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K-OPERATOR FRAME FOR $Hom^{*}_{\mathscr{A}}(\mathscr{X})$

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Abstract. In this work, we introduce the notion of K-operator frame for the set of all adjointable operators $Hom^{*}_{\mathscr{A}}(\mathscr{X})$ on a Hilbert pro-C^{*}-module $\mathscr{X}$. We also study the tensor product of K-operator frame for Hilbert pro-C^{*}-modules. Finally, we establish its dual and some properties.

Keywords: frame;; K-operator frame; pro-C^{*}-algebra; Hilbert pro-C^{*}-modules; tensor product.

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1. Introduction

In 1952, Duffin and Schaeffer [2] introduced the notion of frame in nonharmonic Fourier analysis. In 1986 the work of Duffin and Schaeffer was continued by Grossman and Meyer [6]. After their works, the theory of frame was developed and has been popular.

The notion of frame on Hilbert space has already been successfully extended to pro-C^{*}-algebras and Hilbert modules. In 2008, Joita [8] proposed frames of multipliers in Hilbert

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pro-C^*-modules and showed that many properties of frames in Hilbert C^*-modules are valid for frames of multipliers in Hilbert modules over pro-C^*-algebras.

Operator frames for $B(\mathscr{H})$ is a new notion of frames that Li and Cio introduced in [10] and generalized by Rossafi in [14]. In this article we introduce the notion of K-operator frame for the space $Hom^*_{\mathscr{A}}(\mathscr{X})$ of all adjointable operators on a Hilbert pro-C^*-module for $\mathscr{X}$.

This article is organized as follows: In section 2, we recall some fundamental definitions and notations of Hilbert pro-C^*-modules. In section 3, we give the defintion of K-operator frame and some properties. In section 4, we investigate the tensor product of Hilbert pro-C^*-modules, we show that tensor product of K-operator frames for Hilbert pro-C^*-modules $\mathscr{X}$ and $\mathscr{Y}$, present K-operator frame for $\mathscr{X} \otimes \mathscr{Y}$. Lastly, the dual of K-operator frame and some properties are discussed.

2. Preliminaries

The basic information about pro-C^*-algebras can be found in the works [3, 4, 5, 11, 7, 12, 13].

Recall that a pro-C^*-algebra is a generalization of the notion of a C^*-algebra and it is defined as a complete Hausdorff complex topological $*$-algebra $\mathscr{A}$ whose topology is determined by its continuous C^*-seminorms in the sens that a net $\{a_\alpha\}$ converges to 0 if and only if $p(a_\alpha)$ converges to 0 for all continuous C^*-seminorm p on $\mathscr{A}$ (see [7, 9, 13]), and we have:

1) $p(ab) \leq p(a)p(b)$
2) $p(a^*a) = p(a)^2$

for all $a, b \in \mathscr{A}$

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

We denote by $sp(a)$ the spectrum of a such that: $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathscr{A}} - a$ is not invertible$\}$ for all $a \in \mathscr{A}$. Where $\mathscr{A}$ is unital pro-C^*-algebra with unite $1_{\mathscr{A}}$.

The set of all continuous C^*-seminorms on $\mathscr{A}$ is denoted by $S(\mathscr{A})$. If $\mathscr{A}^+$ denotes the set of all positive elements of $\mathscr{A}$, then $\mathscr{A}^+$ is a closed convex C^*-seminorms on $\mathscr{A}$.

Example 2.1. Every C^*-algebra is a pro-C^*-algebra.

Proposition 2.2. *Let $\mathscr{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathscr{A}}$. Then for any $p \in S(\mathscr{A})$, we have:*

(1) $p(a) = p(a^*)$ for all $a \in A$

(2) $p(1_{\mathscr{A}}) = 1$

(3) If $a, b \in \mathscr{A}^+$ and $a \leq b$, then $p(a) \leq p(b)$

(4) If $1_{\mathscr{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathscr{A}}$

