Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 8288-8305
https://doi.org/10.28919/jmcs/6781
ISSN: 1927-5307

HANKEL AND SLANT HANKEL OPERATORS ON THE BERGMAN SPACE OF THE POLYDISK

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Abstract. In this paper, for $n \geq 1$, we initiate the notion of Hankel operators on the polydisk $\mathbb{D}^n$ and slant Hankel operators on $L^2(\mathbb{T}^n)$ where $\mathbb{T}^n$ denotes the n-torus. We give the necessary and sufficient condition for a bounded operator on $L^2(\mathbb{T}^n)$ to be a slant Hankel operator and study some algebraic properties of slant Hankel operators. Also, we extend our study on the Bergman space.

Keywords: Hankel operator of level n; slant Hankel operator of level n; bounded function; multiplication operator; Bergman space.

2010 AMS Subject Classification: Primary 47B35; Secondary 30H20, 47B38.

1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\mathbb{T}$ denotes the unit circle in the complex plane $\mathbb{C}$. For $n \geq 1$, let $\mathbb{D}^n$ and $\mathbb{T}^n$ be respectively the polydisk in $\mathbb{C}^n$ and n-torus having a distinct boundary of $\mathbb{D}^n$. Let $\varepsilon_j = (y_1, \dots, y_n)$ where $y_i = \delta_{ij}$, $j = 1, 2, \dots, n$. Throughout this paper, z denotes the vector $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, $r = (r_1, r_2, \dots, r_n) \in \mathbb{Z}^n$, $z^r = z_1^{r_1} \cdots z_n^{r_n}$ and $|r| = r_1 + \cdots + r_n$. For $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, m is even if each m_i is even for $i = 1, 2, 3, \dots, n$. Otherwise, m is said to be odd. Also, for $z \in \mathbb{Z}^n$, $\bar{z} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n = z_1^{-1} z_2^{-1} \cdots z_n^{-1} = z^{-1}$. Let $d\sigma$ be the Haar

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Received September 14, 2021

measure on $\mathbb{T}^n$. The space $L^2(\mathbb{T}^n)$ is given by

$$L^2(\mathbb{T}^n) = \left\{ g : \mathbb{T}^n \mapsto \mathbb{C} \mid g(z) = \sum_{r\in\mathbb{Z}^n} f_r z^r, \sum_{r\in\mathbb{Z}^n} | f_r |^2 < \infty \right\}.$$

The inner product for any two functions f, g in $L^2(\mathbb{T}^n)$ is given by $\langle g, h\rangle = \int_{\mathbb{T}^n} g(z)\overline{h(z)}d\sigma(z)$. If $e_m(z) = z^m$ for $m \in \mathbb{Z}^n$, then $\{e_m\}_{m\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$ and $\mathscr{R} = \{e_m\}_{m\in\mathbb{Z}^n_+}$ is an orthonormal basis for $H^2(\mathbb{D}^n)$. The Hardy space $H^2(\mathbb{D}^n)$ consists of all the analytic functions $g \in \mathbb{D}^n$ such that $\sup_{0<a<1} \int_{\mathbb{T}^n} |g(az)|^2 d\sigma(z) < \infty$. Let $\mathscr{R}_{[m_j,\ldots,m_n]} = \{e_{(m_1,\ldots,m_n)} : m_i \in \mathbb{Z}^+, 1 \leq i < j\}$ for $j \in [2,n] \cap \mathbb{Z}_+$ and arbitrarily fixed $m_j, m_{j+1}, \ldots, m_n \in \mathbb{Z}^+$. So, $\mathscr{R}_{[m_j,\ldots,m_n]}$ is an orthonormal basis for $H^2(\mathbb{D}^{j-1})$. For $1 < i \leq j$ and $m_i \in \mathbb{Z}^+$ with $j \leq i \leq n$, $\mathscr{G}_{j-1,(m_j,\ldots,m_n)}$ is used to denote the space $H^2(\mathbb{D}^{j-1})$ to convey that the basis being considering is $\mathscr{R}_{[m_j,\ldots,m_n]}$. Any function $g \in H^2(\mathbb{D}^n)$, the radial limit $\lim_{a\to 1^-} g(az)$ exists for almost every $z \in \mathbb{T}^n$ [12]. The notion of Toeplitz operators was initiated by Toeplitz [16] in the year 1911. Ho [5, 6] defined the slant Toeplitz operators on $L^2(\mathbb{T})$ as those operators whose matrix representation for an orthonormal basis can be obtained by eliminating every other row of a doubly infinite Toeplitz matrix. Hankel operators are the formal companions of Toeplitz operators that have occurred in realization problems for certain discrete-time linear systems and in determining which systems are exactly controllable [11]. The study of Hankel and slant Hankel operators has various applications in Hamburger's moment problem, rational approximation theory, interpolation problems, and stationary process. In 2006, Arora et al. [1] developed the notion of slant Hankel Operators which is motivated by the matricial definition of slant Toeplitz operators on $L^2(\mathbb{T})$ as given by Ho [5, 6]. Recently, Hazarika and Marik [4] initiated the idea of Toeplitz and slant Toeplitz operators in the polydisk and discussed several properties. For relevant results on Hankel, slant Hankel operators, Toeplitz, slant Toeplitz operators, and the concept of polydisk we refer the readers to [2, 3, 7, 10, 12, 13, 14, 18].

Motivated by the works of Ho [5, 6] we introduce Hankel operators of level n on $H^2(\mathbb{D}^n)$ and slant Hankel operators of level n on $L^2(\mathbb{T}^n)$ for $n \geq 1$. We show in this paper that a bounded linear operator $\mathscr{S}$ on $L^2(\mathbb{T}^n)$ is a slant Hankel operator of level n iff $\mathscr{S}$ can be expressed as slant Hankel matrix of level n and discuss the properties of $\mathscr{V}$ and $\mathscr{V}^*$. In the later part, we give the notion of slant Hankel operators of level n on the Bergman space of the polydisk.

Definition 1.1. [4] For $\phi \in L^\infty(\mathbb{T}^n)$, the Laurent operator M_ϕ is defined as $M_\phi f = \phi f \ \forall f \in L^2(\mathbb{T}^n)$.

Remark 1.2. [4]

(i) For $\phi \in L^\infty(\mathbb{T}^n)$, $M_\phi^* = M_{\overline{\phi}}$.

(ii) If $\phi(z) = z_i$ for $1 \leqslant i \leqslant n$, $i \in \mathbb{Z}$, then $M_\phi f = z_i f \ \forall f \in L^2(\mathbb{T}^n)$.

(iii) $M_{z_i}^* = M_{\overline{z}_i}$ and $M_{z_i}^* e_m = e_{m-\varepsilon_j} \ \forall m \in \mathbb{Z}^n$ and $i = 1, 2, 3, \ldots, n$.

