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EXISTENCE RESULTS FOR WEAKLY COUPLED SYSTEM OF ψ- CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, our aim is to develop monotone iterative technique by introducing the notion of coupled upper, lower and quasi solutions and apply it to prove existence of solution of nonlinear boundary value problems for weakly coupled system of ψ-Caputo fractional differential equations. We choose suitable initial iterations and construct two monotone sequences which converge monotonically from above and below to quasi solutions of nonlinear weakly coupled system of ψ-Caputo fractional differential equations. Further we show that these sequences lead to existence of solution of nonlinear boundary value problem for weakly coupled system of ψ-Caputo fractional differential equations.

Keywords: nonlinear weakly coupled system; upper lower and quasi solutions; ψ-Caputo fractional derivative; monotone iterative technique; existence results.

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1. Introduction

Different types of real world phenomena with anomalous dynamics in the field of mathematical, physical,computational, chemical and biological sciences are modeled adequately by various types of fractional differential equations (see [17],[21] and references therein). In literature the qualitative theory of various fractional differential equations are studied by many researchers with the help of different methods such as monotone technique, fixed point technique, method of successive approximation etc and obtained existence, uniqueness results (see [1, 2], [6]-[20],[22]-[32]).Almeida et.al. [3, 4, 5]and Samet et.al.[29] have studied fractional differential equations involving ψ-Caputo fractional derivative.In the year 2020, Derbazi et al. [9]have proved existence and uniqueness of solution of initial value problem (IVP) for ψ- Caputo fractional differential equation via monotone iterative technique.In this paper, we develop monotone technique combined with coupled upper, lower and quasi solutions in which we construct two monotone sequences of iterate and obtain the existence of solution of nonlinear boundary value problem(BVP) for weakly coupled system of ψ-Caputo fractional differential equations.

The paper is organized as follows. In section 2 basic definitions, assumptions and important lemmas are given. Upper, lower and quasi solutions of nonlinear boundary value problems for coupled system of ψ- Caputo fractional differential equations are introduced.Section 3 is devoted for the development of monotone iterative scheme and monotone property. The existence of solution of nonlinear boundary value problems for weakly coupled system of ψ- Caputo fractional differential equations is proved.

2. Definitions and Basic Results

In this section, we introduce some basic definitions, assumptions and important lemmas which are useful for further discussion. Let $J=[0,T]$ be a finite interval on the real axis $\mathbb{R}$ and $\alpha>0$. Fractional integrals and fractional derivatives of a function $x(t)$ with respect to another function ψ are defined as follows [3, 21]. We begin with the definition of left sided ψ-Riemann - Liouville fractional integral.

Definition 2.1. *The left sided ψ- Riemann - Liouville fractional integral of order $\alpha>0$ for an integrable function $x(t):J\to\mathbb{R}$ with respect to another function $\psi:J\to\mathbb{R}$ which is an* *increasing on J and $\psi(t) \in C^1(J)$ such that $\psi'(t) \neq 0$, for all $t \in J$, is defined by*

$$I_{0^+}^{\alpha;\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(t)(\psi(t) - \psi(s))^{\alpha-1} x(s)ds,$$

where Γ is the Gamma function.

Definition 2.2. [3] *Let $n \in \mathbb{N}$ and let $\psi, x \in C^n(J, \mathbb{R})$ be two functions such that ψ is an increasing function on J and $\psi'(t) \neq 0$, for all $t \in J$. The left ψ- Riemann - Liouville fractional derivative of a function $x(t)$ of order $\alpha > 0$ is defined by*

$$\begin{aligned} D_{0^+}^{\alpha;\psi} x(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dx} \right)^n I_{0^+}^{n-\alpha;\psi} x(t), \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dx} \right)^n \int_0^t \psi'(t)(\psi(t) - \psi(s))^{n-\alpha-1} x(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3. [3] *Let $n \in \mathbb{N}$ and let $\psi, x \in C^n(J, \mathbb{R})$ be two functions such that ψ is an increasing function on J and $\psi'(t) \neq 0$, for all $t \in J$. The left ψ-Caputo fractional derivative of a function $x(t)$ of order $\alpha > 0$ is defined by*

$$ {}^cD_{0^+}^{\alpha;\psi} x(t) = I_{0^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dx} \right)^n x(t), \tag{2.1}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}, n = \alpha$ for $\alpha \in \mathbb{N}$.

