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SOME PARTITION ON UNIFORM STRUCTURE OF BE-ALGEBRAS

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Abstract. In this paper, we investigate some relation on a uniform structure of BE-algebras X. Then we prove that for a filter F of X, the set $\{U_F[x] \mid x \in X\}$ is a partition of X.

Keywords: uniform structure; BE-algebras; filter; partition.

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1. INTRODUCTION

The concepts of BE-algebras was first introduced by Iseki and Tanaka [3]. In 2007, Kim and Kim [5] introduced and investigated the notion of BE-algebras as a dualization of a generalization of BCK-algebras. Ahn and So [1] introduced ideals and upper sets in BE-algebras and investigated some properties of ideals. In 2008, Walendziak [10] introduced the notion of commutative BE-algebras and discussed several properties of commutative BE-algebras. In 2009, Kim and Lee [4] generalized the notion of upper sets and introduced the concept of extended upper sets. Algebra and topology are fundamental domains of mathematics. Many of the most important objects of mathematics represent a blend of algebraic objects and topological structures. In 2017, Mehrshad and Golzarpoor [7] studied some properties of uniform topology and

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topological BE-algebras and compare these topologies. Shahdadi and Kouhestani [9] defined (left, right, semi) topological BE-algebras and showed that for each cardinal number α there is at least a topological BE-algebra of order α. Albaracin and Velela [2] studied the topology generated by the family of subsets determined by the right application of BE-ordering of a BE-algebra and investigated some of its properties. In this paper, we investigate some properties of uniform structure on BE-algebras.

2. PRELIMINARIES

Some essential notations and definitions of BE-algebras and ordinary senses in this work has been introduced in this section.

Definition 2.1. [6] Let A be a set and $\mathscr{P}$ be a collection of nonempty subset of A. Then $\mathscr{P}$ is called a partition of A if the following properties are satisfied

(i) for all $B,C \in \mathscr{P}$, either $B = C$ or $B \cap C = \emptyset$;

(ii) $A = \bigcup\limits_{B \in \mathscr{P}} B$.

Definition 2.2. [9]

A BE-algebra is a non empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms, for all $x,y,z \in X$

$(BE1)$ $x * x = 1$,

$(BE2)$ $x * 1 = 1$,

$(BE3)$ $1 * x = x$,

$(BE4)$ $x * (y * z) = y * (x * z)$.

Definition 2.3. [9] Let $(X;*;1)$ be a BE-algebra, and let F be a non-empty subset of X. Then F is said to be a filter of X if the following axioms are satisfies, for all $x,y,z \in X$

$(F1)$ $1 \in F$,

$(F2)$ $x * y \in F$ and $x \in F$ imply $y \in F$.

Example 2.4. [9] Let $X = \{1,a,b,c,d,0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X;*,1)$ is a BE-algebra and $F_1 := \{1,a,b\}$ is a filter of X, but $F_2 := \{1,a\}$ is not a filter of X , since $a*b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Definition 2.5. [6] Let $\sim$ be a binary relation on a set X. Then $\sim$ is called

(i) reflexive if for all $x \in X$, $x{\sim}x$,

(ii) symmetric if for all $x,y \in X$, $x{\sim}y$ implies $y{\sim}x$,

(iii) transitive if for all $x,y,z \in X$, $x{\sim}y$ and $y{\sim}z$ implies $x{\sim}z$.

(iv) compatible if for all $w,x,y,z \in X$, $w{\sim}x$ and $y{\sim}z$ implies $w*y{\sim}x*z$.

We said to be $\sim$ is an equivalent relation if $\sim$ is reflexive, symmetric and transitive. A compatible equivalence on X is called a congruence on X

In 2010, Yong Ho Yon [11] has been introduced a relation as follow : For $\emptyset \neq I \subseteq X$ we define the binary relation $\underset{I}{\sim}$ on X in the following way: $x \underset{I}{\sim} y$ iff $x*y \in I$ and $y*x \in I$ for all $x,y \in X$

Theorem 2.6. [11] *If I be a filter of a BE-algebra X. Then $\underset{I}{\sim}$ is a congruence relation on X.*

Definition 2.7. [6]

