \text{if } x\in[0,1/2] \\ 2/7 & \text{if } x\in(1/2,1]. \end{cases}$$

For $x=1/4$, $y=14/15$, then T is Hardy and Rogers contraction.

Theorem 2.7. *Let P and Q be two non-empty closed subsets of a E-metric space (X,d_E) and $T : P\cup Q \to P\cup Q$ be the Hardy and Rogers contraction. Then T has an unique fixed point in $P\cap Q$.*

Proof. Let us assume that $x\in P$ and $y=Tx\in Q$

From the above definition (2.5)

$$\begin{aligned} d_E(T^{n+1}x,T^nx) &\preccurlyeq \frac{3\theta_5}{1-2\theta_5} d_E(T^nx,T^{n-1}x) \\ &= \eta d_E(T^nx,T^{n-1}x) \end{aligned}$$

Where $\eta=\frac{3\theta_5}{1-2\theta_5}$ and clearly $\eta\in(0,1)$ since $\theta_5\in(0,1/5)$, we have

$$d_E(T^{n+1}x,T^nx)\preccurlyeq \eta^n d_E(Tx,x)$$

Now for $n>m$, consider

$$\begin{aligned} d_E(T^mx,T^nx) &\preccurlyeq d_E(T^mx,T^{m+1}x)+d_E(T^{m+1}x,T^{m+2}x)+\cdots d_E(T^{n-1}x,T^nx) \\ &\preccurlyeq (\eta^m+\eta^{m+1}+\cdots+\eta^{n-1})d_E(Tx,x) \\ &\preccurlyeq \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x). \end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_4 such that $\eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \in \frac{\rho}{2}U_+$, for every $m,n \geq n_4$, therefore $e - \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^m x, T^n x) \preccurlyeq \eta^m(\frac{1-\eta^{n-m}}{1-\eta})d_E(Tx,x) \preceq e \forall n,m \geq n_4.$$

Hence $\{T^n x\}$ is Cauchy sequence. Then there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore, $P \cap Q$ is non-empty.

Now,

$$\begin{aligned}
d_E(Tz,z) &= \lim_{n\to\infty} d_E(Tz, T^{2n}z) \\
&\preccurlyeq \theta_5 \lim_{n\to\infty} [d_E(z, T^{2n-1}x) + d_E(Tz,z) + d_E(T^{2n}x, T^{2n-1}x) \\
&\quad + d_E(Tz, T^{2n-1}x) + d_E(z, T^{2n}x)] \\
&\overset{e}{=} \theta_5[d_E(z,z) + d_E(Tz,z) + d_E(z,z) + d_E(Tz,z) + d_E(z,z)]
\end{aligned}$$

$d_E(Tz,z) \preccurlyeq 2\theta_5 d_E(Tz,z)$ This is not possible, since $\theta_5 \in (0,1/5)$

z is a fixed point of T in $P \cap Q$

Uniqueness

Suppose w is an another fixed point of T in $P \cap Q$ and T is cyclic, we get $w \in P \cap Q$.

$$\begin{aligned}
d_E(z,w) &= d_E(Tz,Tw) \\
&\preccurlyeq \theta_5[d_E(z,w) + d_E(Tz,z) + d_E(Tw,w) + d_E(Tz,w) + d_E(z,Tw)] \\
&= \theta_5[d_E(z,w) + d_E(z,z) + d_E(w,w) + d_E(z,w) + d_E(z,w)] \\
d_E(z,w) &\preccurlyeq 3\theta_5 d_E(z,w) \\
(1-3\theta_5)d_E(z,w) &\preccurlyeq 0_E \\
\Rightarrow\ d_E(z,w) &\overset{e}{=} 0_E \ \text{ since } \theta_5 \in (0,1/5) \\
\Rightarrow \quad z &= w
\end{aligned}$$

Hence z is an unique fixed point of T. □

Definition 2.8. Let P and Q be two non-empty subsets of $E-$metric space (X,d_E). A cyclic map $T : P\cup Q \to P\cup Q$ is said to be a Bianchini contraction if there exists $\theta_6 \in (0,1)$ such that $d_E(Tx,Ty) \preccurlyeq \theta_6 M(x,y)$, for all $x \in P$ and $y \in Q$, where $M(x,y) = \max\{d_E(Tx,x), d_E(Ty,y)\}$.