We denote symbolically $\left(\frac{1}{\psi'(t)} \frac{d}{dx}\right)^n x(t)$ by $x_\psi^{[n]}(t)$ then the left ψ-Caputo fractional derivative of order α of $x(t)$ can be written as

$$ {}^cD_{0^+}^{\alpha;\psi} x(t) = I_{0^+}^{n-\alpha;\psi} x_\psi^{[n]}(t). \tag{2.2}$$

From (2.1) and (2.2), we observe that the left ψ-Caputo fractional derivative of order α of a function $x(t)$ can be expressed as

$$ {}^cD_{0^+}^{\alpha;\psi} x(t) = \begin{cases} \int_0^t \frac{\psi'(t)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} x_\psi^{[n]}(s)ds, & \text{if } \alpha \notin \mathbb{N}, \\ x_\psi^{[n]}(s), & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

Similarly, we can define right ψ-Riemann - Liouville and right ψ-Caputo fractional derivatives. The relation between the ψ- Riemann - Liouville fractional derivative of order α and the ψ-Caputo fractional derivative of order α of a function $x(t)$ is expressed as

$$ {}^{c}D_{0^+}^{\alpha;\psi}x(t) = D_{0^+}^{\alpha;\psi}\left[x(t) - \sum_{k=0}^{n-1}\frac{x_{\psi}^{[n]}(0)}{k!}(\psi(t)-\psi(0))^{k}\right], $$

(See for details [3]).

Consider weakly coupled system of ψ-Caputo fractional differential equations

$$ \begin{aligned} {}^{c}D_{t}^{\alpha;\psi}u_1(t) &= F_1(t,u_1,u_2), \\ {}^{c}D_{t}^{\alpha;\psi}u_2(t) &= F_2(t,u_1,u_2), \end{aligned} \qquad t \in (0,T], \tag{2.3} $$

with nonlinear boundary conditions

$$ G_1(u_1(0),u_1(T)) = 0 = G_2(u_2(0),u_2(T)), \tag{2.4} $$

where ${}^{c}D_{t}^{\alpha;\psi}$ is the ψ- Caputo fractional derivative of order $\alpha \in (0,1]$, $F_i(t,u_1,u_2) : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is real valued continuous function, $i = 1,2$.

Definition 2.4. *: A pair of functions* $\big(u_1(t),u_2(t)\big) \in C([0,T])$ *is solution of nonlinear BVP* (2.3)-(2.4) *if* $(i)\, u_1(t),u_2(t)$ *satisfy the nonlinear weakly coupled system of* ψ*- Caputo fractional differential equations* (2.3) *and* $(ii)\, u_1(t),u_2(t)$ *satisfy the nonlinear boundary conditions* (2.4).

Now we introduce the notion of upper, lower and quasi solutions.

Definition 2.5. *A pair of functions* $(\xi_1,\xi_2) \in C([0,T])$ *is called an upper solution of weakly coupled system of* ψ*- Caputo fractional differential equations* (2.3) *if it satisfies the differential inequalities,*

$$ \begin{aligned} {}^{c}D_{t}^{\alpha;\psi}\xi_1 &\geq F_1(t,\xi_1,\xi_2), \\ {}^{c}D_{t}^{\alpha;\psi}\xi_2 &\geq F_2(t,\xi_1,\xi_2), \end{aligned} \qquad t \in (0,T]. $$

Definition 2.6. *A pair of functions* $(\eta_1,\eta_2) \in C([0,T])$ *is called lower solution of weakly coupled system of* ψ*-Caputo fractional differential equations* (2.3) *if it satisfies the differential inequalities,*

$$ \begin{aligned} {}^{c}D_{t}^{\alpha;\psi}\eta_1 &\leq F_1(t,\eta_1,\eta_2), \\ {}^{c}D_{t}^{\alpha;\psi}\eta_2 &\leq F_2(t,\eta_1,\eta_2), \end{aligned} \qquad t \in (0,T]. $$

Definition 2.7. *Pairs of functions* $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2), \hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2) \in C([0,T])$ *with* $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ *are called ordered upper and lower solutions of the nonlinear BVP* (2.3)-(2.4) *if they satisfy the system of fractional differential inequalities*

$$\left.\begin{aligned} {}^cD_t^{\alpha;\psi}\tilde{u}_1 &\geq F_1(t, \tilde{u}_1, \tilde{u}_2), \\ {}^cD_t^{\alpha;\psi}\tilde{u}_2 &\geq F_2(t, \tilde{u}_1, \tilde{u}_2), \\ {}^cD_t^{\alpha;\psi}\hat{u}_1 &\leq F_1(t, \hat{u}_1, \hat{u}_2), \\ {}^cD_t^{\alpha;\psi}\hat{u}_2 &\leq F_2(t, \hat{u}_1, \hat{u}_2), \end{aligned}\right. \qquad t \in (0,T],$$

and nonlinear boundary conditions,

$$\begin{aligned} G_1(\hat{u}_1(0), \tilde{u}_1(T)) \leq 0 \leq G_1(\tilde{u}_1(0), \hat{u}_1(T)), \\ G_2(\hat{u}_2(0), \tilde{u}_2(T)) \leq 0 \leq G_2(\tilde{u}_2(0), \hat{u}_2(T)). \end{aligned}$$

In what follows, we assume the relation between upper and lower solutions such that

$$(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2), \quad t \in [0,T]. \tag{2.5}$$

We define functional interval as follows.