(5) If $a, b \in \mathscr{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$

(6) If $a, b, c \in \mathscr{A}$ and $a \leq b$ then $c^*ac \leq c^*bc$

(7) If $a, b \in \mathscr{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$

Definition 2.3. [13] A pre-Hilbert module over pro-C^*-algebra $\mathscr{A}$, is a complex vector space E which is also a left $\mathscr{A}$-module compatible with the complex algebra structure, equipped with an $\mathscr{A}$-valued inner product $\langle ., . \rangle$ $E \times E \to \mathscr{A}$ which is $\mathbb{C}$-and $\mathscr{A}$-linear in its first variable and satisfies the following conditions:

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$

2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$

3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$

for every $\xi, \eta \in E$. We say E is a Hilbert $\mathscr{A}$-module (or Hilbert pro-C^*-module over $\mathscr{A}$). If E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)} \quad \xi \in E, p \in S(\mathscr{A})$$

Let $\mathscr{A}$ be a pro-C^*-algebra and let $\mathscr{X}$ and $\mathscr{Y}$ be Hilbert $\mathscr{A}$-modules and assume that I and J be countable index sets. A bounded $\mathscr{A}$-module map from $\mathscr{X}$ to $\mathscr{Y}$ is called an operators from $\mathscr{X}$ to $\mathscr{Y}$. We denote the set of all operator from $\mathscr{X}$ to $\mathscr{Y}$ by $Hom_{\mathscr{A}}(\mathscr{X}, \mathscr{Y})$.

Definition 2.4. An $\mathscr{A}$-module map $T : \mathscr{X} \longrightarrow \mathscr{Y}$ is adjointable if there is a map $T^* : \mathscr{Y} \longrightarrow \mathscr{X}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in \mathscr{X}, \eta \in \mathscr{Y}$, and is called bounded if for all $p \in S(\mathscr{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathscr{Y}}(T\xi) \leq M_p \bar{p}_{\mathscr{X}}(\xi)$ for all $\xi \in \mathscr{X}$.

We denote by $Hom^*_{\mathscr{A}}(\mathscr{X}, \mathscr{Y})$, the set of all adjointable operator from $\mathscr{X}$ to $\mathscr{Y}$ and $Hom^*_{\mathscr{A}}(\mathscr{X}) = Hom^*_{\mathscr{A}}(\mathscr{X}, \mathscr{X})$

Definition 2.5. Let $\mathscr{A}$ be a pro-C^*-algebra and $\mathscr{X},\mathscr{Y}$ be two Hilbert $\mathscr{A}$-modules. The operator $T:\mathscr{X}\to\mathscr{Y}$ is called uniformly bounded below, if there exists $C>0$ such that for each $p\in S(\mathscr{A})$,

$$\bar{p}_{\mathscr{Y}}(T\xi)\leqslant C\bar{p}_{\mathscr{X}}(\xi),\quad \text{for all } \xi\in\mathscr{X}$$

and is called uniformly bounded above if there exists $C'>0$ such that for each $p\in S(\mathscr{A})$,

$$\bar{p}_{\mathscr{Y}}(T\xi)\geqslant C'\bar{p}_{\mathscr{X}}(\xi),\quad \text{for all } \xi\in\mathscr{X}$$

$$\|T\|_\infty=\inf\{M: M \text{ is an upper bound for } T\}$$

$$\hat{p}_{\mathscr{Y}}(T)=\sup\{\bar{p}_{\mathscr{Y}}(T(x)):\xi\in\mathscr{X},\quad \bar{p}_{\mathscr{X}}(\xi)\leqslant 1\}$$

It's clear to see that, $\hat{p}(T)\leqslant\|T\|_\infty$ for all $p\in S(\mathscr{A})$.

Proposition 2.6. [1]. *Let $\mathscr{X}$ be a Hilbert module over pro-C^*-algebra $\mathscr{A}$ and T be an invertible element in $Hom^*_{\mathscr{A}}(\mathscr{X})$ such that both are uniformly bounded. Then for each $\xi\in\mathscr{X}$,*

$$\left\|T^{-1}\right\|_\infty^{-2}\langle\xi,\xi\rangle\leq\langle T\xi,T\xi\rangle\leq\|T\|_\infty^2\langle\xi,\xi\rangle.$$

Lemma 2.7. *Let $\mathscr{X}$ be Hilbert $\mathscr{A}$-module over a pro-C^*-algebra $\mathscr{A}$. Let $T,S\in Hom^*_{\mathscr{A}}(\mathscr{X})$. If $Rang(S)$ is closed, then the following statements are equivalent:*

