Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 8146-8172
https://doi.org/10.28919/jmcs/6783
ISSN: 1927-5307

G-ATOMIC SUBMODULES FOR OPERATORS IN HILBERT C^*-MODULES

FAKHR-DINE NHARI[1], MOHAMED ROSSAFI[2,*]

[1]Laboratory Analysis, Geometry and Applications Department of Mathematics, Faculty Of Sciences, University of Ibn Tofail, Kenitra, Morocco

[2]LaSMA Laboratory Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, B. P. 1796 Fes Atlas, Morocco

Abstract. In this paper, we introduce the notion of $g-$atomic submodule for an adjointable operator and resolution of the identity operator on Hilbert C^*-modules, also we give some properties. Finally, we study the concept of frame operator for a pair of $g-$fusion Bessel sequences.

Keywords: g-fusion frame; K-g-fusion frame; C^*-algebra; Hilbert C^*-modules.

2010 AMS Subject Classification: Primary 41A58; Secondary 42C15.

1. INTRODUCTION

Basis is one of the most important concepts in Vector Spaces study. However, Frames generalise orthonormal bases and were introduced by Duffin and Schaefer [6] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [9] for signal processing. In 2000, Frank-larson [8] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values

*Corresponding author
E-mail address: rossafimohamed@gmail.com
Received September 14, 2021

in the C^*-algebras [12]. Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [11, 13, 14, 15, 16, 17, 18].

The paper is organized as follows, we continue this introductory section we briefly recall the definitions and basic properties of C^*-algebra and Hilbert C^*-modules. In section 2, we introduce the concept of $g-$fusion frame and $K-g-$fusion frame. In section 3, we introduce the concept of resolution of the identity operator on Hilbert C^*-modules and gives some properties. In section 4, we introduce the concept of $g-$atomic submodule for an adjointable operator, also prove some results. Finally in section 5 we study the concept of frame operator for a pair of $g-$fusion bessel sequences.

Throughout this paper, H is considered to be a countably generated Hilbert $\mathscr{A}-$module. Let $\{H_i\}_{i\in I}$ are the collection of Hilbert $\mathscr{A}-$module and $\{W_i\}_{i\in I}$ is a collection of closed orthogonally complemented submodules of H, where I be finite or countable index set. $End^*_{\mathscr{A}}(H,H_i)$ is the set of all adjointable operator from H to H_i. In particular $End^*_{\mathscr{A}}(H)$ denote the set of all adjointable operators on H. P_{W_i} denote the orthogonal projection onto the closed submodule orthogonally complemented W_i of H. Define the module

$$l^2(\{H_i\}_{i\in I}) = \{\{x_i\}_{i\in I} : x_i \in H_i, \|\sum_{i\in I}\langle x_i, x_i\rangle\| < \infty\}$$

with $\mathscr{A}-$valued inner product $\langle x,y\rangle = \sum_{i\in I}\langle x_i,y_i\rangle$, where $x=\{x_i\}_{i\in I}$ and $y=\{y_i\}_{i\in I}$, clearly $l^2(\{H_i\}_{i\in I})$ is a Hilbert $\mathscr{A}-$module.

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathscr{A}$-modules. Our reference for C^*-algebras is [5, 4]. For a C^*-algebra $\mathscr{A}$ if $a\in\mathscr{A}$ is positive we write $a\geq 0$ and $\mathscr{A}^+$ denotes the set of positive elements of $\mathscr{A}$.

Definition 1.1. [4]. If $\mathscr{A}$ is a Banach algebra, an involution is a map $a\to a^*$ of $\mathscr{A}$ into itself such that for all a and b in $\mathscr{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$.

(2) $(ab)^* = b^*a^*$.

(3) $(\alpha a+b)^* = \bar{\alpha}a^* + b^*$.

Definition 1.2. [4]. A C^*-algebra $\mathscr{A}$ is a Banach algebra with involution such that :

$$\|a^*a\| = \|a\|^2$$

for every a in $\mathscr{A}$.

Example 1.3. $\mathscr{B} = B(H)$ the algebra of bounded operators on a Hilbert space, is a C^*-algebra, where for each operator A, A^* is the adjoint of A.

Definition 1.4. [10]. Let $\mathscr{A}$ be a unital C^*-algebra and H be a left $\mathscr{A}$-module, such that the linear structures of $\mathscr{A}$ and U are compatible. H is a pre-Hilbert $\mathscr{A}$-module if H is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle : H \times H \to \mathscr{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x,x \rangle \geq 0$ for all $x \in H$ and $\langle x,x \rangle = 0$ if and only if $x = 0$.

(ii) $\langle ax+y,z \rangle = a\langle x,z \rangle + \langle y,z \rangle$ for all $a \in \mathscr{A}$ and $x,y,z \in H$.

(iii) $\langle x,y \rangle = \langle y,x \rangle^*$ for all $x,y \in H$.

For $x \in H$, we define $||x|| = ||\langle x,x \rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in C^*-algebra $\mathscr{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathscr{A}$-valued norm on H is defined by $|x| = \langle x,x \rangle^{\frac{1}{2}}$ for $x \in H$.

Lemma 1.5. [2]. *Let H and K two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(H,K)$. Then the following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in K$.*

(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x \rangle \geq m' \langle x,x \rangle$ for all $x \in K$.*

Lemma 1.6. [1]. *Let U and H two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(U,H)$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.7. [2] *Let H be a Hilbert $\mathscr{A}$-module over a C^*-algebra $\mathscr{A}$, and $T \in End^*_{\mathscr{A}}(H)$ such that $T^* = T$. The following statements are equivalent:*

(i) *T is surjective.*

(ii) *There are $m, M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$, for all $x \in H$.*

(iii) *There are $m', M' > 0$ such that $m'\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq M'\langle x, x\rangle$ for all $x \in H$.*

Lemma 1.8. [7] *Let $\mathscr{A}$ be a C^*-algebra, U, H and L be Hilbert $\mathscr{A}-$modules. Let $T \in End^*_{\mathscr{A}}(U, L)$ and $T' \in End^*_{\mathscr{A}}(H, L)$ be such that $\overline{\mathscr{R}(T^*)}$ is orthogonally complemented. Then the following statements are equivalent:*

(1) *$T'(T')^* \leq \mu TT^*$ for some $\mu > 0$;*

(2) *There exists $\mu > 0$ such that $||(T')^*z|| \leq \mu||T^*z||$, for any $z \in L$;*

(3) *There exists a solution $X \in End^*_{\mathscr{A}}(H, U)$ of the so-called Douglas equation $T' = TX$;*

(3) *$\mathscr{R}(T') \subseteq \mathscr{R}(T)$.*

2. $K-g-$Fusion Frame in Hilbert C^*-Modules

Definition 2.1. Let $\{W_i\}_{i\in I}$ be a sequence of closed orthogonally complemented submodules of H, $\{v_i\}_{i\in I}$ be a familly of positive weights in $\mathscr{A}$, i.e., each v_i is a positive invertible element from the center of the C^*-algebra $\mathscr{A}$ and $\Lambda_i \in End^*_{\mathscr{A}}(H, H_i)$ for all $i \in I$. We say that $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if and only if there exists two constants $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H. \tag{2.1}$$

The constants A and B are called the lower and upper bounds of $g-$fusion frame, respectively. If $A = B$ then Λ is called tight g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval $g-$fusion frame. If Λ satisfies the inequality

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

then it is called a $g-$fusion bessel sequence with bound B in H.