Definition 2.8. *Let* $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2)$ *and* $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2)$ *be any two functions with* $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ *then we define the (functional interval)sector*

$$\langle \hat{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle = \big\{ (u_1, u_2) \in C([0,T]) : (\hat{u}_1, \hat{u}_2) \leq (u_1, u_2) \leq (\tilde{u}_1, \tilde{u}_2) \big\}.$$

Definition 2.9. *Pairs of functions* $\boldsymbol{v} = (v_1, v_2)$ *,* $\boldsymbol{w} = (w_1, w_2) \in C([0,T])$ *are called coupled quasi solutions of the nonlinear BVP* (2.3)-(2.4) *if they are solutions of the nonlinear, weakly coupled system of* ψ*-Caputo fractional differential equations* (2.3) *and*

$$\begin{aligned} \hat{u}_1 \leq w_1 \leq v_1 \leq \tilde{u}_1, \quad t \in [0,T], \\ \hat{u}_2 \leq w_2 \leq v_2 \leq \tilde{u}_2, \quad t \in [0,T], \end{aligned} \tag{2.6}$$

as well as satisfy nonlinear boundary conditions

$$\begin{aligned} G_1\big(v_1(0), w_1(T)\big) = 0 = G_1\big(w_1(0), v_1(T)\big), \\ G_2\big(v_2(0), w_2(T)\big) = 0 = G_2\big(w_2(0), v_2(T)\big), \end{aligned} \tag{2.7}$$

where $(\tilde{u}_1, \tilde{u}_2), (\hat{u}_1, \hat{u}_2)$ *are coupled upper lower solutions of the nonlinear BVP* (2.3)-(2.4).

Suppose that the nonlinear function $F_i(t,u_1,u_2)$ satisfies following assumptions:

$(\mathbb{A}_1)$ Suppose that there exist constants $\underline{c}_1$ and $\underline{c}_2$ such that for every pair (u_1,u_2), (u_1^*,u_2^*) in the sector $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$, a pair of functions (F_1,F_2) satisfies one sided Lipschitz condition in u_1,u_2;

$$\begin{aligned} &F_1(t,u_1,u_2)-F_1(t,u_1^*,u_2) \geq -\underline{c}_1(u_1-u_1^*) \text{ for } \hat{u_1} \leq u_1^* \leq u_1 \leq \tilde{u_1}, \\ &F_2(t,u_1,u_2)-F_2(t,u_1,u_2^*) \geq -\underline{c}_2(u_2-u_2^*) \text{ for } \hat{u_2} \leq u_2^* \leq u_2 \leq \tilde{u_2}. \end{aligned} \tag{2.8}$$

Note that in view of Lipschitz condition (2.8), the functions H_1 and H_2 given by

$$H_i(t,u_1,u_2) = \underline{c}_i u_i + F_i(t,u_1,u_2), (i=1,2),$$

are monotone nonincreasing in u_1 and u_2 respectively for$(u_1,u_2) \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.

$(\mathbb{A}_2)$ A function $F_i(t,u_1,u_2) \in C(J \times R^2, R)$ is said to be quasimonotone nondecreasing if

$$F_i(t,u_1,u_2) \leq F_i(t,v_1,v_2) \text{ for } u_i = v_i \text{ and } u_j \leq v_j, i \neq j, i=j=1,2 \tag{2.9}$$

hold.

$(\mathbb{A}_3)$ A function $F_i(t,u_1,u_2) \in C(J \times R^2, R)$ is said to be quasimonotone nonincreasing if

$$F_i(t,u_1,u_2) \geq F_i(t,v_1,v_2) \text{ for } u_i = v_i \text{ and } u_j \leq v_j, i \neq j, i=j=1,2 \tag{2.10}$$

hold.

$(\mathbb{A}_4)$ A function $G_i(x,y)$ satisfies following condition:

$$\begin{aligned} &G_1(x,.) \text{is nonincreasing} \forall x \in \mathbb{R} \textit{ and } G_1(.,y) \text{ is nondecreasing} \forall y \in \mathbb{R}, \\ &G_2(x,.) \text{is nonincreasing} \forall x \in \mathbb{R} \textit{ and } G_2(.,y) \text{is nondecreasing} \forall y \in \mathbb{R}. \end{aligned} \tag{2.11}$$

The following lemmas play an important role in further development.

Lemma 2.10. [9] *The initial value problem for $\psi-$ Caputo fractional differential equation*

$$^cD_t^{\alpha;\psi} x(t) + rx(t) = h(t), \quad t \in [0,T],$$

$$x(0) = x_0,$$

has unique solution

$$x(t) = x_0\mathbb{E}_{\alpha,1}(-r(\psi(t)-\psi(0))^{\alpha})+$$
$$\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-r(\psi(t)-\psi(0))^{\alpha})h(s)ds, \quad t\in[0,T],$$

where $\mathbb{E}_{\alpha,\beta}$ is the two-parametric Mittag-Leffler function defined in [21].

