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A CONSTRUCTION OF BALANCED DEGREE-MAGIC GRAPHS

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Abstract. A graph G is called degree-magic if it admits a labelling of the edges by integers $1,2,\dots,|E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1+|E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper, a new construction of balanced degree-magic graphs is introduced.

Keywords: supermagic graphs; degree-magic graphs; cycle graphs.

2010 AMS Subject Classification: 05C78.

1. Introduction

The finite simple graphs and multigraphs without loops and isolated vertices are considered. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and the *size* of G. For any integers p and q, the set of all integers z satisfying $p \leq z \leq q$ is indicated by $[p,q]$.

Let a graph G and a mapping f from $E(G)$ into the set of positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into the set of positive integers defined by

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$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every} \quad v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into the set of positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G).$$

A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. A graph G is said to be *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijective mapping f from $E(G)$ into $[1, |E(G)|]$ is called a *degree-magic labelling* (or only *d-magic labelling*) of a graph G if its index-mapping f^* satisfies

$$f^*(v) = \frac{1+|E(G)|}{2} \deg(v) \quad \text{for all} \quad v \in V(G).$$

A d-magic labelling f of G is called *balanced* if for all $v \in V(G)$ it holds

$$\begin{aligned} |\{e \in E(G) \ &: \eta(v,e) = 1, f(e) \le \lfloor |E(G)|/2 \rfloor \}| \\ &= |\{e \in E(G) : \eta(v,e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor \}|. \end{aligned}$$

A graph G is said to be *degree-magic* (*balanced degree-magic*) (or only *d-magic*) when a d-magic (balanced d-magic) labelling of G exists.

The concept of magic graphs was put forward by Sedl*áč*ek [10]. Later, supermagic graphs were introduced by Stewart [11]. Besides, a new constructuion of supermagic complements of some graphs was recommended [9]. Moreover, the notion of degree-magic graphs was then suggested by Bezegov*á* and Ivan*č*o [1] as an extension of supermagic regular graphs. Recently, numerous papers are published on degree-magic and supermagic graphs, see [2, 3, 4, 5, 6, 7, 8] for more complehensive references.

Let one recall the basic properties of d-magic graphs that will be used in the next.

Theorem 1.1. [1] *Let G be a regular graph. Then G is supermagic if and only if it is d-magic.*

Theorem 1.2. [1] *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is d-magic and H_2 is balanced d-magic, then G is a d-magic graph. Moreover, if H_1 and H_2 are both balanced d-magic, then G is a balanced d-magic graph.*

2. Balanced Degree-Magic Graphs

An injective mapping f from $E(G)$ into the set of positive integers is called a *single-consecutive labelling* (SC-labelling) of a graph G if its index-mapping f^* satisfies

$$f^*(V(G)) = [a, a+|V(G)|-1] \quad \text{for some integer } a.$$

Let $f_i, i \in \{1,2\}$, be a SC-labelling of a graph G_i. The labellings f_1 and f_2 are called *complementary* if $f_1(E(G_1)) \cap f_2(E(G_2)) = \emptyset$ and $f_1(E(G_1)) \cup f_2(E(G_2)) = [1,m]$, where $m = |E(G_1)| + |E(G_2)|$. The complementary labellings f_1 and f_2 are called *balanced* if all pairs of vertices $u \in V(G_1), v \in V(G_2)$ satisfy

$$\begin{aligned} |\{e \in E(G_1) \quad &: \eta(u,e) = 1, f_1(e) \leq \lfloor m/2 \rfloor\}| \\ &+|\{e \in E(G_2) : \eta(v,e) = 1, f_2(e) \leq \lfloor m/2 \rfloor\}| \\ = |\{e \in E(G_1) \quad &: \eta(u,e) = 1, f_1(e) > \lfloor m/2 \rfloor\}| \\ &+|\{e \in E(G_2) : \eta(v,e) = 1, f_2(e) > \lfloor m/2 \rfloor\}|. \end{aligned}$$

Now, one is able to prove the following Proposition.