Lemma 2.2. *let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion bessel sequence for H with bound B. Then for each sequence $\{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$, the series $\sum_{i\in I} v_i P_{W_i}\Lambda_i^* x_i$ is converge unconditionally.*

Proof. let J be a finite subset of I, then

$$\begin{aligned} ||\sum_{i\in J} v_i P_{W_i}\Lambda_i^* x_i|| &= \sup_{||y||=1} ||\langle \sum_{i\in J} v_i P_{W_i}\Lambda_i^* x_i, y\rangle|| \\ &\leq ||\sum_{i\in J}\langle x_i, x_i\rangle||^{\frac{1}{2}} \sup_{||y||=1} ||\sum_{i\in J} v_i^2 \langle \Lambda_i P_{W_i} y, \Lambda_i P_{W_i} y\rangle||^{\frac{1}{2}} \\ &\leq \sqrt{B}||\sum_{i\in J}\langle x_i, x_i\rangle||^{\frac{1}{2}}. \end{aligned}$$

And it follows that $\sum_{j\in I} v_j P_{W_j}\Lambda_j^* f_j$ is unconditionally convergent in H. □

Now, we can define the synthesis operator by lemma 2.2

Definition 2.3. let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion bessel sequence for H. Then the operator $T_\Lambda : l^2(\{H_i\}_{i\in I}) \to H$ defined by

$$T_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I} v_i P_{W_i}\Lambda_i^* x_i, \qquad \forall \{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I}).$$

Is called synthesis operator. We say the adjoint T_Λ^* of the synthesis operator the analysis operator and it is defined by $T_\Lambda^* : \mathscr{H} \to l^2(\{H_i\}_{i\in I})$ such that

$$T_\Lambda^*(x) = \{v_i \Lambda_i P_{W_i}(x)\}_{i\in I}, \qquad \forall x \in H.$$

The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda x = T_\Lambda T_\Lambda^* x = \sum_{j\in I} v_i^2 P_{W_i}\Lambda_i^* \Lambda_i P_{W_i}(x), \qquad \forall x \in H.$$

Is called $g-$fusion frame operator. It can be easily verify that

$$\langle S_\Lambda x, x\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}(x), \Lambda_i P_{W_i}(x)\rangle, \qquad \forall x \in H. \tag{2.2}$$

Furthermore, if Λ is a $g-$fusion frame with bounds A and B, then

$$A\langle x, x\rangle \leq \langle S_\Lambda x, x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

It easy to see that the operator S_Λ is bounded, self-adjoint, positive, now we proof the inversibility of S_Λ. Let $x \in H$ we have

$$||T_\Lambda^*(x)|| = ||\{v_i \Lambda_i P_{W_i}(x)\}_{i\in I}|| = ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}(x), \Lambda_i P_{W_i}(x)\rangle||^{\frac{1}{2}}.$$

Since Λ is $g-$fusion frame then

$$\sqrt{A}||\langle x,x\rangle||^{\frac{1}{2}} \leq ||T_\Lambda^* x||.$$

Then

$$\sqrt{A}||x|| \leq ||T_\Lambda^* x||.$$

Frome lemma 1.5, T_Λ is surjective and by lemma 1.6, $T_\Lambda T_\Lambda^* = S_\Lambda$ is invertible. We now, $AI_H \leq S_\Lambda \leq BI_H$ and this gives $B^{-1}I_H \leq S_\Lambda^{-1} \leq A^{-1}I_H$

Theorem 2.4. *Let H be a Hilbert $\mathscr{A}-$module over C^*-algebra. Then $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if and only if there exist two constants $0 < A \leq B < \infty$ such that for all $x \in H$*

$$A||\langle x,x\rangle|| \leq ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle|| \leq B||\langle x,x\rangle||.$$

Proof. Suppose Λ is $g-$fusion frame for H, since there is $\langle x,x\rangle \geq 0$ then for all $x \in H$,

$$A||\langle x,x\rangle|| \leq ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle|| \leq B||\langle x,x\rangle||$$

Conversely, for each $x \in H$ we have

$$\begin{aligned} ||\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle|| &= ||\sum_{i\in I} \langle v_i \Lambda_i P_{W_i} x, v_i \Lambda_i P_{W_i} x\rangle|| \\ &= ||\langle \{v_i \Lambda_i P_{W_i} x\}_{i\in I}, \{v_i \Lambda_i P_{W_i} x\}_{i\in I}\rangle|| \\ &= ||\{v_i \Lambda_i P_{W_i} x\}_{i\in I}||^2. \end{aligned}$$

We define the operator $L : \mathscr{H} \to l^2(\{H_i\}_{i\in I})$ by $L(x) = \{v_i \Lambda_i P_{W_i} x\}_{i\in I}$, then

$$||L(x)||^2 = ||(v_i \Lambda_i P_{W_i} x)_{i\in I}||^2 \leq B||x||^2.$$

L is $\mathscr{A}-$linear bounded operator, then there exist $C > 0$ sutch that

$$\langle L(x), L(x)\rangle \leq C\langle x,x\rangle, \qquad \forall x \in \mathscr{H}.$$

So

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq C\langle x,x\rangle, \qquad \forall x \in H.$$

Therefore Λ is a $g-$fusion bessel sequence for $\mathscr{H}$. Now we cant define the $g-$fusion frame operator S_Λ on $\mathscr{H}$. So

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle = \langle S_\Lambda x, x\rangle, \qquad \forall x \in H.$$

Since S_Λ is positive, self-adjoint, then

$$\langle S_\Lambda^{\frac{1}{2}}x, S_\Lambda^{\frac{1}{2}}x\rangle = \langle S_\Lambda x, x\rangle, \qquad \forall x \in H.$$

That implies

$$A||\langle x,x\rangle|| \leq ||\langle S_\Lambda^{\frac{1}{2}}x, S_\Lambda^{\frac{1}{2}}x\rangle|| \leq B||\langle x,x\rangle||, \qquad \forall x \in H.$$

Frome lemma 1.7 there exist two canstants $A^{'}, B^{'} > 0$ such that

$$A^{'}\langle x,x\rangle \leq \langle S_\Lambda^{\frac{1}{2}}x, S_\Lambda^{\frac{1}{2}}x\rangle \leq B^{'}\langle x,x\rangle, \qquad \forall f \in H.$$

So

$$A^{'}\langle x,x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq B^{'}\langle x,x\rangle, \qquad \forall x \in H.$$

Hence Λ is a $g-$fusion frame for H. □

Definition 2.5. Let $K \in End_{\mathscr{A}}^{*}(H)$, $\{W_i\}_{i\in I}$ be a sequence of closed orthogonally complemented submodules of H, $\{v_i\}_{i\in I}$ be a famillly of positive weights in $\mathscr{A}$, i.e., each v_i is a positive invertible element from the center of the $C^{*}-$algebra $\mathscr{A}$ and $\Lambda_i \in End_{\mathscr{A}}^{*}(H, H_i)$ for all $i \in I$. We say that $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $K-g-$fusion frame for H if and only if there exists two constants $0 < A \leq B < \infty$ such that

$$A\langle K^{*}x, K^{*}x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq B\langle x,x\rangle, \qquad \forall x \in H. \tag{2.3}$$

The constants A and B are called a lower and upper bounds of $K-g-$fusion frame, respectively.

Proposition 2.6. *Let $K \in End_{\mathscr{A}}^{*}(H)$ and $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion bessel sequence for H. Then Λ is $K-g-$fusion frame for H if and only if there exist a constant $A > 0$ such that $AKK^{*} \leq S_\Lambda$, where S_Λ is the frame operator for Λ.*

Proof. We have for each $x \in H$,

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle = \langle S_\Lambda x, x\rangle.$$

Suppose that Λ is a $K-g-$fusion frame for H, then there exist $A>0$ such that,

$$A\langle K^*x, K^*x\rangle \leq \langle S_\Lambda x, x\rangle,$$

so,

$$AKK^* \leq S_\Lambda.$$

Assume that there exist $A>0$ such that $AKK^* \leq S_\Lambda$, then

$$A\langle K^*x, K^*x\rangle \leq \langle S_\Lambda x, x\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle,$$

since, Λ is $g-$fusion bessel sequence for H, therefore Λ is a $K-g-$fusion frame for H. □

3. Resolution of the Identity Operator in $g-$Fusion Frame

The resolution of the identity operator it was introduced in [3] to study frames of subspaces, similarly we define the resolution of the identity operator for adjointable operators on Hilbert C^*-modules.