Lemma 2.11. [9]*(Comparison result) Suppose that $x(t)\in((0,T],\mathbb{R})$ satisfies*

$$(i) \quad {}^cD_t^{\alpha;\psi}x(t)+rx(t)\geq 0, \quad 0<\alpha\leq 1, \quad t\in[0,T],$$
$$(ii) \quad x(0)\geq 0,$$

where $r\in\mathbb{R}$, then $x(t)\geq 0$ for any $t\in[0,T]$.

3. Existence Result

In this section,we develop monotone iterative scheme and prove the existence of solution of the nonlinear BVP (2.3)-(2.4).

Theorem 3.1. *Let assumptions $(\mathbb{A}_1),(\mathbb{A}_2)$ and $(\mathbb{A}_4)$ hold. Suppose that functions $(\tilde{u}_1,\tilde{u}_2)$ and $(\hat{u}_1,\hat{u}_2)$ are coupled upper and lower solutions of the nonlinear BVP* (2.3)-(2.4)*,such that* (2.5) *holds. Then solution of the nonlinear BVP* (2.3)-(2.4) *exist in the sector $\langle\hat{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle$.*

Proof. In order to prove the result,we proceed as follows. First we find maximal solution (v_1,v_2) and minimal solution (w_1,w_2) of the following IVP for nonlinear weakly coupled system of fractional differential equations

$$\begin{aligned} &{}^cD_t^{\alpha;\psi}u_1 = F_l(t,u_1,u_2),\ t\in(0,T]; \quad u_1(0)=\gamma_1,\\ &{}^cD_t^{\alpha;\psi}u_2 = F_2(t,u_1,u_2),\ t\in(0,T]; \quad u_2(0)=\gamma_2, \end{aligned} \tag{3.1}$$

where $\hat{u_1}(0)\leq\gamma_1\leq\tilde{u_1}(0)$ and $\hat{u_2}(0)\leq\gamma_2\leq\tilde{u_2}(0)$. Further we show that maximal solution (v_1,v_2) and minimal solution (w_1,w_2) are truly coupled quasi solutions of the nonlinear BVP (2.3)-(2.4).Therefore,first we introduce the notion of upper lower solutions of the IVP (3.1) and then propose the monotone iterative scheme to develop monotone technique.

Definition 3.2. *Pairs of functions* $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2), \hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2) \in C([0,T])$ *with* $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ *are called ordered upper and lower solutions of the IVP* (3.1) *if they satisfy the system of fractional differential inequalities*

$$\begin{aligned} {}^cD_t^{\alpha;\psi}\tilde{u}_1 &\geq F_1(t, \tilde{u}_1, \tilde{u}_2), \\ {}^cD_t^{\alpha;\psi}\tilde{u}_2 &\geq F_2(t, \tilde{u}_1, \tilde{u}_2), \\ {}^cD_t^{\alpha;\psi}\hat{u}_1 &\leq F_1(t, \hat{u}_1, \hat{u}_2), \\ {}^cD_t^{\alpha;\psi}\hat{u}_2 &\leq F_2(t, \hat{u}_1, \hat{u}_2), \end{aligned} \qquad t \in (0,T],$$

and initial conditions

$$\begin{aligned} \tilde{u}_1(0) \geq \gamma_1 \geq \hat{u}_1(0), \\ \tilde{u}_2(0) \geq \gamma_2 \geq \hat{u}_2(0). \end{aligned}$$

Further observe that the IVP (3.1) is equivalent to the following IVP

$$\begin{aligned} {}^cD_t^{\alpha;\psi}v_1 + \underline{c}_1 v_1 &= \underline{c}_1 v_1 + F_l(t, v_1, v_2),\ t \in (0,T]; \quad v_1(0) = \gamma_1, \\ {}^cD_t^{\alpha;\psi}v_2 + \underline{c}_2 v_2 &= \underline{c}_2 v_2 + F_2(t, v_1, v_2),\ t \in (0,T]; \quad v_2(0) = \gamma_2. \end{aligned}$$

Monotone Iterative Scheme: Consider the following monotone iterative scheme

$$\begin{aligned} {}^cD_t^{\alpha;\psi}v_1^{(k)} + \underline{c}_1 v_1^{(k)} &= \underline{c}_1 v_1^{(k-1)} + F_l(t, v_1^{(k-1)}, v_2^{(k-1)}),\ t \in (0,T] \\ v_1^{(k)}(0) &= \gamma_1, \\ {}^cD_t^{\alpha;\psi}v_2^{(k)} + \underline{c}_2 v_2^{(k)} &= \underline{c}_2 v_2^{(k-1)} + F_2(t, v_1^{(k)}, v_2^{(k-1)}),\ t \in (0,T] \\ v_2^{(k)}(0) &= \gamma_2. \end{aligned} \tag{3.2}$$

Since for each k, the above set of equations consist of two linear uncoupled fractional differential equations with initial conditions. The existence theory of linear IVP for fractional differential equation is studied in [9]. Therefore, existence of solution $\{v_1^{(k)}, v_2^{(k)}\}$ of the linear IVP (3.2) follows immediately.Choose $(v_1^{(0)}, v_2^{(0)})$ as an initial iteration, we construct a sequence $\{v_1^{(k)}, v_2^{(k)}\}$ from above iterative scheme (3.2). If we choose an initial iteration as an upper solution $(\overline{v}_1^{(0)}, \overline{v}_2^{(0)})$ then we get upper sequence $\{\overline{v}_1^{(k)}, \overline{v}_2^{(k)}\}$ of solution.If we choose an initial iteration as lower solution $(\underline{v}_1^{(0)}, \underline{v}_2^{(0)})$ then we get lower sequence $\{\underline{v}_1^{(k)}, \underline{v}_2^{(k)}\}$ of solution.