(iv) $M_{z_i} e_m = e_{m+\varepsilon_j}$ for $1 \leq i \leq n$ and $m \in \mathbb{Z}^n$.

(v) $M_{z_i} M_{z_i}^* = I = M_{z_i}^* M_{z_i} \ \forall 1 \leq i \leq n$.

Definition 1.3. **Flip operator :** We define the flip operator $J : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ as an operator $(Jf)e_m(z) = (Jf)(z^m) = f(z^{-m})$. It is very easy to show that a flip operator is a self-adjoint operator.

2. Hankel Operator of Level n

Let ϕ be a linear operator on $L^\infty(\mathbb{T}^n)$ and P be the projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. The Hankel operator of level n, $\mathscr{H}_\phi$ on $H^2(\mathbb{D}^n)$ is defined as $\mathscr{H}_\phi = PJM_\phi$ where J is the flip operator on $L^2(\mathbb{T}^n)$.

2.1. Hankel Matrix of level n.

Let $\{u_m\}_{m \in \mathbb{Z}_+^n}$ be a sequence of scalars. A matrix of the form

$$\mathscr{H}^{(1)}_{m_2,\ldots,m_n} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & u_{(0,m_2,\ldots,m_n)} & u_{(-1,m_2,\ldots,m_n)} & u_{(-2,m_2,\ldots,m_n)} & \cdots \\ \cdots & u_{(-1,m_2,\ldots,m_n)} & u_{(-2,m_2,\ldots,m_n)} & u_{(-3,m_2,\ldots,m_n)} & \cdots \\ \cdots & u_{(-2,m_2,\ldots,m_n)} & u_{(-3,m_2,\ldots,m_n)} & u_{(-4,m_2,\ldots,m_n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called Hankel matrix of level 1. A block matrix of the form

$$\mathscr{H}^{(2)}_{m_3,\ldots,m_n} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \mathscr{H}^{(1)}_{0,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-1,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-2,m_3,\ldots,m_n} & \cdots \\ \cdots & \mathscr{H}^{(1)}_{-1,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-2,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-3,m_3,\ldots,m_n} & \cdots \\ \cdots & \mathscr{H}^{(1)}_{-2,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-3,m_3,\ldots,m_n} & \mathscr{H}^{(1)}_{-4,m_3,\ldots,m_n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called Hankel matrix of level 2. While proceeding in this manner, a block matrix

$$\mathscr{H}^{(n)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & \mathscr{H}_0^{(n-1)} & \mathscr{H}_{-1}^{(n-1)} & \mathscr{H}_{-2}^{(n-1)} & \dots \\ \dots & \mathscr{H}_{-1}^{(n-1)} & \mathscr{H}_{-2}^{(n-1)} & \mathscr{H}_{-3}^{(n-1)} & \dots \\ \dots & \mathscr{H}_{-2}^{(n-1)} & \mathscr{H}_{-3}^{(n-1)} & \mathscr{H}_{-4}^{(n-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called Hankel matrix of level n.

Theorem 2.1. *If τ is a bounded linear operator on $H^2(\mathbb{D}^n)$, then τ is a Hankel matrix of level n iff $\langle \tau e_{m-\varepsilon_j}, e_{m'+\varepsilon_j} \rangle = \langle \tau e_m, e_{m'} \rangle \; \forall\, m, m' \in \mathbb{Z}_+^n$ and $1 \leq j \leq n$.*

Proof. Suppose $\langle \tau e_{m-\varepsilon_j}, e_{m'+\varepsilon_j} \rangle = \langle \tau e_m, e_{m'} \rangle \forall\, m, m' \in \mathbb{Z}_+^n$ and $1 \leq j \leq n$.
Let $\{u_{\eta,\zeta}\}_{\eta,\zeta \in \mathbb{Z}_+^n}$ be scalars such that $\tau e_\zeta = \sum_{\eta \in \mathbb{Z}_+^n} u_{\eta,\zeta}\, e_\eta \quad \forall\; \zeta \in \mathbb{Z}_+^n$ and $m = (m_1, m_2, \dots, m_n)$ and $m' = (m'_1, m'_2, \dots, m'_n) \in \mathbb{Z}_+^n$. Fixing $(m_2, \dots, m_n)$, $(m'_2, \dots, m'_n)$ and varying m_1, m'_1

$$\begin{aligned} &\left\langle \tau e_{m-\varepsilon_1}(z), e_{m'+\varepsilon_1}(z) \right\rangle = \left\langle \tau e_m(z), e_{m'}(z) \right\rangle \\ \Rightarrow &\left\langle \sum_{\eta \in \mathbb{Z}_+^n} u_{\eta,m-\varepsilon_1}\, e_\eta(z), e_{m'+\varepsilon_1}(z) \right\rangle = \left\langle \sum_{\eta \in \mathbb{Z}_+^n} u_{\eta,m}\, e_\eta(z), e_{m'}(z) \right\rangle \\ \Rightarrow &\sum_{\eta \in \mathbb{Z}_+^n} u_{\eta,m-\varepsilon_1} \langle e_\eta(z), e_{m'+\varepsilon_1}(z) \rangle = \sum_{\eta \in \mathbb{Z}_+^n} u_{\eta,m} \langle e_\eta(z), e_{m'}(z) \rangle \\ \Rightarrow &\, u_{m'+\varepsilon_1, m-\varepsilon_1} = u_{m',m} \quad \forall m, m' \in \mathbb{Z}_+. \end{aligned}$$

This implies that $\tau : \mathscr{G}_{1,[m_2,\dots,m_n]} \mapsto \mathscr{G}_{1,[m'_2,\dots,m'_n]}$ can be expressed as Hankel matrix of level 1, $\mathscr{H}^{(1)}_{(m'_2,\dots,m'_n)(m_2,\dots,m_n)}$. Again, if we vary m_2, m'_2 and fixing $(m_3, \dots, m_n)$, $(m'_3, \dots, m'_n)$ then

$$\begin{aligned} &\left\langle \tau e_{m-\varepsilon_2}, e_{m'+\varepsilon_2} \right\rangle = \left\langle \tau e_m, e'_m \right\rangle \\ \Rightarrow &\mathscr{H}^{(1)}_{(m'_2+1, m'_3,\dots,m'_n)(m_2-1, m_3,\dots,m_n)} = \mathscr{H}^{(1)}_{(m'_2,\dots,m'_n)(m_2,\dots,m_n)}. \end{aligned}$$