Proposition 2.1. *Let H_1 and H_2 be spanning subgraphs of a graph G which form its decomposition with vertices $v_1, v_2, ..., v_n$. Let f be a SC-labelling of H_1 such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and let g be a SC-labelling of H_2 such that $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. If f and g are complementary, then G is a supermagic graph.*

Proof. Since f is a SC-labelling of H_1 such that $f^*(v_i) = f^*(v_1) + (i-1)$ and g is a SC-labelling of H_2 such that $g^*(v_i) = g^*(v_1) - (i-1)$ for all $i \in [1,n]$, $f^*(v_i) + g^*(v_i) = f^*(v_1) + g^*(v_1)$. Now, consider a mapping φ from $E(G)$ into the set of positive integers defined by

$$\varphi(v_i v_j) = \begin{cases} f(v_i v_j) & : \quad v_i v_j \in E(H_1), \\ g(v_i v_j) & : \quad v_i v_j \in E(H_2). \end{cases}$$

Obviously, $\varphi^*(v_i) = f^*(v_i) + g^*(v_i) = f^*(v_1) + g^*(v_1)$. Since $\varphi(E(G)) = f(E(H_1)) \cup g(E(H_2))$ and the labellings f and g are complementary, φ is a supermagic labelling of G. Therefore, G is a desired graph. □

If the graph G in Proposition 2.1 is regular, then G is d-magic by Therorem 1.1. For balanced d-magic graphs, one can show the following assertion.

Proposition 2.2. *Let H_1 and H_2 be spanning subgraphs of a regular graph G which form its decomposition with vertices $v_1, v_2, ..., v_n$. Let f be a SC-labelling of H_1 such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and let g be a SC-labelling of H_2 such that $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. If f and g are (balanced) complementary, then G is a (balanced) d-magic graph.*

Proof. By using the same proof as Proposition 2.1, G is a supermagic graph. Because G is regular, G is d-magic by Theorem 1.1. Since f and g are balanced complementary, for each vertex $v_i, i \in [1,n]$, of G it holds

$$\begin{aligned}
|\{e \in E(G) &: \eta(v_i,e) = 1, \varphi(e) \le \lfloor |E(G)|/2 \rfloor\}| \\
&= |\{e \in E(H_1) : \eta(v_i,e) = 1, f(e) \le \lfloor |E(G)|/2 \rfloor\}| \\
&+ |\{e \in E(H_2) : \eta(v_i,e) = 1, g(e) \le \lfloor |E(G)|/2 \rfloor\}| \\
= |\{e \in E(H_1) &: \eta(v_i,e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}| \\
&+ |\{e \in E(H_2) : \eta(v_i,e) = 1, g(e) > \lfloor |E(G)|/2 \rfloor\}| \\
&= |\{e \in E(G) : \eta(v_i,e) = 1, \varphi(e) > \lfloor |E(G)|/2 \rfloor\}|.
\end{aligned}$$

Thus, φ is a balanced d-magic labelling of G. That is, G is an expected graph. □

The above two Propositions describe methods to construct supermagic graphs and d-magic graphs by using SC-labellings respectively. In order to use Proposition 2.2, one needs reasonable SC-labellings of some graphs.

Lemma 2.3. *Let G be a cycle graph of order 4 with vertices v_1, v_2, v_3, v_4 and let k, h be positive integers. Then there are a SC-labelling f of G such that $f(E(G)) = \{k, k+1, k+4, k+6\}$ and $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$ and a SC-labelling g of G such that $g(E(G)) = \{h, h+1, h+3, h+5\}$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Moreover, if $k = 1$ and $h = 3$, then the SC-labellings f and g are balanced complementary.*

Proof. Consider a mapping f from $E(G)$ into the set of positive integers given by

$$f(e) = \begin{cases} k & : \; e = v_1v_3, \\ k+6 & : \; e = v_3v_4, \\ k+1 & : \; e = v_4v_2, \\ k+4 & : \; e = v_2v_1. \end{cases}$$

It is easy to see that $f(E(G)) = \{k, k+1, k+4, k+6\}$ and $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$. Hence, f is a desired SC-labelling of G. Moreover, consider a mapping g from $E(G)$ into the set of positive integers defined by

$$g(e) = \begin{cases} h+1 & : \; e = v_1v_3, \\ h+3 & : \; e = v_3v_4, \\ h & : \; e = v_4v_2, \\ h+5 & : \; e = v_2v_1. \end{cases}$$