Monotone Property: We claim that the sequences $\{\overline{v}_1^{(k)}, \overline{v}_2^{(k)}\}$ and $\{\underline{v}_1^{(k)}, \underline{v}_2^{(k)}\}$ possess the monotone property

$$\begin{aligned} &\hat{u}_1 \le \underline{v}_1^{(k)} \le \underline{v}_1^{(k+1)} \le \overline{v}_1^{(k+1)} \le \overline{v}_1^{(k)} \le \tilde{u}_1, \\ &\hat{u}_2 \le \underline{v}_2^{(k)} \le \underline{v}_2^{(k+1)} \le \overline{v}_2^{(k+1)} \le \overline{v}_2^{(k)} \le \tilde{u}_2, \end{aligned} \quad t \in [0,T]. \tag{3.3}$$

We prove our claim by applying principle of mathematical induction: Define $r_1 = \underline{v}_1^{(1)} - \underline{v}_1^{(0)}$ where $\underline{v}_1^{(0)} = \hat{u}_1$. By definition of a lower solution, iterative scheme (3.2) and condition (2.5), we have

$$\begin{aligned} {}^cD_t^{\alpha;\psi} r_1 + \underline{c}_1 r_1 &= {}^c D_t^{\alpha;\psi}(\underline{v}_1^{(1)} - \underline{v}_1^{(0)}) + \underline{c}_1(\underline{v}_1^{(1)} - \underline{v}_1^{(0)}), \\ &= {}^c D_t^{\alpha;\psi}\hat{u}_1 - F_l(t, \hat{u}_1, \hat{u}_2) \ge 0, t \in (0,T] \\ \text{and } r(0) = \underline{v}_1^{(1)}(0) - \underline{v}_1^{(0)}(0) &= \underline{v}_1^{(1)}(0) - \hat{u}_1(0) = \gamma_1 - \hat{u}_1(0) \ge 0. \end{aligned}$$

Applying Lemma 2.11, we get $r_1(t) \ge 0$ implies that $\underline{v}_1^{(1)} \ge \hat{u}_1$ in [0,T]. On similar lines, we can show that $\underline{v}_2^{(1)} \ge \hat{u}_2$. Define $r_2(t) = \overline{v}_2^{(0)} - \overline{v}_2^{(1)}$ where $\overline{v}_2^{(0)} = \tilde{u}_2$. By definition of an upper solution,iterative scheme (3.2),condition (2.5), we have

$$\begin{aligned} {}^cD_t^{\alpha;\psi} r_2 + \underline{c}_2 r_2 &= {}^c D_t^{\alpha;\psi}(\overline{v}_2^{(0)} - \overline{v}_2^{(1)}) + \underline{c}_2(\overline{v}_2^{(0)} - \overline{v}_2^{(1)}), \\ &= {}^c D_t^{\alpha;\psi}\tilde{u}_2 - F_2(t, \tilde{u}_1, \tilde{u}_2) \ge 0, \\ \text{and} \quad r_2(0) &= \overline{v}_2^{(0)}(0) - \overline{v}_2^{(1)}(0) \ge 0. \end{aligned}$$

Applying Lemma 2.11, we get $r_2(t) \ge 0$ implies that $\tilde{u}_2 \ge \overline{v}_2^{(1)}$ in [0,T]. On similar lines, we can show that $\tilde{u}_1 \ge \overline{v}_1^{(1)}$. Define $r_1^{(1)} = \overline{v}_1^{(1)} - \underline{v}_1^{(1)}$. By iterative scheme (3.2) and condition (2.5)and function F_1 is Lipchitzian as well as quasimonotone nondecreasing, we have

$$\begin{aligned} {}^cD_t^{\alpha;\psi} r_1^{(1)} + \underline{c}_1 r_1^{(1)} &= {}^c D_t^{\alpha;\psi}\overline{v}_1^{(1)} + \underline{c}_1\overline{v}_1^{(1)} - [{}^cD_t^{\alpha;\psi}\underline{v}_1^{(1)} + \underline{c}_1\underline{v}_1^{(1)}], \\ &= \underline{c}_1\big(\overline{v}_1^{(0)} - \underline{v}_1^{(0)}\big) + F_l(t, \overline{v}_1^{(0)}, \overline{v}_2^{(0)}) - F_l\big(t, \underline{v}_1^{(0)}, \overline{v}_2^{(0)}\big) + \\ &\qquad F_1\big(t, \underline{v}_1^{(0)}, \overline{v}_2^{(0)}\big) - F_l\big(t, \underline{v}_1^{(0)}, \underline{v}_2^{(0)}\big) \ge 0,\ t \in (0,T], \\ \text{and} \quad r_1^{(1)}(0) &= \overline{v}_1^{(1)}(0) - \underline{v}_1^{(1)}(0) = 0. \end{aligned}$$