Example 2.9. Consider the $E-$metric space $X = \mathbb{R}$. Suppose $P = Q = [0,1]$ and $T : P\cup Q \to P\cup Q$ defined by

$$Tx = \begin{cases} 1/4 & \text{if } x = 1 \\ 1/2 & \text{if } x \in [0,1). \end{cases}$$

For $x = 15/16$, $y = 1$, then T is a Bianchini contraction.

Theorem 2.10. *Let P and Q be two non-empty closed subsets of a e−complete $E-$metric space (X,d_E) and a cyclic map $T : P\cup Q \to P\cup Q$ be a Bianchini contraction. Then T has an unique fixed point in $P\cap Q$.*

Proof. Consider $x \in P, y = Tx \in Q$

By definition (2.8), we have

$$d_E(Tx,Ty) \preccurlyeq \theta_6 M(x,y)$$

Case 1: If $M(x,y) = d_E(Tx,x)$,

$$d_E(Tx,Ty) \preccurlyeq \theta_6 d_E(Tx,x)$$

Put $y = Tx$

$$d_E(Tx,T^2x) \preccurlyeq \theta_6 d_E(x,Tx)$$

$$d_E(T^2x,Tx) \preccurlyeq \theta_6 d_E(x,Tx)$$

$$d_E(T^3x,T^2x) \preccurlyeq \theta_6 d_E(T^2x,Tx)$$

$$d_E(T^3x,T^2x) \preccurlyeq \theta_6^2 d_E(Tx,x)$$

$$\vdots$$

$$d_E(T^{n+1}x,T^nx) \preccurlyeq \theta_6^n d_E(Tx,x)$$

Now for $n > m$, consider

$$\begin{aligned} d_E(T^m x, T^n x) &\precsim d_E(T^m x, T^{m+1}x) + d_E(T^{m+1}x, T^{m+2}x) + \cdots d_E(T^{n-1}x, T^n x) \\ &\precsim (\theta_6^m + \theta_6^{m+1} + \cdots + \theta_6^{n-1}) d_E(Tx, x) \\ &\precsim \theta_6^m \left(\frac{1-\theta_6^{n-m}}{1-\theta_6}\right) d_E(Tx, x). \end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_5 such that $\theta_6^m(\frac{1-\theta_6^{n-m}}{1-\theta_6}) d_E(Tx,x) \in \frac{\rho}{2} U_+$, for every $m, n \geq n_5$, therefore $e - \theta_1^m(\frac{1-\theta_6^{n-m}}{1-\theta_6}) d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^m x, T^n x) \precsim \theta_6^m \left(\frac{1-\theta_6^{n-m}}{1-\theta_6}\right) d_E(Tx, x) \preceq e \, \forall n, m \geq n_5.$$

Hence $\{T^n x\}$ is Cauchy sequence. Then, there exist $z \in P \cap Q$ such that $T^n x \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore $P \cap Q$ is non-empty.

Now

$$\begin{aligned} d_E(Tz, z) &= \lim_{n\to\infty} d_E(Tz, T^{2n}z) \\ &\precsim \theta_6 \lim_{n\to\infty} M(z, T^{2n-1}x) \\ &= \theta_6 d_E(Tz, z) \\ (1-\theta_6) d_E(Tz, z) &\precsim 0_E \\ \Rightarrow d_E(Tz, z) &\stackrel{e}{=} 0_E \\ \Rightarrow Tz &= z \end{aligned}$$

Therefore z is a fixed point of T in $P \cap Q$.