So, $\tau : \mathscr{G}_{2,[m_3,\ldots,m_n]} \mapsto \mathscr{G}_{2,[m'_3,\ldots,m'_n]}$ can be expressed as Hankel matrix of level 2, $\mathscr{H}^{(2)}_{(m'_3,\ldots,m'_n)(m_3,\ldots,m_n)}$. Further, if we vary m_3, m'_3 and fixing $(m_4,\ldots,m_n)$, $(m'_4,\ldots,m'_n)$ then

$$\left\langle \tau e_{m-\varepsilon_3}, e_{m'+\varepsilon_3} \right\rangle = \left\langle \tau e_m, e_{m'} \right\rangle$$
$$\Rightarrow \mathscr{H}^{(2)}_{(m_3+1,m_4,\ldots,m_n)(m'_3-1,m'_4,\ldots,m'_n)} = \mathscr{H}^{(2)}_{(m'_3,\ldots,m'_n)(m_3,\ldots,m_n)}.$$

Hence, $\tau : \mathscr{G}_{3,[m_4,\ldots,m_n]} \mapsto \mathscr{G}_{3,[m'_4,\ldots,m'_n]}$ can be expressed as Hankel matrix of level 3, $\mathscr{H}^{(3)}_{(m'_4,\ldots,m'_n)(m_4,\ldots,m_n)}$. If we continue in this manner, we can conclude that $\tau : L^2(\mathbb{T}^n) \mapsto H^2(\mathbb{D}^n)$ can be expressed as Hankel matrix of level n after n steps. For converse part, suppose τ can be expressed as Hankel matrix of level n. So, for arbitrary $m = (m_1,\ldots,m_n)$ and $m' = (m'_1,\ldots,m'_n)$, $\tau : \mathscr{G}_{j-1,[m_j,\ldots,m_n]} \mapsto \mathscr{G}_{j,[m'_j,\ldots,m'_n]}$ can be represented as a Hankel matrix of level $(j-1)$, $\mathscr{H}^{(j-1)}_{(m'_j,\ldots,m'_n)(m_j,\ldots,m_n)}$ for $j = 1,2,\ldots,n-1$. Now,

$$\begin{aligned}
\left\langle \tau e_{m-\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle &= \left\langle \sum_{\eta\in\mathbb{Z}^n_+} u_{\eta,m-\varepsilon_j} e_\eta(z), e_{m'+\varepsilon_j}(z) \right\rangle \\
&= \sum_{\eta\in\mathbb{Z}^n_+} u_{\eta,m-\varepsilon_j} \langle e_\eta(z), e_{m'+\varepsilon_j}(z) \rangle \\
&= u_{m'+\varepsilon_j,m-\varepsilon_j} \quad [\because e_\eta(z)'s \text{ are orthonormal basis}] \\
&= u_{m',m} \ \ \forall\, j = 1,2,\ldots,n-1. \\
&= \left\langle \tau e_m, e_{m'} \right\rangle.
\end{aligned}$$

Finally, we have to show $\left\langle \tau e_{m-\varepsilon_n}, e_{m'+\varepsilon_n} \right\rangle = \left\langle \tau e_m, e_{m'} \right\rangle$. Let's consider the $n-1$ disk $H^2(\mathbb{D}^{n-1})$. $H^2(\mathbb{D}^{n-1})$ is an isomorphic copy of $\mathscr{G}_{n-1,(m_n)}$ for each $m_n \in \mathbb{Z}_+$ and hence the n disk $H^2(\mathbb{D}^n)$ can be decomposed as $H^2(\mathbb{D}^n) = \oplus_{m_n\in\mathbb{Z}} \mathscr{G}_{n-1,(m_n)}$. So, $\tau : L^2(\mathbb{T}^n) \mapsto H^2(\mathbb{D}^n)$ can be represented as a Hankel matrix where the $(m'_n, m_n)^{th}$ entry is $\mathscr{H}^{(n-1)}_{m'_n,m_n}$ and also we have $\mathscr{H}^{(n-1)}_{m'_n+1,m_n-1} = \mathscr{H}^{(n-1)}_{m'_n,m_n}$. Hence, $\langle \tau e_{m-\varepsilon_n}, e_{m'+\varepsilon_n} \rangle = \langle \tau e_m, e_{m'} \rangle$. □

Theorem 2.2. *An operator τ on $H^2(\mathbb{D}^n)$ is a Hankel operator of level n if it can be expressed as a Hankel matrix of level n.*

Proof. Let $\tau = \mathscr{H}_\phi = PJM_\phi$, $\phi \in L^\infty(\mathbb{T}^n)$. Now, for $m, m' \in \mathbb{Z}_+^n$ and $1 \le j \le n$. We have,

$$
\begin{aligned}
\left\langle \tau e_{m-\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle &= \left\langle \tau e_{m-\varepsilon_j}(z), e_{m'+\varepsilon_j}(z) \right\rangle \\
&= \left\langle PJM_\phi e_{m-\varepsilon_j}(z), e_{m'+\varepsilon_j}(z) \right\rangle \\
&= \left\langle \phi(z) e_{m-\varepsilon_j}(z), J e_{m'+\varepsilon_j}(z) \right\rangle \\
&= \left\langle \sum_{k \in \mathbb{Z}^n} u_k z^k z^{m-\varepsilon_j}, z^{-m'-\varepsilon_j} \right\rangle \\
&= \sum_{k \in \mathbb{Z}^n} u_k \left\langle z^{k+m-\varepsilon_j}, z^{-m'-\varepsilon_j} \right\rangle \\
&= \sum_{k \in \mathbb{Z}^n} u_k \int_{\mathbb{T}^n} z^{k+m-\varepsilon_j} z^{m'+\varepsilon_j} d\sigma(z) \\
&= \sum_{k \in \mathbb{Z}^n} u_k \int_{\mathbb{T}^n} z^{k+m+m'} d\sigma(z). \qquad (2.1)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left\langle \tau e_m(z), e_{m'}(z) \right\rangle &= \left\langle PJM_\phi e_m(z), e_{m'}(z) \right\rangle \\
&= \left\langle \phi(z) e_m(z), J e_{m'}(z) \right\rangle \\
&= \left\langle \sum_{k \in \mathbb{Z}^n} u_k z^k z^m, z^{-m'} \right\rangle \\
&= \sum_{k \in \mathbb{Z}^n} u_k \left\langle z^{k+m}, z^{-m'} \right\rangle \\
&= \sum_{k \in \mathbb{Z}^n} u_k \int_{\mathbb{T}^n} z^{k+m+m'} d\sigma(z). \qquad (2.2)
\end{aligned}
$$

Hence, from (2.1) and (2.2) $\left\langle \tau e_{m-\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle = \left\langle \tau e_m(z), e_{m'}(z) \right\rangle$.
The proof then follows from Theorem 2.1. $\square$

Theorem 2.3. *For every ϕ in $L^\infty(\mathbb{T}^n)$, $\mathscr{H}_\phi^* = \mathscr{H}_{\phi^*}$.*