(i) *$Rang(T)\subseteq Rang(S)$.*

(ii) *$\lambda TT^*\leq SS^*$ for some $\lambda>0$.*

(iii) *There exists $Q\in Hom^*_{\mathscr{A}}(\mathscr{X})$ such that $T=SQ$.*

Similar to C^*-algebra the $*$-homomorphism between two pro-C^*-algebra is increasing

Lemma 2.8. *If $\varphi:\mathscr{A}\longrightarrow\mathscr{B}$ is an $*$-homomorphism between pro-$\mathscr{C}^*$-algebras, then φ is increasing, that is, if $a\leq b$, then $\varphi(a)\leq\varphi(b)$.*

3. K-Operator Frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$

Definition 3.1. Let $\mathscr{X}$ be a Hilbert module over a pro-C^*-algebra $\mathscr{A}$ and let $\{T_i\}_{i\in I}$ be a family of adjointable operators for $\mathscr{X}$. $\{T_i\}_{i\in I}$ is called K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$, if there exists positive constants $A,B>0$ such that

$$A\langle K^*\xi,K^*\xi\rangle\leq\sum_{i\in I}\langle T_i\xi,T_i\xi\rangle\leq B\langle\xi,\xi\rangle,\forall\xi\in\mathscr{X}. \tag{3.1}$$

The numbers A and B are called lower and upper bound of the K-operator frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle = \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle,$$

the K-operator frame is A-tight. If $A = 1$, it is called a normalized tight K-operator frame or a Parseval K-operator frame.

Example 3.2. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbf{N}}, v = \{v_j\}_{j\in\mathbf{N}} \in l^\infty$, we define

$$uv = \{u_j v_j\}_{j\in\mathbf{N}}, u^* = \{\bar{u_j}\}_{j\in\mathbf{N}}, \|u\| = \sup_{j\in\mathbf{N}}|u_j|.$$

Then $\mathscr{A} = \{l^\infty, \|.\|\}$ is a C^*-algebra, as a result $\mathscr{A}$ is pro-C^*-algebra.

Let $\mathscr{X} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathscr{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j\bar{u_j}\}_{j\in\mathbf{N}}.$$

Then $\mathscr{X}$ is a Hilbert $\mathscr{A}$-module.

Now let $\{e_j\}_{j\in\mathbf{N}}$ be the standard orthonormal basis of $\mathscr{X}$. For each $j \in \mathbf{N}$ define the adjointable operator

$$T_j : \mathscr{X} \to \mathscr{X},\ T_j\xi = \langle \xi, e_j\rangle e_j,$$

then for every $\xi \in \mathscr{X}$ we have

$$\sum_{j\in\mathbf{N}}\langle T_j\xi, T_j\xi\rangle = \langle \xi, \xi\rangle.$$

Fix $N \in \mathbf{N}^*$ and define

$$K : \mathscr{X} \to \mathscr{X},\ Ke_j = \begin{cases} je_j & \text{if } j \le N, \\ 0 & \text{if } j > N. \end{cases}$$

It is easy to check that K is adjointable and satisfies

$$K^*e_j = \begin{cases} je_j & \text{if } j \le N, \\ 0 & \text{if } j > N. \end{cases}$$

For any $\xi \in \mathscr{X}$ we have

$$\frac{1}{N^2}\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in\mathbf{N}}\langle T_j\xi, T_j\xi\rangle = \langle \xi, \xi\rangle.$$

This shows that $\{T_j\}_{j\in\mathbf{N}}$ is a K-operator frame with bounds $\dfrac{1}{N^2}, 1$.

Let $\{T_i\}_{i\in I}$ be a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. Define an operator

$$R : \mathscr{X} \to l^2(\mathscr{X}) \;\; by \;\; R\xi = \{T_i\xi\}_{i\in I}, \forall \xi \in \mathscr{X}.$$

The operator R is called the analysis operator of the K-operator frame $\{T_i\}_{i\in I}$.