Uniqueness

Suppose w is an another fixed point of T in $P \cap Q$ and T is cyclic, we get $w \in P \cap Q$

$$\begin{aligned} d_E(z, w) &= d_E(Tz, Tw) \\ &\precsim \theta_6 M(z, w) \\ &\precsim \theta_6 d_E(Tz, w) \end{aligned}$$

$$(1-\theta_6)d_E(z,w) \preccurlyeq 0_E$$

$$\Rightarrow d_E(z,w) \stackrel{e}{=} 0_E \text{ since } \theta_6 \in (0,1)$$

$$\Rightarrow \quad z = w$$

Hence z is an unique fixed point of T.

Case 2:

$$M(x,y) = d_E(Ty,y)$$

$$d_E(Tx,Ty) \preccurlyeq \theta_6 M(x,y)$$

$$= \theta_6 d_E(Ty,y)$$

Put $y = Tx$

$$d_E(Tx,T^2x) \preccurlyeq \theta_6 d_E(T^2,Tx)$$

$$d_E(T^2x,Tx) \preccurlyeq \theta_6 d_E(T^2x,Tx)$$

Which is impossible, since $\theta_6 \in (0,1)$.

This completes the proof. □

Corollary 2.11. *Let P and Q be two non-empty closed subsets of an $e-$complete $E-$metric space (X,d_E). The cyclic map $T : P \cup Q \to P \cup Q$ and*

$$d_E(Tx,Ty) \preccurlyeq \theta_6 d_E(Tx,x), \textit{for some } \theta_6 \in (0,1)$$

. Then T has an unique fixed point in $P \cap Q$.

Proof. The proof is followed by taking $M(x,y) = d_E(Tx,x)$ in the above theorem □

Theorem 2.12. *Let P and Q be two non-empty closed subsets of $e-$complete $E-$metric space (X,d_E). Suppose that $T : P \cup Q \to P \cup Q$ is a map satisfying $T(P) \subset Q$ and $T(Q) \subset P$ and there exists $\theta_7 \in (0,1)$ such that*

$$d_E(Tx,Ty) \preccurlyeq \theta_7 M(x,y)$$

where $M(x,y) = \max\{d_E(Tx,y), d_E(Ty,x)\}$ for all $x \in P$ and $y \in Q$. Then T has unique fixed point in $P \cap Q$.

Proof. Fix $x \in P$

$d_E(Tx,Ty) \preccurlyeq \theta_7 M(x,y)$ where $M(x,y) = \max\{d_E(Tx,y), d_E(Ty,x)\}$ for all $x \in P$ and $y \in Q$

Case (1) $M(x,y) = d_E(Tx,x)$

$$d_E(Tx,Ty) \preccurlyeq \theta_7 d_E(Tx,x)$$

$$d_E(Tx,T^2x) \preccurlyeq \theta_7 d_E(x,Tx)$$

$$d_E(T^2x,Tx) \preccurlyeq \theta_7 d_E(x,Tx)$$

$$d_E(T^3x,T^2x) \preccurlyeq \theta_7 d_E(T^2x,Tx)$$

$$d_E(T^3x,T^2x) \preccurlyeq \theta_7^2 d_E(Tx,x)$$

$$\vdots$$

$$d_E(T^{n+1}x,T^nx) \preccurlyeq \theta_7^n d_E(Tx,x)$$

Now for $n > m$, consider

$$\begin{aligned} d_E(T^mx,T^nx) &\preccurlyeq d_E(T^mx,T^{m+1}x) + d_E(T^{m+1}x,T^{m+2}x) + \cdots d_E(T^{n-1}x,T^nx) \\ &\preccurlyeq (\theta_7^m + \theta_7^{m+1} + \cdots + \theta_7^{n-1}) d_E(Tx,x) \\ &\preccurlyeq \theta_7^m \left(\frac{1-\theta_7^{n-m}}{1-\theta_7}\right) d_E(Tx,x). \end{aligned}$$