Proof. Let f and g be any two functions in $H^2(\mathbb{D}^n)$. Then,

$$\begin{aligned}
\langle \mathscr{H}_\phi^* f, g\rangle &= \langle f, \mathscr{H}_\phi g\rangle = \langle f, PJM_\phi g\rangle = \langle f, \mathscr{H}_\phi g\rangle = \langle M_\phi^* Jf, g\rangle = \langle M_{\overline{\phi}} Jf, g\rangle \\
&= \langle JM_{\phi^*} f, g\rangle \qquad \left(\because M_{\overline{\phi}} J = JM_{\phi^*}\right) \\
&= \langle PJM_{\phi^*} f, g\rangle \\
&= \langle \mathscr{H}_{\phi^*} f, g\rangle \qquad \forall f \in H^2(\mathbb{D}^n).
\end{aligned}$$

Hence, $\mathscr{H}_\phi^* = \mathscr{H}_{\phi^*}$ □

3. Slant Hankel Operator of Level n

Let $\{u_m\}_{m\in\mathbb{Z}^n}$ be a sequence of scalars. A matrix of the expression

$$\mathscr{S}^{(1)}_{m_2,\ldots,m_n} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & u_{(0,m_2,\ldots,m_n)} & u_{(-1,m_2,\ldots,m_n)} & u_{(-2,m_2,\ldots,m_n)} & \cdots \\ \cdots & u_{(-2,m_2,\ldots,m_n)} & u_{(-3,m_2,\ldots,m_n)} & u_{(-4,m_2,\ldots,m_n)} & \cdots \\ \cdots & u_{(-4,m_2,\ldots,m_n)} & u_{(-5,m_2,\ldots,m_n)} & u_{(-6,m_2,\ldots,m_n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called a slant Hankel matrix of level 1. A block matrix of the form

$$\mathscr{S}^{(2)}_{m_3,\ldots,m_n} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \mathscr{S}^{(1)}_{0,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-1,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-2,m_3,\ldots,m_n} & \cdots \\ \cdots & \mathscr{S}^{(1)}_{-2,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-3,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-4,m_3,\ldots,m_n} & \cdots \\ \cdots & \mathscr{S}^{(1)}_{-4,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-5,m_3,\ldots,m_n} & \mathscr{S}^{(1)}_{-6,m_3,\ldots,m_n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called a slant Hankel matrix of level 2. Continuing in this way, we get the slant Hankel matrix of level n as

$$\mathscr{S}^{(n)} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \mathscr{S}^{(n-1)}_{0} & \mathscr{S}^{(n-1)}_{-1} & \mathscr{S}^{(n-1)}_{-2} & \cdots \\ \cdots & \mathscr{S}^{(n-1)}_{-2} & \mathscr{S}^{(n-1)}_{-3} & \mathscr{S}^{(n-1)}_{-4} & \cdots \\ \cdots & \mathscr{S}^{(n-1)}_{-4} & \mathscr{S}^{(n-1)}_{-5} & \mathscr{S}^{(n-1)}_{-6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.1. *A bounded linear operator $\mathscr{S}$ on $L^2(\mathbb{T}^n)$ is a slant Hankel matrix of level n iff* $\left\langle \mathscr{S} e_{m-2\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle = \left\langle \mathscr{S} e_m, e_{m'} \right\rangle \ \forall \ m, m' \in \mathbb{Z}^n$.

Proof. The proof is trivial and it follows directly from Theorem 2.1. □

Theorem 3.2. *A bounded linear operator $\mathscr{S}$ on $L^2(\mathbb{T}^n)$ is slant Hankel operator of level n iff* $M_{z_j}\mathscr{S} = \mathscr{S} M_{z_j^{-2}} \quad \forall\, j = 1, 2, \ldots, n$.

Proof. We have,

$$\begin{aligned}
& M_{z_j}\mathscr{S} = \mathscr{S} M_{z_j^{-2}} \quad \forall\, j = 1, 2, \ldots, n \\
\Leftrightarrow\ & \left\langle M_{z_j}\mathscr{S} e_m, e_{m'} \right\rangle = \left\langle \mathscr{S} M_{z_j^{-2}}\, e_m, e_{m'} \right\rangle \quad \forall \ m, m' \in \mathbb{Z}^n \\
\Leftrightarrow\ & \left\langle \mathscr{S}\, e_m, M^*_{z_j}\, e_{m'} \right\rangle = \left\langle \mathscr{S} M_{z_j^{-2}}\, e_m, e_{m'} \right\rangle \ \forall \ m, m' \in \mathbb{Z}^n \\
\Leftrightarrow\ & \left\langle \mathscr{S}\, e_m, e_{m'-\varepsilon_j} \right\rangle = \left\langle \mathscr{S}\, e_{m-2\varepsilon_j}, e_{m'} \right\rangle \ \forall \ m, m' \in \mathbb{Z}^n \\
\Leftrightarrow\ & \left\langle \mathscr{S}\, e_m, e_{m'} \right\rangle = \left\langle \mathscr{S}\, e_{m-2\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle \ \forall \ m, m' \in \mathbb{Z}^n.
\end{aligned}$$

The proof now follows from Theorem 3.1. □

Definition 3.3. Let $\mathscr{V}$ be a linear operator defined on $L^2(\mathbb{T}^n)$, then for $m \in \mathbb{Z}^n$

$$\mathscr{V}\, e_m = \begin{cases} e_{\frac{-m}{2}} & \text{if } m \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $h(z) = \sum_{k\in\mathbb{Z}^n} u_k z^k \in L^2(\mathbb{T}^n)$, then $\mathscr{V} h(z) = \sum_{k\in\mathbb{Z}^n} u_{2k} z^{-k}$. Further, the adjoint of $\mathscr{V}$ is given by $\mathscr{V}^*\, e_m = e_{-2m}$ for each $m \in \mathbb{Z}^n$. Therefore, $\mathscr{V}^* h(z) = \sum_{k\in\mathbb{Z}^n} u_k z^{-2k} = f(z^{-2})$ for $h(z) = \sum_{k\in\mathbb{Z}^n} u_k z^k \in L^2(\mathbb{T}^n)$. For $j \in \mathbb{Z}^n$, let P_e be the projection on the closed span of $\left\{e_{2m} : m \in \mathbb{Z}^n\right\}$ in $L^2(\mathbb{T}^n)$. Then,

$$P_e\, e_j = \begin{cases} e_j & \text{if } j \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Also, for $m, m' \in \mathbb{Z}^n$,

$$\left\langle \mathscr{V} e_m, e_{m'} \right\rangle = \begin{cases} 1 & \text{if } 2m' = -m; \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Hence, $\|\mathscr{V}\| = 1$.