Definition 3.1. A family of adjointable operators $\{T_i\}_{i\in I}$ on H is called a resolution of identity operator on H if for all $x \in H$ we have $x = \sum_{i\in I} T_i x$, provided the series converges unconditionally for all $x \in H$.

Theorem 3.2. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H with frame bounds C, D and S_Λ be its associated $g-$fusion frame operator. Then the famiily $\{v_i^2 P_{W_i} \Lambda_i^* T_i\}_{i\in I}$ is a resolution of the identity operator on H, where $T_i = \Lambda_i P_{W_i} S_\Lambda^{-1}$, for all $i \in I$. Furthermore, for each $x \in H$, we have*

$$\frac{C}{D^2}\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle T_i x, T_i x\rangle \leq \frac{D}{C^2}\langle x, x\rangle.$$

Proof. Since Λ is a $g-$fusion frame for H, then for all $x \in H$,

$$x = \sum_{i\in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} S^{-1} x = \sum_{i\in I} v_i^2 P_{W_i} \Lambda_i^* T_i x,$$

so, $\{v_i^2 P_{W_i} \Lambda_i^* T_i\}_{i\in I}$ is a resolution of the identity operator on H.

And we have for each $x \in H$,

$$\begin{aligned}\sum_{i\in I} v_i^2 \langle T_i x, T_i x\rangle &= \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} S_\Lambda^{-1} x, \Lambda_i P_{W_i} S_\Lambda^{-1} x\rangle \\ &\leq D\langle S_\Lambda^{-1}x, S_\Lambda^{-1}x\rangle \\ &\leq D\|S_\Lambda^{-1}\|^2 \langle x,x\rangle \\ &\leq \frac{D}{C^2}\langle x,x\rangle. \qquad (3.1)\end{aligned}$$

On the other hand, for each $x \in H$,

$$\langle x,x\rangle = \langle S_\Lambda S_\Lambda^{-1}x, S_\Lambda S_\Lambda^{-1}x\rangle \leq \|S_\Lambda\|^2 \langle S_\Lambda^{-1}x, S_\Lambda^{-1}x\rangle,$$

then,

$$\begin{aligned}\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} S_\Lambda^{-1} x, \Lambda_i P_{W_i} S_\Lambda^{-1} x\rangle &\geq C\langle S_\Lambda^{-1}x, S_\Lambda^{-1}x\rangle \\ &\geq C\|S_\Lambda\|^{-2}\langle x,x\rangle \\ &\geq \frac{C}{D^2}\langle x,x\rangle. \qquad (3.2)\end{aligned}$$

From inequality (3.1) and (3.2), we have for each $x \in H$,

$$\frac{C}{D^2}\langle x,x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} S_\Lambda^{-1} x, \Lambda_i P_{W_i} S_\Lambda^{-1} x\rangle \leq \frac{D}{C^2}\langle x,x\rangle.$$

□

Theorem 3.3. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H with frame bounds C, D and S_Λ be its associated $g-$fusion frame operator and let $T_i : H \to H_i$ be a adjointable operator such that $\{v_i^2 P_{W_i} \Lambda_i^* T_i\}_{i\in I}$ is a resolution of the identity operator on H. Then,*

$$\frac{1}{D}\|\sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^* T_i x\|^2 \leq \|\sum_{i\in I} v_i^2 \langle T_i x, T_i x\rangle\|, \quad \forall x \in H.$$

Proof. Asume $J \subset I$ with $|J| < \infty$, let $x \in H$ and set $y = \sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^* T_i x$. Then,

$$\begin{aligned}\|y\|^4 &= \|\langle y, y\rangle\|^2\\ &= \|\langle y, \sum_{i\in J} v_i^2 P_{W_i}\Lambda_i^* T_i x\rangle\|^2\\ &= \|\sum_{i\in J}\langle v_i\Lambda_i P_{W_i} y, v_i T_i x\rangle\|^2\\ &\le \|\sum_{i\in J} v_i^2\langle\Lambda_i P_{W_i} y, \Lambda_i P_{W_i} y\rangle\| \times \|\sum_{i\in J} v_i^2\langle T_i x, T_i x\rangle\|\\ &\le D\|y\|^2 \times \|\sum_{i\in J} v_i^2\langle T_i x, T_i x\rangle\|,\end{aligned}$$

so,

$$\frac{1}{D}\|y\|^2 \le \|\sum_{i\in J} v_i^2\langle T_i x, T_i x\rangle\|,$$

then,

$$\frac{1}{D}\|\sum_{i\in J} v_i^2 P_{W_i}\Lambda_i^* T_i x\|^2 \le \|\sum_{i\in J} v_i^2\langle T_i x, T_i x\rangle\|.$$

Since the inequality holds for any finite subset $J \subset I$, we have

$$\frac{1}{D}\|\sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^* T_i x\|^2 \le \|\sum_{i\in I} v_i^2\langle T_i x, T_i x\rangle\|.$$

□

Theorem 3.4. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H with frame bounds C, D and let $T_i : H \to H_i$ be a adjointable operator such that $\{v_i^2 P_{W_i}\Lambda_i^* T_i\}_{i\in I}$ is a resolution of the operator on H. If $T_i^*\Lambda_i P_{W_i} = T_i$, then*

$$\frac{1}{D}\|x\|^2 \le \|\sum_{i\in I} v_i^2\langle T_i x, T_i x\rangle\| \le DE\|x\|^2, \qquad \forall x \in H,$$

where $E = \sup_{i\in I}\|T_i\|^2 < \infty$

Proof. We have for each $x \in H$, $x = \sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^* T_i x$.

Let $x \in H$, we get by theorem 3.3

$$\begin{aligned}\frac{1}{D}\|x\|^2 &\leq \|\sum_{i\in I} v_i^2 \langle T_i x, T_i x\rangle\| \\ &\leq \|\sum_{i\in I} v_i^2 \|T_i\|^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \\ &\leq \|E\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \\ &\leq ED\|x\|^2\end{aligned}$$

□

Theorem 3.5. *Let $\{W_i\}_{i\in I}$ be a collection of closed orthogonally complemented submodules of H and $\{v_i\}_{i\in I}$ be a collection of bounded weights and $\Lambda_i \in End^*_{\mathscr{A}}(H, H_i)$ for each $i \in I$. Then $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if the following conditions are hold:*

(1) *For all $x \in H$, there exists $A > 0$ such that*

$$\|\sum_{i\in I} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \leq \frac{1}{A}\|x\|^2.$$

(2) *$\{v_i P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}\}_{i\in I}$ is a resolution of the identity operator on H.*

Proof. We have for each $x \in H$, $x = \sum_{i\in I} v_i P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} x$, then

$$\begin{aligned}\|x\|^4 &= \|\langle x, x\rangle\|^2 \\ &= \|\langle x, \sum_{i\in I} v_i P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} x\rangle\|^2 \\ &\leq \|\sum_{i\in I} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \times \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \\ &\leq \frac{1}{A}\|x\|^2 \times \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|,\end{aligned}$$

so,

$$A\|x\|^2 \leq \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|.$$

On the other hand,

$$\begin{aligned}\|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| &\leq B\|\sum_{i\in I} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\| \\ &\leq \frac{B}{A}\|x\|^2,\end{aligned}$$

where $B = \sup_{i\in I}\{v_i^2\}$.