Let X be a set and $\mathfrak{P}$ be a collection of nonempty subsets of X. Then $\mathfrak{P}$ is called a partition of X if the following properties are satisfied :

(i) for all $A,B \in \mathfrak{P}$, either $A = B$ or $A \cap B = \emptyset$

(ii) $X = \bigcup\limits_{A \in \mathfrak{P}} A$

3. Uniform Topology on BE-Algebras

In this section , M. Mohamadhasani and M. Haveshki [8] in 2010, introduce the notion on a BE-algebra and investigates some of its properties as follow :

Definition 3.1. Let $(X;*,1)$ be a BE-algebra and $U,V \subseteq X \times X$ define

$U \circ V = \{(x,y) \in X \times X \mid (z,y) \in U \;\; (x,z) \in V \;\; z \in X\}$

$U^{-1} = \{(x,y) \in X \times X \mid (y,x) \in U\}$

$\triangle = \{(x,x) \in X \times X \mid x \in X\}$

Definition 3.2. Let $(X;*,1)$ be a BE-algebra and $\mathscr{K} \subseteq X \times X$,we said to be $(X,\mathscr{K})$ is a uniform structure if it satisfies the following axioms :

(U_1) $\triangle \subseteq U$ for all $U \in \mathscr{K}$;

(U_2) If $U \in \mathscr{K}$ then $U^{-1} \in \mathscr{K}$;

(U_3) If $U \in \mathscr{K}$ there exsite $V \in \mathscr{K}$ such that $V \circ V \subseteq U$

(U_4) If $U,V \in \mathscr{K}$ then $U \cap V \in \mathscr{K}$

(U_5) If $U \in \mathscr{K}$ and $U \subseteq V \subseteq X \times X$ then $V \in \mathscr{K}$

Theorem 3.3. *Let Λ be an arbitrary family of filters of a BE-algebra X which is closed under intersection. If $U_F = \{(x,y) \in X \times X \mid x \underset{F}{\sim} y\}$ and $\mathscr{K}^* = \{U_F \mid F \in \Lambda\}$, then $\mathscr{K}^*$ satisfies the conditions $(U_1) - (U_4)$*

Theorem 3.4. *Let $(X;*,1)$ be a BE-algebra,then $U[x]$ is an open neighborhood of x, and then the set $T = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathscr{K}, U[x] \subseteq G\}$ is a topology on X.*

Clearly $\emptyset$ and the set X belong to T ,also that T is closed under arbitrary union and finite intersection.

Definition 3.5. Let $(X,\mathscr{K})$ be a uniform structure. Then the topology $T = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathscr{K}, U[x] \subseteq G\}$ is called a uniform topology on X induced by $\mathscr{K}$

Example 3.6. Let $X = \{1,a,b,c,d\}$ and a binary operation "$*$" define as follow :

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X;*,1)$ is a *BE*-algebra.

Clearly $F = \{1,a,c\}$ is a filter of X and let $\Lambda = \{F\}$ By theorem 3.3 we have $\mathscr{K}^* = \{U_F\} = \{(x,y) \mid x \underset{F}{\sim} y\} = \{(1,1),(a,1),(1,a),(1,c),(c,1),(a,a),(a,c),(c,a),(b,b),$ $(b,d),(d,b),(c,c),(d,d)\}$

Then $(X,\mathscr{K})$ is a uniform space, where $\mathscr{K} = \{U \mid U_F \subseteq U\}$.

Open neighborhoods are: $U_F[1] = \{1,a,c\}, U_F[a] = \{1,a,c\}$, $U_F[b] = \{b,d\}$,$U_F[c] = \{1,a,c\}$, $U_F[d] = \{b,d\}$. By theoren 3.5 we get $T = \{\{1,a,c\},\{b,d\},\{1,a,b,c,d\},\emptyset\}$ and hence (X,T) is a uniform topological space.