Definition 3.4. A slant Hankel operator of level n is defined as $\mathscr{S}_\phi = \mathscr{V} M_\phi$ where $\phi \in L^\infty(\mathbb{T}^n)$.

Theorem 3.5. $\|\mathscr{S}_\phi\| \leq \|\phi\|_\infty$ *for* $\phi \in L^\infty(\mathbb{T}^n)$.

Theorem 3.6. *A slant Hankel operator of level n,* $\mathscr{S}_\phi$ *on* $L^\infty(\mathbb{T}^n)$ *has the property*

$$\left\langle \mathscr{S}_\phi e_{m-2\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle = \left\langle \mathscr{S}_\phi e_m, e_{m'} \right\rangle \quad \forall\, m, m' \in \mathbb{Z}^n, 1 \leq j \leq n.$$

Proof. We have,

$$\begin{aligned} LHS &= \left\langle \mathscr{S}_\phi e_{m-2\varepsilon_j}(z), e_{m'+\varepsilon_j}(z) \right\rangle \\ &= \left\langle \mathscr{V} M_\phi e_{m-2\varepsilon_j}(z), e_{m'+\varepsilon_j}(z) \right\rangle \\ &= \left\langle \phi(z) z^{m-2\varepsilon_j}, \mathscr{V}^*(z^{m'+\varepsilon_j}) \right\rangle \\ &= \left\langle \sum_{k \in \mathbb{Z}^n} u_k z^k z^{m-2\varepsilon_j}, z^{-2m'-2\varepsilon_j} \right\rangle \\ &= \sum_{k \in \mathbb{Z}^n} u_k \left\langle z^{k+m-2\varepsilon_j}, z^{-2m'-2\varepsilon_j} \right\rangle \\ &= \sum_{k \in \mathbb{Z}^n} u_k \int_{\mathbb{T}^n} z^{k+m+2m'} d\sigma(z) \\ &= \sum_{k \in \mathbb{Z}^n} u_k \left\langle z^{k+m}, z^{-2m'} \right\rangle \\ &= \left\langle \phi(z) z^m, \mathscr{V}^*(z^{m'}) \right\rangle \\ &= \left\langle \mathscr{V} M_\phi e_m(z), e_{m'}(z) \right\rangle \\ &= \left\langle \mathscr{S}_\phi e_m(z), e_{m'}(z) \right\rangle = RHS. \end{aligned} \tag{3.3}$$

Hence, the theorem is proved. □

Theorem 3.7. *A bounded linear operator $\mathscr{S}$ on $L^2(\mathbb{T}^n)$ is a slant Hankel operator of level n iff $\mathscr{S}$ can be represented as a slant Hankel matrix of level n.*

Proof. Let $\mathscr{S}$ be a slant Hankel operator of level n on $L^2(\mathbb{T}^n)$. Then, by Definition 3.4, $\mathscr{S} = \mathscr{S}_\phi$ for some $\phi \in L^\infty(\mathbb{T}^n)$. If $(\beta_{m,m'})_{m,m' \in \mathbb{Z}^n}$ is the matrix representation of $\mathscr{S}_\phi$ with respect to the given orthonormal basis, then by Theorem 3.6

$$\left\langle \mathscr{S}_\phi e_{m-2\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle = \left\langle \mathscr{S}_\phi e_m, e_{m'} \right\rangle \quad \forall\, m, m' \in \mathbb{Z}^n, 1 \le j \le n.$$

Therefore, $(\beta_{m,m'})_{m,m' \in \mathbb{Z}^n}$ is a slant Hankel matrix of level n.

Conversely, let the matrix $(\beta_{m,m'})_{m,m' \in \mathbb{Z}^n}$ of $\mathscr{S}$ be a slant Hankel matrix of level n.Then,

$$\begin{aligned} \left\langle \mathscr{S} e_m, e_{m'} \right\rangle &= (\beta_{m,m'}) \\ &= (\beta_{m'+\varepsilon_j, m-2\varepsilon_j}) \\ &= \left\langle \mathscr{S} e_{m-2\varepsilon_j}, e_{m'+\varepsilon_j} \right\rangle. \end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned} \left\langle M_{z_j} \mathscr{S} e_m, e_{m'} \right\rangle &= \left\langle \mathscr{S} e_m, M_{\bar{z}_j} e_{m'} \right\rangle \\ &= \left\langle \mathscr{S} e_m, e_{m'-\varepsilon_j} \right\rangle \\ &= \left\langle \mathscr{S} e_{m-2\varepsilon_j}, e_{m'} \right\rangle \\ &= \left\langle \mathscr{S} M_{z_j^{-2}} e_m, e_{m'} \right\rangle \end{aligned} \tag{3.5}$$

$$\Rightarrow M_{z_j} \mathscr{S} e_m = \mathscr{S} M_{z_j^{-2}} e_m \quad \forall\; m \in \mathbb{Z}^n$$

$$\Rightarrow M_{z_j} \mathscr{S} = \mathscr{S} M_{z_j^{-2}}. \tag{3.6}$$

Hence, by Theorem 3.2 $\mathscr{S}$ is a slant Hankel operator. □

4. PROPERTIES OF $\mathscr{V}$ AND $\mathscr{V}^*$

Lemma 4.1. [4] *Let $S = \{(l_1, \ldots, l_n) \in \mathbb{Z}^n :$ each l_i is either 0 or $1\}$. Then for $m \in \mathbb{Z}^n$, m odd, there exists unique $p \in \mathbb{Z}^n$ and $l \neq 0 \in S$ such that $m = 2p + l$.*

Lemma 4.2. [4] *For $m, l \in S$ with $m \neq 0, l \neq 0$, we have $m + l$ is even iff $m = l$.*

Theorem 4.3. *For all $m \in \mathbb{Z}^n$, $\mathscr{V} M_{z^{-2m}} \mathscr{V}^* = M_{z^m}$ and $\mathscr{V} M_{z^{-l}} \mathscr{V}^* = 0$ if $l \in \mathbb{Z}^n$ is odd.*

Theorem 4.4. *Let $\xi, \zeta \in L^2(\mathbb{T}^n)$ such that $\xi\zeta \in L^2(\mathbb{T}^n)$. Then,*

(a) $\mathscr{V}^*(\xi\zeta) = \mathscr{V}^*(\xi)\mathscr{V}^*(\zeta)$.

(b) $\mathscr{V}(\xi\zeta) = \mathscr{V}(\xi)\mathscr{V}(\zeta)$.

(c) $\mathscr{V}[\mathscr{V}^*(\xi)\mathscr{V}^*(\zeta)] = \xi\,\zeta$.