Applying Lemma 2.11, we get $r_1^{(1)}(t) \geq 0$. This implies that $\overline{v}_1^{(1)} \geq \underline{v}_1^{(1)}$ in [0,T]. On similar lines, we can show that $\overline{v}_2^{(1)} \geq \underline{v}_2^{(1)}$. Thus we have

$$\underline{u}_1^{(0)} \leq \underline{v}_1^{(1)} \leq \overline{v}_1^{(1)} \leq \overline{u}_1^{(0)}, t \in [0,T],$$

$$\underline{u}_2^{(0)} \leq \underline{v}_2^{(1)} \leq \overline{v}_2^{(1)} \leq \overline{u}_2^{(0)}, t \in [0,T].$$

The result is true for k = 1.Assume by induction that the result is true for k

$$\hat{u}_1 \leq \underline{v}_1^{(k)} \leq \overline{v}_1^{(k)} \leq \tilde{u}_1, t \in [0,T],$$

$$\hat{u}_2 \leq \underline{v}_2^{(k)} \leq \overline{v}_2^{(k)} \leq \tilde{u}_2, t \in [0,T].$$

We prove that the result is true for k+1, i.e.

$$\underline{v}_1^{(k)} \leq \underline{v}_1^{(k+1)} \leq \overline{v}_1^{(k+1)} \leq \overline{v}_1^{(k)}, t \in [0,T],$$

$$\underline{v}_2^{(k)} \leq \underline{v}_2^{(k+1)} \leq \overline{v}_2^{(k+1)} \leq \overline{v}_2^{(k)}, t \in [0,T].$$

Define $r_1^{(k)} = \underline{v}_1^{(k+1)} - \underline{v}_1^{(k)}$. By iterative scheme (3.2), condition (2.5) and function f_1 is Lipschitzian as well as quasimonotone nondecreasing, we have

$$\begin{aligned} {}^cD_t^{\alpha;\psi} r_1^{(k)} + \underline{c}_1 r_1^{(k)} &= {}^c D_t^{\alpha;\psi} \underline{v}_1^{(k+1)} + \underline{c}_1 \underline{v}_1^{(k+1)} - [{}^cD_t^{\alpha;\psi} \underline{v}_1^{(k)} + \underline{c}_1 \underline{v}_1^{(k)}], \\ &= \underline{c}_1 [\underline{v}_1^{(k)} - \underline{v}_1^{(k-1)}] + F_l(\underline{v}_1^{(k)}, \underline{v}_2^{(k)}) - F_l(\underline{v}_1^{(k-1)}, \underline{v}_2^{(k)}) + \\ &\qquad F_l(\underline{v}_1^{(k-1)}, \underline{v}_2^{(k)}) - F_l(\underline{v}_1^{(k-1)}, \underline{v}_2^{(k-1)}) \geq 0,\ t \in (0,T]. \end{aligned}$$

$$\text{Also} \quad r_1^{(k)}(0) = \underline{v}_1^{(k+1)}(0) - \underline{v}_1^{(k)}(0) = 0.$$

Applying Lemma 2.11, we get $r_1^{(k)}(t) \geq 0$. This implies that $\underline{v}_1^{(k+1)} \geq \underline{v}_1^{(k)}$ in [0,T].On similar lines we can show that $\underline{v}_2^{(k+1)} \geq \underline{v}_2^{(k)}$, $\overline{v}_2^{(k)} \geq \overline{v}_2^{(k+1)}$, $\overline{v}_1^{(k)} \geq \overline{v}_1^{(k+1)}$. We can show $\underline{v}_1^{(k+1)} \leq \overline{v}_1^{(k+1)}$ and $\underline{v}_2^{(k+1)} \leq \overline{v}_2^{(k+1)}$. The monotone property (3.3)follows by induction for all k. From monotone property (3.3), we conclude that the sequence $\{\overline{v}_1^{(k)}, \overline{v}_2^{(k)}\}$ is monotone nonincreasing and bounded from below hence converges to some limit function. Also the sequence $\{\underline{v}_1^{(k)}, \underline{v}_2^{(k)}\}$ monotone nondecreasing and bounded from above hence converges to some limit function.So the point wise limits

$$\lim_{k\to\infty} \overline{v}_1^{(k)}(t) = v_1(t); \quad \lim_{k\to\infty} \overline{v}_2^{(k)}(t) = v_2(t)$$

and

$$\lim_{k\to\infty} \underline{v}_1^{(k)}(t) = w_1(t); \quad \lim_{k\to\infty} \underline{v}_2^{(k)}(t) = w_2(t),$$

exist and these limits are denoted by (v_1, v_2) and (w_1, w_2) .They are called maximal and minimal solutions respectively and satisfy the monotone property

$$\hat{u}_1 \leq \underline{v}_1^{(k)} \leq \underline{v}_1^{(k+1)} \leq w_1 \leq v_1 \leq \overline{v}_1^{(k+1)} \leq \overline{v}_1^{(k)} \leq \tilde{u}_1, t \in [0,T],$$
$$\hat{u}_2 \leq \underline{v}_2^{(k)} \leq \underline{v}_2^{(k+1)} \leq w_2 \leq v_2 \leq \overline{v}_2^{(k+1)} \leq \overline{v}_2^{(k)} \leq \tilde{u}_2, t \in [0,T].$$