We conclude that Λ is a $g-$fusion frame for H. □

4. $g-$Atomic Submodule

We begin this section with the following lemma

Lemma 4.1. *Let $\{W_i\}_{i\in I}$ be a sequence of orthogonally complemented closed submodules of H and $T \in End^*_{\mathscr{A}}(H)$ invertible, if $T^*TW_i \subseteq W_i$ for each $i \in I$, then $\{TW_i\}_{i\in I}$ is a sequence of orthogonally complemented closed submodules and $P_{W_i}T^* = P_{W_i}T^*P_{TW_i}$.*

Proof. Firstly for each $i \in I$, $T : W_i \to TW_i$ is invertible, so each TW_i is a closed submodule of H. We show that $H = TW_i \oplus T(W_i^{\perp})$. Since $H = TH$, then for each $x \in H$, there exists $y \in H$ sutch that $x = Ty$. On the other hand $y = u+v$, for some $u \in W_i$ and $v \in W_i^{\perp}$. Hence $x = Tu+Tv$, where $Tu \in TW_i$ and $Tv \in T(W_i^{\perp})$, plainly $TW_i \cap T(W_i^{\perp}) = (0)$, therefore $H = TW_i \oplus T(W_i^{\perp})$. Hence for every $y \in W_i$, $z \in W_i^{\perp}$ we have $T^*Ty \in W_i$ and therefore $\langle Ty, Tz\rangle = \langle T^*Ty, z\rangle = 0$, so $T(W_i^{\perp}) \subset (TW_i)^{\perp}$ and consequently $T(W_i^{\perp}) = (TW_i)^{\perp}$ witch implies that TW_i is orthogonally complemented.
Let $x \in H$ we have $x = P_{TW_i}x + y$, for some $y \in (TW_i)^{\perp}$, then $T^*x = T^*P_{TW_i}x + T^*y$. Let $v \in W_i$ then $\langle T^*y, v\rangle = \langle y, Tv\rangle = 0$ then $T^*y \in W_i^{\perp}$ and we have $P_{W_i}T^*x = P_{W_i}T^*P_{TW_i}x + P_{W_i}T^*y$, then $P_{W_i}T^*x = P_{W_i}T^*P_{TW_i}x$ thus implies that for each $i \in I$ we have $P_{W_i}T^* = P_{W_i}T^*P_{TW_i}$. □

Definition 4.2. Let $K \in End^*_{\mathscr{A}}(H)$ and $\{W_i\}_{i\in I}$ be a collection of closed submodules orthogonally complemented of H, let $\{v_i\}_{i\in I}$ be a collection of positive weights in $\mathscr{A}$, i.e., each v_i is a positive invertible element from the center of the C^*-algebra $\mathscr{A}$ and $\Lambda_i \in End^*_{\mathscr{A}}(H, H_i)$ for each $i \in I$. Then the family $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is said to be a $g-$atomic submodule of H with respect to K if the following statements hold:

(1) Λ is a $g-$fusion bessel sequence in H.
(2) For every $x \in H$ there exists $\{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$ such that

$$K(x) = \sum_{i\in I} v_i P_{W_i}\Lambda_i^* x_i \quad \textit{and} \quad \|\{x_i\}_{i\in I}\| \le C\|x\|$$

for some $C > 0$.

Theorem 4.3. *Let $K \in End_{\mathscr{A}}^{*}(H)$ and $\{W_i\}_{i\in I}$ be a collection of closed submodules orthogonally complemented of H, let $\{v_i\}_{i\in I}$ be a collection of positive weights, $\Lambda_i \in End_{\mathscr{A}}^{*}(H,H_i)$ for each $i \in I$ and suppose that the operator $L : H \to l^2(\{H_i\}_{i\in I})$ define by $L(x) = \{v_i\Lambda_i P_{W_i}x\}_{i\in I}$ such that $\overline{\mathscr{R}(L)}$ is orthogonally commplemented, then the following statements are equivalent:*

(1) *$\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$atomic submodules of H with respect to K.*

(2) *Λ is a $K-g-$fusion frame for H.*

Proof. $(1) \Rightarrow (2)$ We have Λ is a $g-$fusion bessel sequence. Now let $x \in H$,

$$\begin{aligned}\|\langle K^*x, K^*x\rangle\| &= \|K^*x\|^2\\ &= \sup_{\|y\|=1}\|\langle K^*x, y\rangle\|\\ &= \sup_{\|y\|=1}\|\langle x, K(y)\rangle\|.\end{aligned}$$

Since $y \in H$ there exits $\{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$ such that

$$K(y) = \sum_{i\in I} v_i P_{W_i}\Lambda_i^* y_i \quad and \quad \|\{y_i\}_{i\in I}\| \leq C\|y\|$$

for some $C > 0$. So, for each $x \in H$,

$$\begin{aligned}\|K^*x\|^2 &= \sup_{\|y\|=1}\|\langle x, \sum_{i\in I} v_i P_{W_i}\Lambda_i^* y_i\rangle\|^2\\ &= \sup_{\|y\|=1}\|\sum_{i\in I}\langle v_i\Lambda_i P_{W_i}, y_i\rangle\|^2\\ &\leq \sup_{\|y\|=1}\|\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle\|\|\sum_{i\in I}\langle y_i, y_i\rangle\|\\ &\leq C^2\|\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle\|,\end{aligned}$$

hence,

$$\frac{1}{C^2}\|K^*x\|^2 \leq \|\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle\|,$$

therefore, Λ is a $K-g-$fusion frame for H.

$(2) \Rightarrow (1)$ Suppose that Λ is a $K-g-$fusion frame for H, then Λ is a $g-$fusion bessel sequence for H. Let $x \in H$, we have

$$A\langle K^*x, K^*x\rangle \leq \langle Lx, Lx\rangle,$$

so,

$$AKK^* \leq L^*L,$$

then by lemma 1.8 there exists $G \in End^*_{\mathscr{A}}(H, l^2(\{H_i\}_{i\in I}))$ define by $Gx = \{x_i\}_{i\in I}$ such that $K = L^*G$, hence for each $x \in H$

$$\begin{aligned} K(x) &= L^*Gx \\ &= L^*(\{x_i\}_{i\in I}) \\ &= \sum_{i\in I} v_i P_{W_i} \Lambda_i^* x_i, \end{aligned}$$

and

$$\|\{x_i\}_{i\in I}\| = \|Gx\| \leq C\|x\|,$$

for some $C > 0$. We conclude that Λ is a $g-$atomic submodule of H with respect to K. □

Theorem 4.4. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H. Then Λ is a $g-$atomic submodule of H with respect to its $g-$fusion frame operator S_Λ.*

Proof. We have Λ is a $g-$fusion bessel sequence for H, and we have for each $x \in H$,

$$\begin{aligned} S_\Lambda x &= \sum_{i\in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} x \\ &= \sum_{i\in I} v_i P_{W_i} \Lambda_i^* (v_i \Lambda_i P_{W_i} x), \end{aligned}$$

now we put $x_i = v_i \Lambda_i P_{W_i} x$, for each $i \in I$, hence,

$$\begin{aligned} \|\{x_i\}_{i\in I}\| &= \|\{v_i \Lambda_i P_{W_i} x\}_{i\in I}\| \\ &= \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|^{\frac{1}{2}} \\ &\leq \sqrt{B}\|x\|. \end{aligned}$$

Therefore, Λ is a $g-$atomic submodule of H with respect to its $g-$fusion frame operator S_Λ. □

Theorem 4.5. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ and $\Gamma = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be two $g-$atomic submodules of H with respect to $K \in End^*_{\mathscr{A}}(H)$. If $U, V \in End^*_{\mathscr{A}}(H)$ such that $U+V$ is invertible operator on H with $K(U+V) = (U+V)K$, suppose that the operator $L : H \to l^2(\{H_i\}_{i\in I})$ define by*

$L(x) = \{v_i(\Lambda_i + \Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I}$ *such that* $\overline{\mathscr{R}(L)}$ *is orthogonally complemented, then*

$$\{(U+V)W_i, (\Lambda_i + \Gamma_i)P_{W_i}(U+V)^*, v_i\}_{i\in I}$$

is a $g-$*atomic submodule of* H *with respect to* K.