One can see that $g(E(G)) = \{h, h+1, h+3, h+5\}$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Thus, g is a required SC-labelling of G. Now, consider the case $k = 1$ and $h = 3$, one then has

$$f(e) = \begin{cases} 1 & : \; e = v_1v_3, \\ 7 & : \; e = v_3v_4, \\ 2 & : \; e = v_4v_2, \\ 5 & : \; e = v_2v_1, \end{cases}$$

and

$$g(e) = \begin{cases} 4 & : \; e = v_1v_3, \\ 6 & : \; e = v_3v_4, \\ 3 & : \; e = v_4v_2, \\ 8 & : \; e = v_2v_1. \end{cases}$$

Clearly, f and g are balanced complementary labellings. □

Lemma 2.4. *Let G be a cycle graph of odd order $n \geq 3$ with vertices $v_1, v_2, \ldots, v_n$ and let k, h be positive integers. Then there exist a SC-labelling f of G such that $f(E(G)) = [k, k+n-1]$ and*

$f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and a SC-labelling g of G such that $g(E(G)) = [h, h+n-1]$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. Moreover, if $k = 1$ and $h = n+1$, then the SC-labellings f and g are balanced complementary.

Proof. Consider a mapping f from $E(G)$ into the set of positive integers given by

$$f(e) = \begin{cases} k+(n-1)/2 \ : \ e = v_n v_1, \\ k \ : \ e = v_1 v_2, \\ k+(n-1)/2+1 \ : \ e = v_2 v_3, \\ k+1 \ : \ e = v_3 v_4, \\ k+(n-1)/2+2 \ : \ e = v_4 v_5, \\ k+2 \ : \ e = v_5 v_6, \\ \cdots \\ k+(n-1)/2-1 \ : \ e = v_{n-2} v_{n-1}, \\ k+n-1 \ : \ e = v_{n-1} v_n. \end{cases}$$

One is able to check that $f(E(G)) = [k, k+n-1]$ and $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$. Thus, f is a desired SC-labelling of G. Besides, consider a mapping g from $E(G)$ into the set of positive integers defined by

$$g(e) = \begin{cases} h+(n-1)/2 \ : \ e = v_1 v_n, \\ h \ : \ e = v_n v_{n-1}, \\ h+(n-1)/2+1 \ : \ e = v_{n-1} v_{n-2}, \\ h+1 \ : \ e = v_{n-2} v_{n-3}, \\ h+(n-1)/2+2 \ : \ e = v_{n-3} v_{n-4}, \\ h+2 \ : \ e = v_{n-4} v_{n-5}, \\ \cdots \\ h+(n-1)/2-1 \ : \ e = v_3 v_2, \\ h+n-1 \ : \ e = v_2 v_1. \end{cases}$$

One can get that $g(E(G)) = [h, h+n-1]$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. Hence, g is a required SC-labelling of G. Now, consider the case $k = 1$ and $h = n+1$, one then gets

$$f(e) = \begin{cases} 1+(n-1)/2 \ : \ e = v_n v_1, \\ 1 \ : \ e = v_1 v_2, \\ 2+(n-1)/2 \ : \ e = v_2 v_3, \\ 2 \ : \ e = v_3 v_4, \\ 3+(n-1)/2 \ : \ e = v_4 v_5, \\ 3 \ : \ e = v_5 v_6, \\ \cdots \\ (n-1)/2 \ : \ e = v_{n-2} v_{n-1}, \\ n \ : \ e = v_{n-1} v_n, \end{cases}$$

and

$$g(e) = \begin{cases} n+1+(n-1)/2 \ : \ e = v_1 v_n, \\ n+1 \ : \ e = v_n v_{n-1}, \\ n+2+(n-1)/2 \ : \ e = v_{n-1} v_{n-2}, \\ n+2 \ : \ e = v_{n-2} v_{n-3}, \\ n+3+(n-1)/2 \ : \ e = v_{n-3} v_{n-4}, \\ n+3 \ : \ e = v_{n-4} v_{n-5}, \\ \cdots \\ n+(n-1)/2 \ : \ e = v_3 v_2, \\ 2n \ : \ e = v_2 v_1. \end{cases}$$

Evidently, f and g are balanced complementary labellings. □

In the next results, one is able to prove some sufficient conditions for balanced d-magic graphs.