Now we prove that the maximal solution (v_1, v_2) and minimal solution (w_1, w_2) are solutions of IVP(3.1). Let $v_1^{(k)}$ be either $\overline{v}_1^{(k)}$ or $\underline{v}_1^{(k)}$ and $v_2^{(k)}$ be either $\overline{v}_2^{(k)}$ or $\underline{v}_2^{(k)}$. The integral representation of solution of linear problem (3.2) is

$$\begin{aligned} v_1^{(k)}(t) = {} & \gamma_1 E_{\alpha,1}\big(-\underline{c}_1[\psi(t)-\psi(0)]^{\alpha}\big)+ \\ & \int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} E_{\alpha,\alpha}\big(-\underline{c}_1[\psi(t)-\psi(0)]^{\alpha}\big) H_1(v_1^{(k-1)}, v_2^{(k-1)})(s)ds, \\ & \text{where} \quad H_1(v_1^{(k-1)}, v_2^{(k-1)})(t) = \underline{c}_1 v_1^{(k-1)}(t) + F_1(t, v^{(k-1)}, v_2^{(k-1)}). \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} v_2^{(k)}(t) = {} & \gamma_2 E_{\alpha,1}\big(-\underline{c}_2[\psi(t)-\psi(0)]^{\alpha}\big)+ \\ & \int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} E_{\alpha,\alpha}\big(-\underline{c}_2[\psi(t)-\psi(0)]^{\alpha}\big) H_2(v_1^{(k-1)}, v_2^{(k-1)})(s)ds, \\ & \text{where} \quad H_2(v_1^{(k-1)}, v_2^{(k-1)})(t) = \underline{c}_2 v_1^{(k-1)}(t) + F_1(t, v^{(k-1)}, v_2^{(k-1)}). \end{aligned} \tag{3.5}$$

The functions H_1 and $H_2$are continuous and monotone nondecreasing in v_1 and v_2 respectively. The monotone convergence of $(v_1^{(k)}, v_2^{(k)})$ to (v_1, v_2) implies that $\big(H_1(v_1^{(k-1)}, v_2^{(k-1)}), H_2(v_1^{(k)}, v_2^{(k-1)})\big)$ converges to $\big(H_1(v_1, v_2), H_2(v_1, v_2)\big)$. Taking limit as $k \to \infty$, in the equations (3.4) and (3.5)and applying the dominated convergence theorem, we observe that the function (v_1, v_2) satisfies the integral equations

$$\begin{aligned} v_1(t) = {} & \gamma_1 E_{\alpha,1}\big(-\underline{c}_1[\psi(t)-\psi(0)]^{\alpha}\big)+ \\ & \int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} E_{\alpha,\alpha}\big(-\underline{c}_1[\psi(t)-\psi(0)]^{\alpha}\big) H_1(v_1, v_2)(s)ds, \end{aligned}$$

and

$$\begin{aligned} v_2(t) = {} & \gamma_2 E_{\alpha,1}\big(-\underline{c}_2[\psi(t)-\psi(0)]^{\alpha}\big)+ \\ & \int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} E_{\alpha,\alpha}\big(-\underline{c}_2[\psi(t)-\psi(0)]^{\alpha}\big) H_2(v_1, v_2)(s)ds. \end{aligned}$$

We conclude that $(v_1, v_2) \in C([0,T])$ is the integral representation of solution of IVP (3.1)and upper sequence $\{\overline{v}_1^{(k)}, \overline{v}_2^{(k)}\}$ converges monotonically from above to a solution (v_1, v_2) of IVP

(3.1) as well as lower sequence $\{\underline{v}_1^{(k)}, \underline{v}_2^{(k)}\}$ converges monotonically from below to a solution (w_1, w_2) of IVP (3.1) and satisfy the relation $(v_1(t), v_2(t)) \geq (w_1(t), w_2(t)), t \in [0,T]$. Now we prove that maximal and minimal solutions (v_1, v_2) and (w_1, w_2) satisfy the nonlinear boundary conditions(2.7). We know that function $G_1(x, .)$ is nonincreasing in x

$$G_1\big(v_1(0), w_1(T)\big) \leq G_1\big(\hat{u}_1(0), w_1(T)\big), \quad \text{for} \quad \hat{u}_1(0) \leq v_1(0), \tag{3.6}$$

$$G_1\big(v_1(0), w_1(T)\big) \geq G_1\big(\tilde{u}_1(0), w_1(T)\big), \quad \text{for} \quad v_1(0) \leq \tilde{u}_1(0). \tag{3.7}$$