Proof. By theorem 4.3, Λ and Γ are $K-g-$fusion frame for H, so for each $x \in H$ there exist positive constants (A_1, B_1) and (A_2, B_2) such that

$$A_1\|K^*x\|^2 \leq \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle\| \leq B_1\|x\|^2,$$

and

$$A_2\|K^*x\|^2 \leq \|\sum_{i\in I} v_i^2 \langle \Gamma_i P_{W_i}x, \Gamma_i P_{W_i}x\rangle\| \leq B_2\|x\|^2.$$

Since $U+V$ is invertible, then

$$\begin{aligned}\langle K^*x, K^*x\rangle &= \langle ((U+V)^*)^{-1}(U+V)^*K^*x, ((U+V)^*)^{-1}(U+V)^*K^*x\rangle \\ &\leq \|(U+V)^{-1}\|^2\langle (U+V)^*K^*x, (U+V)^*K^*x\rangle.\end{aligned}$$

Now, for each $x \in H$ we have

$$\begin{aligned}&\|\sum_{i\in I} v_i\langle(\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x, (\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x\rangle\|^{\frac{1}{2}} \\ &= \|\{v_i(\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I}\| \\ &= \|\{v_i\Lambda_iP_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I} + \{v_i\Gamma_iP_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I}\| \\ &\leq \|\{v_i\Lambda_iP_{W_i}(U+V)^*x\}_{i\in I}\| + \|\{v_i\Gamma_iP_{W_i}(U+V)^*x\}_{i\in I}\| \\ &\leq \sqrt{B_1}\|\langle(U+V)^*x, (U+V)^*x\rangle\|^{\frac{1}{2}} + \sqrt{B_2}\|\langle(U+V)^*x, (U+V)^*x\rangle\|^{\frac{1}{2}} \\ &\leq (\sqrt{B_1}+\sqrt{B_2})\|\langle(U+V)^*x, (U+V)^*x\rangle\|^{\frac{1}{2}} \\ &\leq (\sqrt{B_1}+\sqrt{B_2})\|(U+V)\|\|\langle x, x\rangle\|^{\frac{1}{2}} \\ &= (\sqrt{B_1}+\sqrt{B_2})\|(U+V)\|\|x\|. \qquad (4.1)\end{aligned}$$

On the other hand

$$\|\sum_{i\in I} v_i\langle(\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x,(\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*P_{(U+V)W_i}x\rangle\|^{\frac{1}{2}}$$
$$=\|\{v_i\Lambda_iP_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I}+\{v_i\Gamma_iP_{W_i}(U+V)^*P_{(U+V)W_i}x\}_{i\in I}\|$$
$$\geq\|\{v_i\Lambda_iP_{W_i}(U+V)^*x\}_{i\in I}\|$$
$$\geq\sqrt{A_1}\|\langle((U+V)K)^*x,((U+V)K)^*x\rangle\|^{\frac{1}{2}}$$
$$\geq A_1\|(U+V)^{-1}\|^{-1}\|\langle K^*x,K^*x\rangle\|^{\frac{1}{2}} \tag{4.2}$$

From (4.1) and (4.2), we conclude that $\{(U+V)W_i,(\Lambda_i+\Gamma_i)P_{W_i}(U+V)^*,v_i\}_{i\in I}$ is a $K-g-$fusion frame for H, therefore Λ is a $g-$atomic submodule of H with respect to K. □

Theorem 4.6. *Let $\Lambda=\{W_i,\Lambda_i,v_i\}_{i\in I}$ is a $g-$atomic submodule for $K\in End^*_{\mathscr{A}}(H)$ and S_Λ be the frame operator of Λ. if $U\in End^*_{\mathscr{A}}(H)$ is a positive and invertible operator on H, suppose that the operator $L:H\to l^2(\{H_i\}_{i\in I})$ define by $L(x)=\{v_i\Lambda_iP_{W_i}(I_H+U)^*P_{(I_H+U)W_i}x\}_{i\in I}$ such that $\overline{\mathscr{R}(L)}$ is orthogonally complemented, then $\theta=\{(I_H+U)W_i,\Lambda_iP_{W_i}(I_H+U)^*,v_i\}_{i\in I}$ is a $g-$atomic submodule of H with respect to K. Moreover, for any natural number n, $\theta'=\{(I_H+U^n)W_i,\Lambda_iP_{W_i}(I_H+U^n)^*,v_i\}_{i\in I}$ is a $g-$atomic submodule of H with respect to K.*

Proof. We have for each $x\in H$,

$$\sum_{i\in I} v_i^2\langle\Lambda_iP_{W_i}(I_H+U)^*P_{(I_H+U)W_i}(x),\Lambda_iP_{W_i}(I_H+U)^*P_{(I_H+U)W_i}(x)\rangle$$
$$=\sum_{i\in I} v_i^2\langle\Lambda_iP_{W_i}(I_H+U)^*(x),\Lambda_iP_{W_i}(I_H+U)^*(x)\rangle$$
$$\leq B\langle(I_H+U)^*(x),(I_H+U)^*(x)\rangle$$
$$\leq B\|(I_H+U)\|^2\langle x,x\rangle,$$

Thus, θ is a $g-$bessel sequence in H, Also, for each $x\in H$ we have

$$\sum_{i\in I} P_{(I_H+U)W_i}(\Lambda_iP_{W_i}(I_H+U)^*)^*\Lambda_iP_{W_i}(I_H+U)^*P_{(I_H+U)W_i}(x)$$
$$=\sum_{i\in I} v_i^2P_{(I_H+U)W_i}(I_H+U)P_{W_i}\Lambda_i^*\Lambda_iP_{W_i}(I_H+U)^*P_{(I_H+U)W_i}(x)$$

$$= \sum_{i\in I} v_i^2 (P_{W_i}(I_H+U)^* P_{(I_H+U)W_i})^* \Lambda_i^* \Lambda_i P_{W_i}(I_H+U)^* P_{(I_H+U)W_i}(x)$$

$$= \sum_{i\in I} v_i^2 (P_{W_i}(I_H+U)^*)^* \Lambda_i^* \Lambda_i P_{W_i}(I_H+U)^*(x)$$

$$= \sum_{i\in I} v_i^2 (I_H+U) P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}(I_H+U)^*(x)$$

$$= (I_H+U) \sum_{i\in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i}(I_H+U)^*(x)$$

$$= (I_H+U) S_\Lambda (I_H+U)^*(x).$$

This shows that the frame operator of θ is $(I_H+U)S_\Lambda(I_H+U)^*$. Since U and S_Λ are positive, we have

$$(I_H+U)S_\Lambda(I_H+U)^* \geq S_\Lambda \geq AKK^*,$$

Then by proposition 2.6, we can conclude that θ is a $K-g-$fusion frame for H, so by theorem 4.3, θ is a $g-$atomic submodule of H with respect to K. According to the preceding procedure, for any natural number n, the frame operator of $\theta^{'}$ is $(I_H+U^n)S_\Lambda(I_H+U^n)^*$ and similary, it can be shown that $\theta^{'}$ is a $g-$atomic submodule of H with respect to K. □

5. Frame Operator for a Pair of $g-$Fusion Bessel Sequences

Definition 5.1. Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ and $\Gamma = \{V_i, \Gamma_i, w_i\}_{i\in I}$ be two $g-$fusion bessel sequences in H with bounds B_1 and B_2. Then the operator $S_{\Gamma\Lambda} : H \to H$, defined by

$$S_{\Gamma\Lambda}(x) = \sum_{i\in I} v_i w_i P_{V_i} \Gamma_i^* \Lambda_i P_{W_i}(x), \quad \forall x \in H,$$

is called the frame operator for the pair of $g-$fusion bessel sequences Λ and Γ.