Lemma 4.5. *Let $\xi(z) = \sum_{m \in \mathbb{Z}^n} u_m z^{-m}$, $\xi \in L^2(\mathbb{T}^n)$. If $\xi_l(z) = \sum_{m \in \mathbb{Z}^n} u_{2m+l}\, z^{-m}$ for $l \in S$, then $\xi(z) = \sum_{l \in S} z^{-l}\xi_l(z^2)$. Also, $\xi_0(z^2) = P_e\xi(z)$ and $\xi_o(\bar{z}) = \mathscr{V}\xi(z)$.*

Proof. Let $\xi(z) = \sum_{m \in \mathbb{Z}^n} u_m z^{-m}$. If $\zeta(z) = \sum_{\substack{m \in \mathbb{Z}^n \\ m \text{ is even}}} u_m z^{-m}$ and $\eta(z) = \sum_{\substack{m \in \mathbb{Z}^n \\ m \text{ is odd}}} u_m z^{-m}$.
Then, $\xi(z) = \zeta(z) + \eta(z)$. For $l \in S$, we define $\xi_l(z) = \sum_{m \in \mathbb{Z}^n} u_{2m+l}\, z^{-m}$.
As $\xi \in L^2(\mathbb{T}^n)$, so $\xi_l \in L^2(\mathbb{T}^n)\ \ \forall\ l\ \in S$. We have,

$$\zeta(z) = \sum_{\substack{m \in \mathbb{Z}^n \\ m \text{ is even}}} u_m z^{-m} = \sum_{m \in \mathbb{Z}^n} u_{2m} z^{-2m} = \xi_0(z^2).$$

$$\begin{aligned}
\eta(z) = \sum_{\substack{m \in \mathbb{Z}^n \\ m \text{ is odd}}} u_m z^{-m} &= \sum_{0 \neq l \in S} \sum_{m \in \mathbb{Z}^n} u_{2m+l} z^{-(2m+l)} \\
&= \sum_{0 \neq l \in S} z^{-l} \sum_{m \in \mathbb{Z}^n} u_{2m+l} (z^2)^{-m} \\
&= \sum_{0 \neq l \in S} z^{-l} \xi_l(z^2).
\end{aligned}$$

Thus, $\xi(z) = \xi_0(z^2) + \sum_{0 \neq l \in S} z^{-l}\xi_l(z^2) = \sum_{l \in S} z^{-l}\xi_l(z^2)$. Now, by Definition 3.3

$$\mathscr{V}\zeta(z) = \mathscr{V} \sum_{\substack{m \in \mathbb{Z}^n \\ m \text{ is even}}} u_m z^{-m} = \sum_{m \in \mathbb{Z}^n} u_{2m} z^m = \xi_0(\bar{z}), \quad \mathscr{V}\eta(z) = 0$$

$$\Rightarrow \xi_o(\bar{z}) = \mathscr{V}\xi(z) \text{ and}$$

$P_e\,\zeta(z) = \zeta(z)$ by (3.1). Also, $P_e\,\eta(z) = 0$.

Therefore, $P_e\,\xi(z) = P_e\,\zeta(z) = \xi_0(z^2)$.

$\square$

Theorem 4.6. *If one of ζ and η is in $L^\infty(\mathbb{T}^n)$ for $\zeta,\eta \in L^2(\mathbb{T}^n)$, then $\mathscr{V}(\zeta,\eta) = (\mathscr{V}\zeta)(\mathscr{V}\eta) + \sum_{\substack{l\in S\\ l\neq 0}} z^l(\mathscr{V} z^l\zeta)(\mathscr{V} z^l\eta)$.*

Proof. We have, $\zeta(z) = \zeta_0(z^2) + \sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)$ [by Lemma 4.5] and
$\eta(z) = \eta_0(z^2) + \sum_{0\neq m\in S} z^{-m}\eta_m(z^2)$ [by Lemma 4.5]
where $\zeta_0(z^2) = P_e\zeta(z)$ and $\eta_0(z^2) = P_e\,\eta(z)$. So,

$$\begin{aligned}\zeta\eta = \zeta_0(z^2)\eta_0(z^2) + \zeta_0(z^2)\sum_{0\neq m\in S} z^{-m}\eta_m(z^2) + \eta_0(z^2)\sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)\\ + \sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)\sum_{0\neq m\in S} z^{-m}\eta_m(z^2).\end{aligned} \tag{4.1}$$

Now,

$$\begin{aligned}\mathscr{V}\Big[\big\{\zeta_0(z^2)\big\}\ \big\{\eta_0(z^2)\big\}\Big] &= \mathscr{V}\Big[\big\{\mathscr{V}^*\zeta_0(\bar{z})\big\}\ \big\{\mathscr{V}^*\eta_0(\bar{z})\big\}\Big] \quad \text{[by Definition 3.3]}\\ &= \zeta_0(\bar{z})\ \ \eta_0(\bar{z}) \quad \text{[by Theorem 4.4(b)]}\\ &= \big\{\mathscr{V}\zeta(z)\big\}\ \big\{\mathscr{V}\eta(z)\big\} \quad \text{[by Lemma 4.5]}.\end{aligned} \tag{4.2}$$

Since, $\zeta_0(z^2)\sum_{0\neq m\in S} z^{-m}\eta_m(z^2)$ and $\eta_0(z^2)\sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)$ are the expressions which involves only the powers of z that are odd. Therefore, by Definition 3.3

$$\mathscr{V}\Big[\zeta_0(z^2)\sum_{0\neq m\in S} z^{-m}\eta_m(z^2)\Big] = 0 \ \text{ and } \ \mathscr{V}\Big[\eta_0(z^2)\sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)\Big] = 0. \tag{4.3}$$

Also,

$$\sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)\sum_{0\neq m\in S} z^{-m}\eta_m(z^2) = \sum_{0\neq l\in S}\sum_{0\neq m\in S} z^{-(l+m)}\zeta_l(z^2)\eta_m(z^2).$$

If $l+m$ is odd then the above expression is 0. But by Lemma 4.2, $l+m$ is even iff $l=m$. Thus, if $l+m$ is even then so is $-(l+m)$.