Theorem 2.5. *Let G be a graph which can be decomposable into two spanning cycle subgraphs of order* 4*. Then G is a balanced d-magic graph.*

Proof. Suppose that two spanning cycle subgraphs of G have vertices v_1, v_2, v_3, v_4. Thus, by Lemma 2.3, there are two balanced complementary SC-labellings f, g of these cycles such that $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Since these two cycles are regular and form its decomposition, G is a regular graph. Therefore, according to Proposition 2.2, G is a balanced d-magic graph. □

Combining Theorem 1.2 and Theorem 2.5, one immediately has

Corollary 2.6. *For any positive integer k, if a graph G can be decomposable into $2k$ spanning cycle subgraphs of order 4, then G is a balanced d-magic graph.*

Joining Theorem 1.1 and Corollary 2.6, one absolutely has

Corollary 2.7. *For any positive integer k, if a graph G can be decomposable into $2k$ spanning cycle subgraphs of order 4, then G is a supermagic graph.*

Theorem 2.8. *Let G be a graph which can be decomposable into two spanning cycle subgraphs of odd order $n \geq 3$. Then G is a balanced d-magic graph.*

Proof. Assume that two spanning cycle subgraphs of G of odd order $n \geq 3$ have vertices $v_1, v_2, ..., v_n$. Hence by Lemma 2.4, there are two balanced complementary SC-labellings f, g of these cycles such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. It is clear that these two cycles are regular and they form its decomposition, so G is a regular graph. Therefore, according to Proposition 2.2, G is a balanced d-magic graph. □

Combining Theorem 1.2 and Theorem 2.8, one suddenly has

Corollary 2.9. *For any positive integer k, if a graph G can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then G is a balanced d-magic graph.*

Joining Theorem 1.1 and Corollary 2.9, one certainly has

Corollary 2.10. *For any positive integer k, if a graph G can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then G is a supermagic graph.*

Notice that there exist SC-labellings f and g of a cycle graph of order 8 with vertices $v_1, v_2, ..., v_8$ such that $f(E(G)) = [k, k+3] \cup \{k+8, k+9, k+11, k+12\}$ and $g(E(G)) = [h, h+3] \cup \{h+6, h+9, h+10, h+11\}$ for any positive integers h, k. Moreover, if $k = 1$ and

$h=5$, then the SC-labellings f and g are balanced complementary. These SC-labellings f and g are shown as follows.

$$f(e)=\begin{cases} k & : \quad e=v_1v_4, \\ k+11 & : \quad e=v_4v_6, \\ k+2 & : \quad e=v_6v_7, \\ k+12 & : \quad e=v_7v_8, \\ k+3 & : \quad e=v_8v_5, \\ k+9 & : \quad e=v_5v_3, \\ k+1 & : \quad e=v_3v_2, \\ k+8 & : \quad e=v_2v_1, \end{cases}$$

and

$$g(e)=\begin{cases} h+11 & : \quad e=v_1v_4, \\ h & : \quad e=v_4v_6, \\ h+9 & : \quad e=v_6v_7, \\ h+1 & : \quad e=v_7v_8, \\ h+6 & : \quad e=v_8v_5, \\ h+2 & : \quad e=v_5v_3, \\ h+10 & : \quad e=v_3v_2, \\ h+3 & : \quad e=v_2v_1. \end{cases}$$

One can prove that $f^*(v_1)<f^*(v_2)<\cdots<f^*(v_8)$ while $g^*(v_1)>g^*(v_2)>g^*(v_3)>g^*(v_4)>g^*(v_7)>g^*(v_6)>g^*(v_5)>g^*(v_8)$. Furthermore, consider the case $k=1$ and $h=5$, one then obtains

$$f(e) = \begin{cases} 1 & : \; e = v_1v_4, \\ 12 & : \; e = v_4v_6, \\ 3 & : \; e = v_6v_7, \\ 13 & : \; e = v_7v_8, \\ 4 & : \; e = v_8v_5, \\ 10 & : \; e = v_5v_3, \\ 2 & : \; e = v_3v_2, \\ 9 & : \; e = v_2v_1, \end{cases}$$

and

$$g(e) = \begin{cases} 16 & : \; e = v_1v_4, \\ 5 & : \; e = v_4v_6, \\ 14 & : \; e = v_6v_7, \\ 6 & : \; e = v_7v_8, \\ 11 & : \; e = v_8v_5, \\ 7 & : \; e = v_5v_3, \\ 15 & : \; e = v_3v_2, \\ 8 & : \; e = v_2v_1. \end{cases}$$

Obviously, f and g are balanced complementary labellings. However, by the method of Proposition 2.2, one can not construct a balanced d-magic graph by using two balanced complementary labellings of a cycle subgraph of order 8 upwardly because the condition does not hold.