Theorem 5.2. *The frame operator $S_{\Gamma\Lambda}$ for the pair of $g-$fusion bessel sequences Λ and Γ is bounded and $S_{\Gamma\Lambda}^* = S_{\Lambda\Gamma}$.*

Proof. We have for each $x \in H$,

$$
\begin{aligned}
\|S_{\Gamma\Lambda}x\| &= \sup_{\|y\|=1} \|\langle \sum_{i\in I} v_i w_i P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}(x), y\rangle \\
&= \sup_{\|y\|=1} \|\sum_{i\in I} v_i w_i \langle \Lambda_i P_{W_i}x, \Gamma_i P_{V_i}y\rangle\| \\
&\leq \sup_{\|y\|=1} \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle\|^{\frac{1}{2}} \|\sum_{i\in I} w_i^2 \langle \Gamma_i P_{V_i}y, \Gamma_i P_{V_i}y\rangle\|^{\frac{1}{2}} \\
&\leq \sqrt{B_1 B_2}\|x\|,
\end{aligned}
$$

then $S_{\Gamma\Lambda}$ is a bounded with $\|S_{\Gamma\Lambda}\| \leq \sqrt{B_1 B_2}$.

Also, for each $x, y \in H$ we have

$$
\begin{aligned}
\langle S_{\Gamma\Lambda}x, y\rangle &= \langle \sum_{i\in I} v_i w_i P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}(x), y\rangle \\
&= \sum_{i\in I} v_i w_i \langle x, P_{W_i}\Lambda_i^*\Gamma_i P_{V_i}(y)\rangle \\
&= \langle x, \sum_{i\in I} P_{W_i}\Lambda_i^*\Gamma_i P_{V_i}(y)\rangle = \langle x, S_{\Lambda\Gamma}y\rangle
\end{aligned}
$$

□

Theorem 5.3. *Let $S_{\Gamma\Lambda}$ be the frame operator for a pair of $g-$fusion bessel sequences Λ and Γ with bounds B_1 and B_2, respectively. And $\overline{\mathscr{R}(S_{\Gamma\Lambda})}$ is orthogonally complemented. Then the following statements are equivalent:*

(1) *$S_{\Gamma\Lambda}$ is bounded below.*
(2) *there exists $K \in End_{\mathscr{A}}^*(H)$ such that $\{T_i\}_{i\in I}$ is a resolution of the identity operator on H, where $T_i = v_i w_i K P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}$, $i \in I$.*

If one of the given conditions holds, then Λ is a $g-$fusion frame.

Proof. $(1) \Rightarrow (2)$ Suppose that $S_{\Gamma\Lambda}$ is bounded below. Then for each $x \in H$ there exists $A > 0$ such that

$$A\|x\| \leq \|S_{\Lambda}x\|,$$

hence,

$$A\|\langle x, x\rangle\| \leq \|\langle S_{\Gamma\Lambda}^* S_{\Gamma\Lambda}x, x\rangle\|,$$

then,

$$I_H I_H^* \leq \frac{1}{A} S_{\Gamma\Lambda}^* S_{\Gamma\Lambda},$$

so, by lemma 1.8, there exists $K \in End_{\mathscr{A}}^*(H)$ such that $I_H = KS_{\Gamma\Lambda}$, therefore for each $x \in H$ we have

$$\begin{aligned} x &= KS_{\Gamma\Lambda}x \\ &= K\sum_{i\in I} v_i w_i P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}x \\ &= \sum_{i\in I} v_i w_i K P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}x \\ &= \sum_{i\in I} T_i x, \end{aligned}$$

thus $\{T_i\}_{i\in I}$ is a resolution of the identity operator on H, where $T_i = v_i w_i K P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}$.

$(2) \Rightarrow (1)$ we have for each $x \in H$,

$$\begin{aligned} \|x\| &= \|\sum_{i\in I} v_i w_i K P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}x\| \\ &= \|K\sum_{i\in I} v_i w_i P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}x\| \\ &= \|KS_{\Gamma\Lambda}x\| \\ &\leq \|K\| \times \|S_{\Gamma\Lambda}x\|, \end{aligned}$$

then,

$$\|K\|^{-1}\|x\| \leq \|S_{\Gamma\Lambda}x\|.$$

Hence, $S_{\Gamma\Lambda}$ is bounded below.

Last part: Suppose that $S_{\Gamma\Lambda}$ is bounded below. Then for all $x \in H$ there exists $A > 0$ such that $A\|x\| \leq \|S_{\Gamma\Lambda}x\|$ and this implies that

$$\begin{aligned} A\|x\| &\leq \sup_{\|y\|=1} \|\langle S_{\Gamma\Lambda}x, y\rangle\| \\ &= \sup_{\|y\|=1} \|\langle \sum_{i\in I} v_i w_i P_{V_i}\Gamma_i^*\Lambda_i P_{W_i}x, y\rangle\| \\ &= \sup_{\|y\|=1} \|\sum_{i\in I} v_i w_i \langle \Lambda_i P_{W_i}x, \Gamma_i P_{V_i}x\rangle\| \end{aligned}$$

$$\leq \sup_{\|y\|=1} \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|^{\frac{1}{2}} \|\sum_{i\in I} w_i \langle \Gamma_i P_{V_i} x, \Gamma_i P_{V_i} x\rangle\|^{\frac{1}{2}}$$

$$\leq B_2 \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|^{\frac{1}{2}},$$

hence,

$$\frac{A^2}{B_2}\|x\|^2 \leq \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|.$$

So, Λ is a $g-$fusion frame for H. □

Theorem 5.4. *Let $S_{\Lambda\Gamma}$ be the frame operator for a pair of $g-$fusion bessel sequences Λ and Γ with bounds B_1 and B_2, respectively. Suppose $\lambda_1 < 1$, $\lambda_2 > -1$ such that each $x \in H$, $\|x - S_{\Gamma\Lambda} x\| \leq \lambda_1 \|x\| + \lambda_2 \|S_{\Gamma\Lambda} x\|$. Then Λ is a $g-$fusion frame for H.*

Proof. We have for each $x \in H$,

$$\|x\| - \|S_{\Gamma\Lambda}\| \leq \|x - S_{\Gamma\Lambda} x\| \leq \lambda_1 \|x\| + \lambda_2 \|S_{\Gamma\Lambda} x\|,$$

then,

$$\left(\frac{1-\lambda_1}{1+\lambda_2}\right)\|x\| \leq \|S_{\Gamma\Lambda} x\|$$

$$\leq \sqrt{B_2}\|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|^{\frac{1}{2}},$$

Hence,

$$\frac{1}{B_2}\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \|x\| \leq \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|. \tag{5.1}$$

Thus, Λ is a $g-$fusion frame for H with bounds $\frac{1}{B_2}\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2$ and B_1. □

Theorem 5.5. *Let $S_{\Gamma\Lambda}$ be the frame operator for a pair of $g-$fusion bessel sequences Λ and Γ of bounds B_1 and B_2, repectively. Assume $\lambda \in [0,1)$ such that*

$$\|x - S_{\Gamma\Lambda} x\| \leq \lambda \|x\|, \quad \forall x \in H.$$

Then Λ and Γ are $g-$fusion frames for H.