Therefore,

$$\mathscr{V}\left[\left\{\sum_{0\neq l\in S} z^{-l}\zeta_l(z^2)\right\}\left\{\sum_{0\neq l\in S} z^{-l}\eta_l(z^2)\right\}\right]$$

$$= \sum_{0\neq l\in S} \mathscr{V}(z^{-2l})\mathscr{V}\{\zeta_l(z^2)\}\mathscr{V}\{\eta_l(z^2)\}$$

$$= \sum_{0\neq l\in S} z^l\mathscr{V}\{\zeta_l(z^2)\}\mathscr{V}\{\eta_l(z^2)\}. \tag{4.4}$$

As, $\zeta(z) = \sum_{m\in S} z^{-m}\zeta_m(z^2)$, so for any $0\neq l\in S$, we have

$$z^{-l}\zeta_l(z^2) = \zeta(z) - \sum_{\substack{m\in S\\ m\neq l}} z^{-m}\zeta_m(z^2)$$

$$\Rightarrow \zeta_l(z^2) = z^l\zeta(z) - \sum_{\substack{m\in S\\ m\neq l}} z^{-m+l}\zeta_m(z^2).$$

Hence,

$$\mathscr{V}\{\zeta_l(z^2)\} = \mathscr{V}\{z^l\zeta(z)\}. \tag{4.5}$$

Similarly,

$$\mathscr{V}\{\eta_l(z^2)\} = \mathscr{V}\{z^l\eta(z)\}. \tag{4.6}$$

Combining equations (4.1) - (4.6), we get

$$\mathscr{V}(\zeta,\eta) = (\mathscr{V}\zeta)(\mathscr{V}\eta) + \sum_{\substack{l\in S\\ l\neq 0}} z^l(\mathscr{V}z^l\zeta)(\mathscr{V}z^l\eta).$$

□

5. Slant Hankel Operator of Level n on the Bergman Space

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let v be the Lebesgue volume measure on $\mathbb{D}^n$, normalized so that $v(\mathbb{D}^n) = 1$. For $0 < p < \infty$, the Bergman space $A^p(\mathbb{D}^n)$ consists of analytic functions f in $L^p(\mathbb{D}^n)$. For simplicity, we consider the case of $p = 2$, which is a Banach space of analytic functions in $L^2(\mathbb{D}^n)$. Let ϕ_μ be the linear fractional transformation on $\mathbb{D}^n$ given by $\phi_\mu(w) = \phi_{\mu_1}(w_1)\ldots\phi_{\mu_n}(w_n)$ where $\phi_\mu(w) = \frac{\mu-w}{1-\bar{\mu}w}$, μ and $w\in\mathbb{D}^n$. The reproducing kernel in

$A^2(\mathbb{D}^n)$ is given by $K_w(z) = \prod_{j=1}^{n} \frac{1}{(1-\overline{w}_j z_j)^2}$ and $f(z) = \langle f, K_z \rangle$ for all $f \in A^2(\mathbb{D}^n)$, z, $w \in \mathbb{D}^n$. The orthogonal projection of P of $A^2(\mathbb{D}^n)$ onto $L^2(\mathbb{D}^n)$ is given by

$$(Pg)(w) = \langle g, K_w \rangle = \int_{\mathbb{D}^n} g(z) \prod_{j=1}^{n} \frac{1}{(1-\overline{w}_j z_j)^2} dv(z)$$

for $g \in L^2(\mathbb{D}^n)$ and $w \in \mathbb{D}^n$. For a function $f \in L^\infty(\mathbb{D}^n)$ and $g \in A^2(\mathbb{D}^n)$, the Hankel operator of level n is defined as

$$\mathscr{H}_f(g) = (I-P)(fg) = \int_{\mathbb{D}^n} \big[f(w) - f(z)\big] h(z) \prod_{j=1}^{n} \frac{1}{(1-\overline{w}_j z_j)^2} dv(z).$$

The operator W on $A^2(\mathbb{D}^n)$ is defined by

$$W\ e_m = \begin{cases} e_{\frac{m}{2}} & if \quad m \text{ is even;} \\ 0 & \text{otherwise.} \end{cases}$$

If $\phi \in L^\infty(\mathbb{D}^n)$ and $\mathscr{H}_\phi$ is the Hankel operator of level n on $A^2(\mathbb{D}^n)$ then the slant Hankel operator of level n on $A^2(\mathbb{D}^n)$ is defined as $\mathscr{S}_\phi = W \mathscr{H}_\phi$. For related results on Bergman space we refer the readers to [8, 9, 15, 17]

Lemma 5.1. *W is a bounded linear operator, and $\|W\|_2 = \sqrt{2}$.*

Theorem 5.2. *The operator W^* on $A^2(\mathbb{D}^n)$ has the property $W^*(e_m) = \frac{2m+\varepsilon_j}{m+\varepsilon_j} e_{2m}$ where $m \in \mathbb{Z}_n^+$ and $\|f\|_2 \leq \|W^* f\|_2 \leq 2\|f\|_2$.*

Proof. Let $f(z) = \sum_{m' \in \mathbb{Z}_+^n} a_{m'} z^{m'} \in A^2(\mathbb{D}^n)$. Then,

$$\begin{aligned} \big\langle W^* e_m(z), f(z) \big\rangle &= \Big\langle z^m, W\Big(\sum_{m' \in \mathbb{Z}_+^n} a_{m'} z^{m'}\Big)\Big\rangle \\ &= \Big\langle z^m, \sum_{m' \in \mathbb{Z}_+^n} a_{2m'} z^{m'} \Big\rangle \\ &= \overline{a}_{2m} \big\langle z^m, z^m \big\rangle \\ &= \frac{\overline{a}_{2m}}{m+\varepsilon_j}. \end{aligned}$$

Also,

$$\begin{aligned}\left\langle \frac{2m+\varepsilon_j}{m+\varepsilon_j} e_{2m}(z), f(z) \right\rangle &= \frac{2m+\varepsilon_j}{m+\varepsilon_j} \left\langle z^{2m}, \sum_{m' \in \mathbb{Z}_+^n} a_{m'} z^{m'} \right\rangle \\ &= \frac{2m+\varepsilon_j}{m+\varepsilon_j} \overline{a}_{2m} \langle z^{2m}, z^{2m} \rangle \\ &= \frac{\overline{a}_{2m}}{m+\varepsilon_j}.\end{aligned}$$

Hence, $W^*(e_m) = \frac{2m+\varepsilon_j}{m+\varepsilon_j} e_{2m}$. Now,

$$\begin{aligned}\|W^* f\|_2^2 &= \Big\|W^* \sum_{m \in \mathbb{Z}_+^n} a_m z^m \Big\|_2^2 \\ &= \| \sum_{m \in \mathbb{Z}_+^n} a_m \frac{2m+\varepsilon_j}{m+\varepsilon_j} z^{2m} \|_2^2 \\ &= \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{|2m+\varepsilon_j|}{|m+\varepsilon_j|^2} \\ &\geq \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{1}{|m+\varepsilon_j|} \\ &= \|f\|_2^2. \qquad (5.1)\end{aligned}$$

$$\begin{aligned}\|W^* f(z)\|_2^2 &= \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \left\langle \frac{2m+\varepsilon_j}{m+\varepsilon_j} z^{2m}, \frac{2m+\varepsilon_j}{m+\varepsilon_j} z^{2m} \right\rangle \\ &= \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \left| \frac{2m+\varepsilon_j}{m+\varepsilon_j} \right|^2 \langle z^{2m}, z^{2m} \rangle \\ &= \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{|2m+\varepsilon_j|}{|m+\varepsilon_j|^2} \\ &= 4 \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{|2m+\varepsilon_j|}{|2m+2\varepsilon_j|^2} \\ &\leq 4 \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{1}{|2m+\varepsilon_j|} \\ &\leq 4 \sum_{m \in \mathbb{Z}_+^n} |a_m|^2 \frac{1}{|m+\varepsilon_j|} \\ &= 4\|f\|_2^2. \qquad (5.2)\end{aligned}$$