Also we know that function $G_1(., y)$ is nondecreasing in y

$$G_1\big(\hat{u}_1(0), w_1(T)\big) \leq G_1\big(\hat{u}_1(0), \tilde{u}_1(T)\big), \quad \text{for} \quad w_1(T) \leq \tilde{u}_1(T), \tag{3.8}$$

$$G_1\big(\tilde{u}_1(0), w_1(T)\big) \geq G_1\big(\tilde{u}_1(0), \hat{u}_1(T)\big), \quad \text{for} \quad \hat{u}_1(T) \leq w_1(T). \tag{3.9}$$

From equations (3.6),(3.8) and equations (3.7), (3.9), we have

$$G_1\big(v_1(0), w_1(T)\big) \leq G_1\big(\hat{u}_1(0), w_1(T)\big) \leq G_1\big(\hat{u}_1(0), \tilde{u}_1(T)\big) \leq 0, \tag{3.10}$$

$$G_1\big(v_1(0), w_1(T)\big) \geq G_1\big(\tilde{u}_1(0), w_1(T)\big) \geq G_1\big(\tilde{u}_1(0), \hat{u}_1(T)\big) \geq 0. \tag{3.11}$$

From inequalities (3.10) and (3.11), we get

$$G_1\big(v_1(0), w_1(T)\big) = 0. \tag{3.12}$$

On similar lines we prove

$$G_2\big(v_2(0), w_2(T)\big) = 0. \tag{3.13}$$

Now we prove

$$G_2\big(w_2(0), v_2(T)\big) = 0.$$

We know that function $G_2(x, .)$ is nonincreasing in x

$$G_2\big(w_2(0), v_2(T)\big) \leq G_2\big(\hat{u}_2(0), v_2(T)\big), \quad \text{for} \quad w_2(0) \geq \hat{u}_2(0), \tag{3.14}$$

$$G_2\big(w_2(0), v_2(T)\big) \geq G_2\big(\tilde{u}_2(0), v_2(T)\big), \quad \text{for} \quad w_2(0) \leq \tilde{u}_2(0). \tag{3.15}$$

Also we know that function $G_2(.,y)$ is nondecreasing in y

$$G_2\big(\hat{u}_2(0),v_2(T)\big) \le G_2\big(\hat{u}_2(0),\tilde{u}_2(T)\big), \quad \text{for} \quad v_2(T) \le \tilde{u}_2(T), \tag{3.16}$$

$$G_2\big(\tilde{u}_2(0),v_2(T)\big) \ge G_2\big(\tilde{u}_2(0),\hat{u}_2(T)\big), \quad \text{for} \quad v_2(T) \ge \hat{u}_2(T). \tag{3.17}$$

From equations (3.14),(3.16) and equations (3.15), (3.17), we have

$$G_2\big(w_2(0),v_2(T)\big) \le G_2\big(\hat{u}_2(0),v_2(T)\big) \le G_2\big(\hat{u}_2(0),\tilde{u}_2(T)\big) \le 0, \tag{3.18}$$

$$G_2\big(w_2(0),v_2(T)\big) \ge G_2\big(\tilde{u}_2(0),v_2(T)\big) \ge G_2\big(\tilde{u}_2(0),\hat{u}_2(T)\big) \ge 0. \tag{3.19}$$

From inequalities (3.18) and (3.19), we get

$$G_2\big(w_2(0),v_2(T)\big) = 0. \tag{3.20}$$

On similar lines we prove

$$G_1\big(w_1(0),v_1(T)\big) = 0. \tag{3.21}$$

From equations (3.12) and (3.21) we get

$$G_1\big(v_1(0),w_1(T)\big) = 0 = G_1\big(w_1(0),v_1(T)\big).$$

From equations (3.13) and (3.20) we get

$$G_2\big(v_2(0),w_2(T)\big) = 0 = G_2\big(w_2(0),v_2(T)\big).$$

Thus the maximal solution (v_1,v_2) and minimal solution (w_1,w_2) are coupled quasi solutions of the nonlinear BVP (2.3)-(2.4). The proof is complete. □

Theorem 3.3. *Let assumptions* $(\mathbb{A}_1),(\mathbb{A}_3)$ *and* $(\mathbb{A}_4)$ *hold.Suppose that the functions* $(\tilde{u}_1,\tilde{u}_2)$ *and* $(\hat{u}_1,\hat{u}_2)$ *are coupled upper and lower solutions of the nonlinear BVP* (2.3)-(2.4)*,such that* (2.5) *holds. Then solution of the nonlinear BVP* (2.3)-(2.4) *exist in the sector* $\langle \hat{\boldsymbol{u}},\tilde{\boldsymbol{u}}\rangle$.

Proof: It is similar to Theorem 3.1.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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