Let $e \succeq 0_E$ be given, choose $\rho > 0$ such that $e - \rho U_+ \subseteq E^+$ and there is a positive integer n_6 such that $\theta_7^m(\frac{1-\theta_7^{n-m}}{1-\theta_7}) d_E(Tx,x) \in \frac{\rho}{2}U_+$, for every $m,n \geq n_6$, therefore $e - \theta_7^m(\frac{1-\theta_7^{n-m}}{1-\theta_7}) d_E(Tx,x) - \frac{\rho}{2}U_+ \subseteq e - \rho U_+ \subseteq E^+$, hence

$$d_E(T^mx,T^nx) \preccurlyeq \theta_7^m \left(\frac{1-\theta_7^{n-m}}{1-\theta_7}\right) d_E(Tx,x) \preceq e \, \forall n,m \geq n_6.$$

Hence $\{T^nx\}$ is Cauchy sequence. Then, there exist $z \in P \cap Q$ such that $T^nx \xrightarrow{e} z$. Notice that $\{T^{2n}x\}$ is a sequence in P and $\{T^{2n-1}x\}$ is a sequence in Q having the same limit z. As P and Q are closed $z \in P \cap Q$. Therefore $P \cap Q$ is non empty.

Now

$$
\begin{aligned}
d_E(Tz,z) &= \lim_{n\to\infty} d_E(Tz,T^{2n}z) \\
&\preccurlyeq \theta_7 \lim_{n\to\infty} M(z,T^{2n-1}x) \\
&\stackrel{e}{=} \theta_7 d_E(Tz,z) \\
(1-\theta_7)d_E(Tz,z) &\preccurlyeq 0_E \\
\Rightarrow d_E(Tz,z) &\stackrel{e}{=} 0_E \\
\Rightarrow Tz &= z
\end{aligned}
$$

Therefore z is a fixed point of T in $P \cap Q$.

Uniqueness

Suppose w is an another fixed point of T in $P \cap Q$ and T is a cyclic, we get $w \in P \cap Q$

$$
\begin{aligned}
d_E(z,w) &= d_E(Tz,Tw) \\
&\preccurlyeq \theta_7 M(z,w) \\
&\preccurlyeq \theta_7 d_E(Tz,w) \\
(1-\theta_7)d_E(z,w) &\preccurlyeq 0_E \\
\Rightarrow\ d_E(z,w) &\stackrel{e}{=} 0_E \ \text{ since } \theta_7 \in (0,1) \\
\Rightarrow \quad z &= w
\end{aligned}
$$

Hence z is an unique fixed point of T.

Case (2)

$$M(x,y) = d_E(Tx,y)$$

$$d_E(Tx,Ty) \preccurlyeq \theta_7 d_E(x,y)$$

$$\text{Put } y = Tx$$

$$d_E(Tx,T^2x) \preccurlyeq \theta_7 d_E(Tx,Tx)$$

$$d_E(Tx,T^2x) = 0_E$$

$$T^2x = Tx$$

$$T(Tx) = Tx$$

$$Ty = y$$

$$Ty = Tx$$

$$x = y$$

Since $Tx = x$ and $Ty = y$

T has unique fixed point. □

Corollary 2.13. *Let P and Q be two non-empty closed subsets of $e-$complete $E-$metric space (X, d_E) with closed positive cone E^+ such that non-empty semi-interior points of E^+. Let for some positive integer n, the mapping $T : P \cup Q \to P \cup Q$ is a map satisfying $T(P) \subset Q$ and $T(Q) \subset P$ and there exists $\theta_7 \in (0,1)$ such that*

$$d_E(T^n x, T^n y) \preccurlyeq \theta_7 \max\{d_E(Tx, y), d_E(Ty, x)\}$$

for all $x \in P$ and $y \in Q$. Then T has unique fixed point in $P \cap Q$.

ETHICAL APPROVAL

This article does not contain any studies with human participants or animals performed by any of the authors.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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