For the last result, two balanced complementary of SC-labellings of some cycle graphs and their associated balanced d-magic graphs are presented as follows.

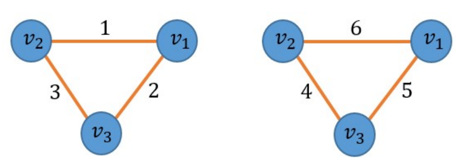

FIGURE 1. Two balanced complementary SC-labellings of a cycle graph C_3.

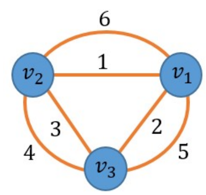

FIGURE 2. A balanced d-magic graph constructed by two spanning cycle subgraphs C_3.

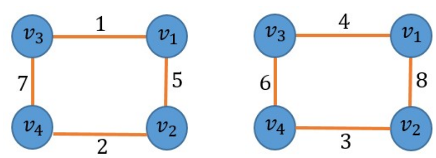

FIGURE 3. Two balanced complementary SC-labellings of a cycle graph C_4.

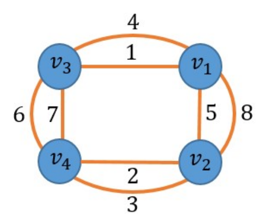

FIGURE 4. A balanced d-magic graph constructed by two spanning cycle subgraphs C_4.

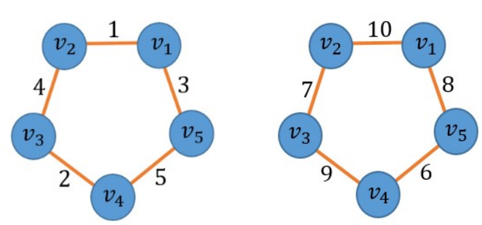

FIGURE 5. Two balanced complementary SC-labellings of a cycle graph C_5.

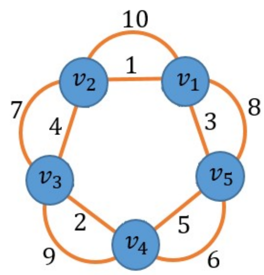

FIGURE 6. A balanced d-magic graph constructed by two spanning cycle subgraphs C_5.

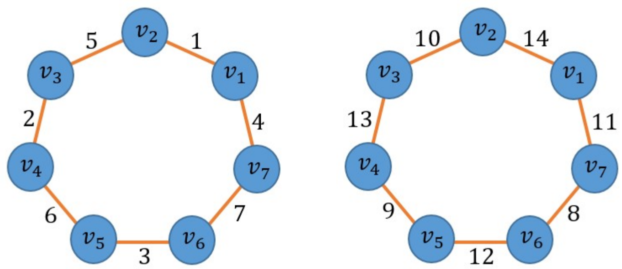

FIGURE 7. Two balanced complementary SC-labellings of a cycle graph C_7.

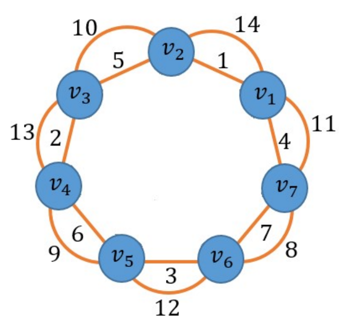

FIGURE 8. A balanced d-magic graph constructed by two spanning cycle subgraphs C_7.

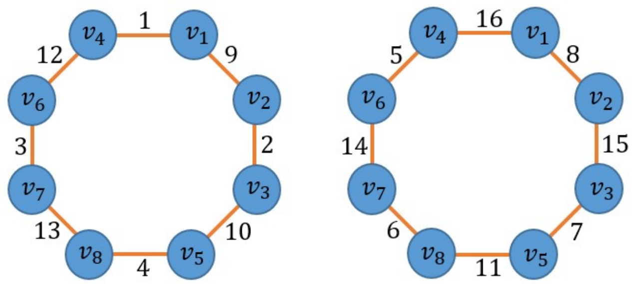

FIGURE 9. Two balanced complementary SC-labellings of a cycle graph C_8.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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