Proof. We put $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in (5.1), then

$$\frac{(1-\lambda)^2}{B_2}\|x\|^2 \leq \|\sum_{i\in I} v_i^2 \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle\|,$$

therefore, Λ is a $g-$fusion frame for H. Now for each $x \in H$, we have

$$\begin{aligned}\|x - S_{\Gamma\Lambda}^*\| &= \|(I_H - S_{\Gamma\Lambda})^* x\| \\ &\leq \|I_H - S_{\Gamma\Lambda}\|\|x\| \\ &\leq \lambda\|x\|,\end{aligned}$$

then,

$$\|x\| - \|S_{\Gamma\Lambda}^* x\| \leq \lambda\|x\|,$$

hence,

$$\begin{aligned}(1-\lambda)\|x\| &\leq \|S_{\Gamma\Lambda}^* x\| \\ &= \sup_{\|y\|=1} \|\langle S_{\Gamma\Lambda}^* x, y\rangle\| \\ &= \sup_{\|y\|=1} \|\langle v_i w_i P_{W_i} \Lambda_i^* \Gamma_i P_{V_i} x, y\rangle\| \\ &= \sup_{\|y\|=1} \|\sum_{i\in I} \langle w_i \Gamma_i P_{V_i} x, v_i \Lambda_i P_{W_i} y\rangle\| \\ &\leq \|\sum_{i\in I} w_i^2 \langle \Gamma_i P_{V_i} x, \Gamma_i P_{V_i} x\rangle\|^{\frac{1}{2}} \|\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} y, \Lambda_i P_{W_i} y\rangle\|^{\frac{1}{2}} \\ &\leq \sqrt{B_1}\|\sum_{i\in I} w_i^2 \langle \Gamma_i P_{V_i} x, \Gamma_i P_{V_i} x\rangle\|^{\frac{1}{2}}\|,\end{aligned}$$

so,

$$\frac{(1-\lambda)^2}{B_1}\|x\|^2 \leq \|\sum_{i\in I} w_i^2 \langle \Gamma_i P_{V_i} x, \Gamma_i P_{V_i} x\rangle\|.$$

We conclude that Γ is a $g-$fusion frame for H with bounds $\frac{(1-\lambda)^2}{B_1}$ and B_2. □

Definition 5.6. Let H and X be two Hilbert C^*-modules. Define

$$H \oplus X = \{(x, y) : x \in H, y \in X\}.$$

Then $H \oplus X$ forms a Hilbert C^*-module with respect to point-wise operations and inner $\mathscr{A}-$valued defined by

$$\langle (x,y),(x^{'},y^{'})\rangle = \langle x,x^{'}\rangle_H + \langle y,y^{'}\rangle_X \quad \forall x,x^{'} \in H \quad and \quad \forall y,y^{'} \in X.$$

Now, if $U \in End^*_{\mathscr{A}}(H,Z)$, $V \in End^*_{\mathscr{A}}(X,Y)$, then for all $x \in H$, $y \in X$ we define

$$U \oplus V \in End^*_{\mathscr{A}}(H \oplus X, Z \oplus Y) \quad by \quad (U \oplus V)(x,y) = (Ux, Vy),$$

and $(U \oplus V)^* = U^* \oplus V^*$, where Z,Y are Hilbert C^*-modules and also we define $P_{M\oplus N}(x,y) = (P_M x, P_N y)$, where P_M, P_N and $P_{M\oplus N}$ are orthogonal projections onto the closed orthogonally complemented submodules $M \subset H$, $N \subset X$ and $M \oplus N \subset H \oplus X$, respectively.

From here we assume that for each $i \in I$, $W_i \oplus V_i$ are the closed orthogonally complemented submodules of $H \oplus X$ and $\Gamma_i \in End^*_{\mathscr{A}}(X, X_i)$, where $\{X_i\}_{i\in I}$ is the collection of Hilbert C^*-modules and $\Lambda_i \oplus \Gamma_i \in End^*_{\mathscr{A}}(H \oplus X, H_i \oplus X_i)$.

Theorem 5.7. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H with frame bounds A,B and $\Gamma_i = \{V_i, \Gamma_i, v_i\}_{i\in I}$ be a $g-$fusion frame for X with frame bounds C,D. Then $\Lambda \oplus \Gamma = \{W_i \oplus V_i, \Lambda_i \oplus \Gamma_i, v_i\}_{i\in I}$ is a $g-$fusion frame for $H \oplus X$. Furthermore, if S_Λ, S_Γ and $S_{\Lambda\oplus\Gamma}$ are $g-$fusion frame operators for Λ, Γ and $\Lambda \oplus \Gamma$, respectively, then we have $S_{\Lambda\oplus\Gamma} = S_\Lambda \oplus S_\Gamma$.*

Proof. Let $x \in H$ and $y \in X$, we have

$$\begin{aligned}
&\sum_{i\in I} v_i^2 \langle (\Lambda_i \oplus \Gamma_i) P_{W_i\oplus V_i}(x,y), (\Lambda_i \oplus \Gamma_i) P_{W_i\oplus V_i}(x,y)\rangle \\
&= \sum_{i\in I} v_i^2 \langle (\Lambda_i \oplus \Gamma_i)(P_{W_i}x, P_{V_i}y), (\Lambda_i \oplus \Gamma_i)(P_{W_i}x, P_{V_i}y)\rangle \\
&= \sum_{i\in I} v_i^2 \langle (\Lambda_i P_{W_i}x, \Gamma_i P_{V_i}y), (\Lambda_i P_{W_i}x, \Gamma_i P_{V_i}y)\rangle \\
&= \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle_H + \sum_{i\in I} v_i^2 \langle \Gamma_i P_{V_i}y, \Gamma_i P_{V_i}y\rangle_X \\
&\leq B\langle x,x\rangle_H + D\langle y,y\rangle_X \\
&\leq \max(B,D)(\langle x,x\rangle_H + \langle y,y\rangle_X) \\
&= \max(B,D)\langle (x,y),(x,y)\rangle \qquad (5.2)
\end{aligned}$$

Simalary, it can be shown that

$$\min(A,C)\langle(x,y),(x,y)\rangle \leq \sum_{i\in I} v_i^2\langle(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y)\rangle. \tag{5.3}$$

From inequality (5.2) and (5.3), we conclude that $\Lambda\oplus\Gamma$ is a $g-$fusion frame for $H\oplus X$.

Furthermore, for $(x,y)\in H\oplus X$ we have

$$\begin{aligned}
S_{\Lambda\oplus\Gamma}(x,y) &= \sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y)\\
&= \sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i P_{W_i}x,\Gamma_i P_{V_i}y)\\
&= \sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i^*\oplus\Gamma_i^*)(\Lambda_i P_{W_i}x,\Gamma_i P_{V_i}y)\\
&= \sum_{i\in I} v_i^2 (P_{w_i}\Lambda_i^*\Lambda_i P_{W_i}x, P_{V_i}\Gamma_i^*\Gamma_i P_{V_i}y)\\
&= \left(\sum_{i\in I} v_i^2 P_{w_i}\Lambda_i^*\Lambda_i P_{W_i}x, \sum_{i\in I} P_{V_i}\Gamma_i^*\Gamma_i P_{V_i}y\right)\\
&= (S_\Lambda x, S_\Gamma y)\\
&= (S_\Lambda\oplus S_\Gamma)(x,y).
\end{aligned}$$

Therefore, $S_{\Lambda\oplus\Gamma} = S_\Lambda\oplus S_\Gamma$. □

Theorem 5.8. *Let $\Lambda\oplus\Gamma=\{W_i\oplus V_i,\Lambda_i\oplus\Gamma_i,v_i\}_{i\in I}$ be a $g-$fusion frame for $H\oplus X$ with frame operator $S_{\Lambda\oplus\Gamma}$. Then*

$$\alpha=\{S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(W_i\oplus V_i),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}},v_i\}_{i\in I}$$

is a Parseval $g-$fusion frame for $H\oplus X$.