The adjoint of the analysis operator R,

$$R^*(\{\xi_i\}_{i\in I}) : l^2(\mathscr{X}) \to \mathscr{X}$$

is defined by

$$R^*(\{\xi_i\}_{i\in J}) = \sum_{i\in I} T_i^*\xi_i, \forall \{\xi_i\}_{i\in I} \in l^2(\mathscr{X}).$$

The operator R^* is called the synthesis operator of the K-operator frame $\{T_i\}_{i\in I}$.

By composing R and R^*, the frame operator $S_T : \mathscr{X} \to \mathscr{X}$ for the K-operator frame is given by

$$S_T(\xi) = R^*R\xi = \sum_{i\in I} T_i^*T_i\xi.$$

Proposition 3.3. *Let $\{T_i\}_{i\in I}$ be a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bounds A and B. Then $\{T_i\}_{i\in I}$ is an operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ if K is bounded, surjective and $K = K^*$.*

Proof. Since K is surjective, there exists $m > 0$ such that

$$\langle K^*\xi, K^*\xi\rangle \geq m\langle \xi, \xi\rangle, \; \forall \xi \in \mathscr{X}.$$

Also, since $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$, we have

$$mA\langle \xi, \xi\rangle \leq A\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle, \forall \xi \in \mathscr{X}.$$

Hence $\{T_i\}_{i\in I}$ is an operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bounds mA and B. □

Theorem 3.4. *For an operator Bessel sequence $\{T_i\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$, the following statements are equivalent:*

(1) *$\{T_i\}_{i\in I}$ is K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.*

(2) *There exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{T_i\}_{i\in I}$.*

(3) *$K = S^{\frac{1}{2}}Q$, for some $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$.*

Proof. (1) $\Rightarrow$ (2) Note that $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bounds A and B and frame operator S if and only if

$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in J}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle, \forall \xi \in \mathscr{X}.$$

Thus, we have

$$\langle AKK^*\xi, \xi\rangle \leq \langle S\xi, \xi\rangle \leq \langle B\xi, \xi\rangle, \forall \xi \in \mathscr{X}.$$

Hence $S \geq AKK^*$.

(2) $\Rightarrow$ (3) Suppose there is $A > 0$ such that $AKK^* \leq S$.

This give $AKK^* \leq S^{\frac{1}{2}}S^{\frac{1}{2}^*}$. Then by the Lemma 2.7, $K = S^{\frac{1}{2}}Q$, for some $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$.

(3) $\Rightarrow$ (1) Let $K = S^{\frac{1}{2}}Q$, for some $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$. Then by the Lemma 2.7, there exists $A > 0$ such that $AKK^* \leq S^{\frac{1}{2}}S^{\frac{1}{2}^*}$. This give $AKK^* \leq S$. Hence $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.5. *Let $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$ an invertible map such that both are uniformly bounded and $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. Then $\{T_iQ\}_{i\in I}$ is a Q^*K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.*

Proof. Note that $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bounds A and B and frame operator S if and only if

$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle, \forall \xi \in \mathscr{X}.$$

Thus, we have

$$A\langle K^*Q\xi, K^*Q\xi\rangle \leq \sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle \leq B\langle Q\xi, Q\xi\rangle, \forall x \in \mathscr{X}.$$

This give

$$A\langle (Q^*K)^*\xi, (Q^*K)^*\xi\rangle \leq \sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle \leq B\|Q\|^2_\infty\langle \xi, \xi\rangle, \forall \xi \in \mathscr{X}.$$

Hence $\{T_iQ\}_{i\in I}$ is a Q^*K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.6. *Let $K \in Hom^*_{\mathscr{A}}(\mathscr{X})$ and $\{T_i\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$ is a tight K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A_1. Then $\{T_i\}_{i\in I}$ is a tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A_2 if and only if $K^{-1} = \frac{A_1}{A_2}K^*$.*

Proof. Suppose that $\{T_i\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$ is a tight K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A_1. If $\{T_i\}_{i\in I}$ is a tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A_2. Then

$$\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle = A_2\langle \xi,\xi\rangle, \forall \xi \in \mathscr{X}$$

So, for each $\xi \in \mathscr{X}$, we have $A_1\langle K^*\xi, K^*\xi\rangle = A_2\langle \xi,\xi\rangle$. This gives

$$\langle KK^*\xi,\xi\rangle = \langle \frac{A_2}{A_1}\xi,\xi\rangle, \forall \xi \in \mathscr{X}.$$

Then $KK^* = \frac{A_2}{A_1}I$, Hence $K^{-1} = \frac{A_1}{A_2}K^*$.