4. PARTITION OF UNIFORM STRUCTURE ON *BE*-ALGEBRAS

By theorem 3.3 , Clearly $U_F \subseteq U$ and by definition 3.5, $U_F[x] = \{y \in X \mid (x,y) \in U_F\}$. Then we have the following theorem :

Theorem 4.1. *Let $(X;*,1)$ be a BE-algebra and F is a filter of X , then $U_F[x] = F$ for all $x \in F$*

Proof. Let $x \in F$ and $a \in U_F[x]$,we have $(x,a) \in U_F$ and hence $x*a \in F$ and $a*x \in F$, by definition 2.3 we get $a \in F$. Thus $U_F[x] \subseteq F$. Next, let $a \in F$, we have now $1,a,x \in F$ implies that $(x,a) \in U_F$ and hence $a \in U_F[x]$. Therefore $U_F[x] = F$.

□

Example 4.2. In Example 3.7, a *BE*-algebra $(X;*,1)$ with a filter $F = \{1,a,b\}$ in X.

By theorem 3.3 we have $\mathscr{K}^* = \{U_F\} = \{(x,y) \mid x \underset{F}{\sim} y\} = \{(1,1),(1,a),(a,1),(1,b),(b,1),$ $(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$ and hence ;

$$U_F[1] = \{1,a,b\} = F$$

$$U_F[a] = \{1,a,b\} = F$$

$$U_F[b] = \{1,a,b\} = F$$

$$U_F[c] = \{c,d\}$$

$$U_F[d] = \{c,d\}$$

We see that $U_F[x] = F$ for all $x \in F$.

Another example : for a filter $J = \{1,b\}$ in X we have

$$\begin{aligned}\mathscr{K}^* = \{U_J\} &= \{(x,y) \mid x \underset{J}{\sim} y\} \\ &= \{(1,1),(1,b),(b,1),(a,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}\end{aligned}$$

and hence $U_J[1] = \{1,b\} = J = U_J[b]$, but then $U_J[a] = \{a\}$ and $U_J[c] = \{c,d\} = U_J[d]$. So that $U_J[x] = J$ for all $x \in J$.

Corollary 4.3. *Let $(X;*,1)$ be a BE-algebra and F is a filter of X , then $1 \in U_F[x]$ for all $x \in F$.*

Proof. Clearly by theorem 4.1. □

Theorem 4.4. *Let $(X;*,1)$ be a BE-algebra and F is a filter of X, then $U_F[x] \cap F = \emptyset$ for all $x \in X - F$.*

Proof. let $x \in X - F$ and suppose that $U_F[x] \cap F \neq \emptyset$, there exist $a \in U_F[x]$ and $a \in F$ such that $(x,a) \in U_F$, thus $x * a \in F$ and $a * x \in F$ implies that $x \in F$ by definition 2.3 , that contradiction. Therefore $U_F[x] \cap F = \emptyset$ □

Example 4.5. In example 4.2 , a BE-algebra $X = \{1,a,b,c,d\}$ with a filter $F = \{1,a,b\}$, we have $X - F = \{c,d\}$ and thus

$$U_F[c] \cap F = \{c,d\} \cap \{1,a,b\} = \emptyset$$

$$U_F[d] \cap F = \{c,d\} \cap \{1,a,b\} = \emptyset$$

That is for $x \in X - F$ implies that $U_F[x] \cap F = \emptyset$

Theorem 4.6. *Let $(X;*,1)$ be a BE-algebra and F is a filter of X , then the set $P = \{U_F[x] \mid x \in X\}$ is a partition of X.*

Proof. Clearly $x \in U_F[x]$ for all $x \in F$, $U_F[x] \neq \emptyset$. Let $U_F[x], U_F[y] \in P$ such that $U_F[x] \neq U_F[y]$. Suppose that $U_F[x] \cap U_F[y] \neq \emptyset$, there is $a \in U_F[x]$ and $a \in U_F[y]$ that is $(x,a) \in U_F$ and $(y,a) \in U_F$ implies that $x \underset{F}{\sim} a$ and $y \underset{F}{\sim} a$. Since $\underset{F}{\sim}$ is a congruence relation , we have $x \underset{F}{\sim} y$ and $y \underset{F}{\sim} x$, hence $U_F[x] = U_F[y]$, that's contradiction. So $U_F[x] \cap U_F[y] = \emptyset$ .Clearly $x \in U_F[x]$ for all $x \in X$, thus $\bigcup_{x \in X} U_F[x] = X$. Therefore P is a partition of X.

□

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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