Combining equations (5.1) and (5.2), we get $\|f\|_2 \leq \|W^* f\|_2 \leq 2\,\|f\|_2$. □

Definition 5.3. The Berezin transform of a bounded linear operator $\mathscr{B}$ on $A^2(\mathbb{D}^n)$ is defined to be the function $\widetilde{\mathscr{B}}$ defined on $\mathbb{D}^n$ by $\widetilde{\mathscr{B}}(w) = \langle \mathscr{B}k_w, k_w\rangle$, where k_w is the normalized reproducing kernel for $w \in \mathbb{D}^n$. The kernel property of K_w implies $\|K_w\|_2^2 = \langle K_w, K_w\rangle = K_w(w) = \prod_{j=1}^{n} \frac{1}{1-|w_j|^2}$. Thus, $k_w(z) = \prod_{j=1}^{n} \frac{1-|w_j|^2}{(1-\overline{w}_j z_j)^2} = \prod_{j=1}^{n} 1-|w_j|^2 \sum_{m=o}^{\infty}(m+1)\overline{w}_j^m z_j^m$ for $z, w \in \mathbb{D}^n$. Therefore, $\widetilde{\mathscr{B}}(w) = \prod_{j=1}^{n}(1-|w_j|^2)\langle \mathscr{B}K_w, K_w\rangle \quad \forall w \in \mathbb{D}^n$.

Let $J_z^2(\varsigma) = \sum_{n=0}^{\infty}(2n+1)\overline{z}^n\varsigma^{2n} = \frac{1+\overline{z}\varsigma^2}{(1-\overline{z}\varsigma^2)^2} \quad \forall z, \varsigma \in \mathbb{D}^n$. It is clear that $J_z^2(\varsigma) \in H^\infty(\mathbb{D}^n)$.

Theorem 5.4. *$Wk_z(\varsigma) = \prod_{j=1}^{n}\big(1-|z_j|^2\big)J^2_{\overline{\varsigma}_j}(\overline{z}_j)$ and $\widetilde{W}k_z(\varsigma) = \prod_{j=1}^{n}\big(1-|z_j|^2\big)^2 J^2_{\overline{z}_j}(\overline{z}_j)\,\forall z, \varsigma \in \mathbb{D}^n$.*

Proof. We have,

$$\begin{aligned}
Wk_z(\varsigma) &= W\left(\prod_{j=1}^{n}(1-|z_j|^2)\sum_{m=0}^{\infty}(m+1)\overline{z}_j^m\varsigma_j^m\right)\\
&= \prod_{j=1}^{n}(1-|z_j|^2)\sum_{m=0}^{\infty}(2m+1)\overline{z}_j^{2m}\varsigma_j^m\\
&= \prod_{j=1}^{n}\big(1-|z_j|^2\big)J^2_{\overline{\varsigma}_j}(\overline{z}_j).
\end{aligned}$$

Also,

$$\begin{aligned}
\widetilde{W}k_z(\varsigma) &= \langle Wk_z(\varsigma), k_z(\varsigma)\rangle\\
&= \left\langle \prod_{j=1}^{n}\big(1-|z_j|^2\big)J^2_{\overline{\varsigma}_j}(\overline{z}_j), \prod_{j=1}^{n}(1-|z_j|^2)\sum_{m=0}^{\infty}(m+1)\overline{z}_j^m\varsigma_j^m\right\rangle\\
&= \prod_{j=1}^{n}(1-|z_j|^2)^2\left\langle J^2_{\overline{\varsigma}_j}(\overline{z}_j), \sum_{m=0}^{\infty}(m+1)\overline{z}_j^m\varsigma_j^m\right\rangle\\
&= \prod_{j=1}^{n}(1-|z_j|^2)^2\sum_{m=0}^{\infty}(2m+1)\overline{z}_j^{2m}\big\langle \varsigma_j^m, (m+1)\overline{z}_j^m\varsigma_j^m\big\rangle\\
&= \prod_{j=1}^{n}(1-|z_j|^2)^2\sum_{m=0}^{\infty}(2m+1)\overline{z}_j^{2m}z_j^m\\
&= \prod_{j=1}^{n}(1-|z_j|^2)^2 J^2_{\overline{z}_j}(\overline{z}_j).
\end{aligned}$$

$\square$

Theorem 5.5. *Let $\phi \in L^\infty(\mathbb{D}^n)$ be a bounded linear operator on $A^2(\mathbb{D}^n)$, then the slant Hankel operator of level n, $\mathscr{S}_\phi$ has the property $\|\mathscr{S}_\phi\| \leq \sqrt{2}\|\phi\|_\infty$.*

Proof. We have, $\|\mathscr{S}_\phi\| = \|W\mathscr{H}_\phi\| \leq \|W\|\|\mathscr{H}_\phi\| \leq \sqrt{2}\|\phi\|_\infty$. □

Lemma 5.6. [9] *Let f and g be in $L^2(\mathbb{D}^n)$, if there is a positive constant ε such that $\sup_{w\in\mathbb{D}^n} \|f \circ \phi_w - P(f \circ \phi_w)\|_{2+\varepsilon}\|g \circ \phi_w - P(g \circ \phi_w)\|_{2+\varepsilon} < \infty$, then the product $\mathscr{H}_f\mathscr{H}_g^*$ is bounded.*

Theorem 5.7. *Let f and g be in $L^2(\mathbb{D})^n$, if there is a positive constant ε such that $\sup_{w\in\mathbb{D}^n} \|f \circ \phi_w - P(f \circ \phi_w)\|_{2+\varepsilon}\|g \circ \phi_w - P(g \circ \phi_w)\|_{2+\varepsilon} < \infty$, then the product $\mathscr{S}_f\mathscr{S}_g^*$ is bounded.*

Proof. We have,

$$\begin{aligned}
\|\mathscr{S}_f\mathscr{S}_g^*\| &= \|W\mathscr{H}_f(W\mathscr{H}_g)^*\| = \|W\mathscr{H}_f\mathscr{H}_g^*W^*\| \\
&\leq \|W\|\|\mathscr{H}_f\mathscr{H}_g^*\|\|W^*\| \\
&< \infty \quad \left[\text{by Lemma 5.1, Lemma 5.6 and Theorem 5.2}\right]
\end{aligned}$$

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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