Proof. Since $S_{\Lambda\oplus\Gamma}$ is a positive invertible, then $S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}S_{\Lambda\oplus\Gamma}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}=I_{H\oplus X}$, hence

$$\begin{aligned}
(x,y) &= S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}S_{\Lambda\oplus\Gamma}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(x,y)\\
&= \sum_{i\in I} v_i^2 S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(x,y),
\end{aligned}$$

so,

$$\begin{aligned}\langle (x,y),(x,y)\rangle &= \langle \sum_{i\in I} v_i^2 S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}} P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(x,y),(x,y)\rangle\\ &= \sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(x,y)\rangle\\ &= \sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}P_{S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(W_i\oplus V_i)}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}P_{S_{\Lambda\oplus\Gamma}^{-\frac{1}{2}}(W_i\oplus V_i)}(x,y)\rangle.\end{aligned}$$

This shows that α is a Parseval $g-$fusion frame for $H\oplus X$. □

Theorem 5.9. *Let $\Lambda\oplus\Gamma=\{W_i\oplus V_i,\Lambda_i\oplus\Gamma_i,v_i\}_{i\in I}$ be a $g-$fusion frame for $H\oplus X$ with frame bounds A,B and $S_{\Lambda\oplus\Gamma}$ be the corresponding frame operator. Then*

$$\alpha=\{S_{\Lambda\oplus\Gamma}^{-1}(W_i\oplus V_i),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1},v_i\}_{i\in I}$$

is a $g-$fusion frame for $H\oplus X$ with frame operator $S_{\Lambda\oplus\Gamma}^{-1}$.

Proof. For each $x\in H$ and $y\in X$ we have

$$\begin{aligned}(x,y) &= S_{\Lambda\oplus\Gamma}S_{\Lambda\oplus\Gamma}^{-1}(x,y)\\ &= \sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1}(x,y).\end{aligned}$$

We have for each $(x,y)\in H\oplus X$,

$$\begin{aligned}&\|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_i\oplus V_i)}(x,y),\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1}P_{S_{\Lambda\oplus\Gamma}^{-1}(W_i\oplus V_i)}(x,y)\rangle\|\\ &=\|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S_{\Lambda\oplus\Gamma}^{-1}(x,y)\rangle\|\\ &\leq B\|S_{\Lambda\oplus\Gamma}^{-1}\|^2\|(x,y)\|^2.\end{aligned}$$

On the other hand for each $(x,y)\in H\oplus X$ we have

$$\begin{aligned}
\|(x,y)\|^4 &= \|\langle (x,y),(x,y)\rangle\|^2\\
&= \|\langle \sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y),(x,y)\rangle\|^2\\
&= \|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y)\rangle\|^2\\
&\leq \|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y)\|\\
&\qquad\qquad \times \|\sum_{i\in I} v_i^2(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}(x,y)\rangle\|\\
&\leq B\|(x,y)\|^2\|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y)\rangle\|,
\end{aligned}$$

then,

$$B^{-1}\|(x,y)\|^2 \leq \|\sum_{i\in I} v_i^2\langle (\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y),(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y)\rangle\|.$$

Therefore, α is a $g-$fusion frame for $H\oplus X$. Let S_α be the $g-$fusion frame for α and take $G_i=\Lambda_i\oplus\Gamma_i$. Now, for each $(x,y)\in H\oplus X$.

$$\begin{aligned}
S_\alpha(x,y) &= \sum_{i\in I} v_i^2 P_{S^{-1}_{\Lambda\oplus\Gamma}(W_i\oplus V_i)}(G_iP_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma})^*(G_iP_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma})P_{S^{-1}_{\Lambda\oplus\Gamma}(W_i\oplus V_i)}(x,y)\\
&= \sum_{i\in I} v_i^2(P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}P_{S^{-1}_{\Lambda\oplus\Gamma}(W_i\oplus V_i)})^*G_i^*G_i(P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}P_{S^{-1}_{\Lambda\oplus\Gamma}(W_i\oplus V_i)})(x,y)\\
&= \sum_{i\in I} v_i^2(P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma})^*G_i^*G_i(P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma})(x,y)\\
&= \sum_{i\in I} v_i^2 S^{-1}_{\Lambda\oplus\Gamma}P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)(P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma})(x,y)\\
&= S^{-1}_{\Lambda\oplus\Gamma}\left(\sum_{i\in I} v_i^2 P_{W_i\oplus V_i}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)P_{W_i\oplus V_i}S^{-1}_{\Lambda\oplus\Gamma}(x,y)\right)\\
&= S^{-1}_{\Lambda\oplus\Gamma}S_{\Lambda\oplus\Gamma}(S^{-1}_{\Lambda\oplus\Gamma}(x,y))\\
&= S^{-1}_{\Lambda\oplus\Gamma}(x,y).
\end{aligned}$$

Therefore, $S_\alpha = S^{-1}_{\Lambda\oplus\Gamma}$. □

ACKNOWLEDGMENTS

It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Alijani, M. Dehghan, $*$-frames in Hilbert $\mathscr{C}^*$modules, U. P. B. Sci. Bull. Ser. A 73 (2011), 89-106.

[2] Lj. Arambašić, On frames for countably generated Hilbert C^*-modules, Proc. Amer. Math. Soc. 135 (2007), 469-478.

[3] P. G. Casazza, G. Kutyniok: Frames of subspaces. Wavelets, Frames and Operator Theory. Contemporary Mathematics 345. American Mathematical Society, Providence, 2004, pp. 87-114.

[4] J. B. Conway, A Course In Operator Theory, American Mathematical Society, Vol. 21, 2000.

[5] F. R. Davidson, $\mathscr{C}^*$-algebra by example, Fields Ins. Monog. 1996.

[6] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic fourier series, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[7] X. Fang, J. Yu and H. Yao, Solutions to operator equations on Hilbert C^*-modules, Linear Algebra Appl. 431 (2009), 2142-2153.

[8] M. Frank, D. R. Larson, $\mathscr{A}$-module frame concept for Hilbert $\mathscr{C}^*$-modules, functinal and harmonic analysis of wavelets, Contempt. Math. 247 (2000), 207-233.

[9] D. Gabor, Theory of communications, J. Elec. Eng. 93 (1946), 429-457.

[10] I. Kaplansky, Modules over operator algebras, Amer. J. Math. 75 (1953), 839-858.

[11] A. Khorsavi, B. Khorsavi, Fusion frames and g-frames in Hilbert $\mathscr{C}^*$-modules, Int. J. Wavelet. Multiresolution Inform. Proc. 6 (2008), 433-446.

[12] E. C. Lance, Hilbert C^*-Modules: A Toolkit for Operator Algebraist, London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1995.

[13] M. Rossafi, S. Kabbaj, $*$-K-operator Frame for $End^*_{\mathscr{A}}(\mathscr{H})$, Asian-Eur. J. Math. 13 (2020), 2050060.

[14] M. Rossafi, S. Kabbaj, Operator Frame for $End^*_{\mathscr{A}}(\mathscr{H})$, J. Linear Topol. Algebra, 8 (2019), 85-95.

[15] S. Kabbaj, M. Rossafi, $*$-operator Frame for $End^*_{\mathscr{A}}(\mathscr{H})$, Wavelet Linear Algebra, 5 (2) (2018), 1-13.

[16] M. Rossafi, S. Kabbaj, $*$-K-g-frames in Hilbert $\mathscr{A}$-modules, J. Linear Topol. Algebra, 7 (2018), 63-71.

[17] M. Rossafi, S. Kabbaj, $*$-g-frames in tensor products of Hilbert C^*-modules, Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 17-25.

[18] M. Rossafi, S. Kabbaj, Generalized Frames for $B(\mathscr{H},\mathscr{K})$, Iran. J. Math. Sci. Inf. accepted.