Conversely, suppose that $K^{-1} = \frac{A_1}{A_2}K^*$. Then $KK^* = \frac{A_2}{A_1}I$. Thus

$$\langle KK^*\xi,\xi\rangle = \langle \frac{A_2}{A_1}\xi,\xi\rangle, \forall \xi \in \mathscr{X}.$$

Since $\{T_i\}_{i\in I}$ is a tight K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$, we have

$$\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle = A_2\langle \xi,\xi\rangle, \forall \xi \in \mathscr{X}$$

Hence $\{T_i\}_{i\in I}$ is a tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Remark 3.7. Let $K \in Hom^*_{\mathscr{A}}(\mathscr{X})$.

1) If $\{T_i\}_{i\in I}$ is a K-tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A, then $\{T_i(K^N)^*\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$ is K^{N+1}-tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A.
2) If $\{T_i\}_{i\in I}$ is a tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A, then, for all $K \in Hom^*_{\mathscr{A}}(\mathscr{X})$ an ivertible element such that both are uniformly bounded $\{T_iK^*\}_{i\in I}$ is K-tight operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame bound A.

Next, we show that K-operator frame for $\mathscr{X}$ is invariant under a adjointable operator, provided K^* commutes with the inverse of a given operator. A relation between the best bounds of a given K-operator frame and the best bounds of K-operator frame obtained by the action of adjointable operator is given in the following theorem

Theorem 3.8. *Let $\{T_i\}_{i\in I}$ be a K-operator frame for $\mathscr{X}$ with best frame bounds A and B. If $Q : \mathscr{X} \to \mathscr{X}$ is a adjointable and inversible operator such that both are uniformly bounded and*

Q^{-1} commutes with K^* , then $\{T_iQ\}_{i\in I}$ is a K-operator frame for $\mathscr{X}$ with best frame bounds C and D satisfying the inequalities

$$A\|Q^{-1}\|_\infty^{-2} \leq C \leq A\|Q\|_\infty^2 \text{ and } B\|Q^{-1}\|_\infty^{-2} \leq D \leq B\|Q\|_\infty^2 \tag{3.2}$$

Proof. Since B is an upper bound for $\{T_i\}_{i\in J}$, for all $\xi \in \mathscr{X}$, we have

$$\sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle \leq B\langle Q\xi, Q\xi\rangle \leq B\|Q\|_\infty^2\langle \xi,\xi\rangle, \xi \in \mathscr{X}.$$

Also, we have

$$\begin{aligned} A\langle K^*\xi, K^*\xi\rangle &= A\langle K^*Q^{-1}Q\xi, K^*Q^{-1}Q\xi\rangle \\ &= A\langle Q^{-1}K^*Q\xi, Q^{-1}K^*Q\xi\rangle \\ &\leq \|Q^{-1}\|_\infty^2\sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle, \xi \in \mathscr{X}. \end{aligned}$$

Therefore, we obtain

$$A\|Q^{-1}\|_\infty^{-2}\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle \leq B\|Q\|_\infty^2\langle \xi,\xi\rangle$$

Hence, $\{T_iQ\}_{i\in I}$ is a K-operator frame for $\mathscr{X}$ with bounds $A\|Q^{-1}\|_\infty^{-2}$ and $B\|Q\|_\infty^2$. Now let C and D be the best bounds of the K-operator frame $\{T_iQ\}_{i\in I}$. Then

$$A\|Q^{-1}\|_\infty^{-2} \leq C \text{ and } D \leq B\|Q\|_\infty^2 \tag{3.3}$$

Also, $\{T_iQ\}_{i\in I}$ is a K-operator frame for $Hom_{\mathscr{A}}^*(\mathscr{X})$ with frame bounds C and D and

$$\langle K^*\xi, K^*\xi\rangle = \langle QQ^{-1}K^*\xi, QQ^{-1}K^*\xi\rangle \leq \|Q\|_\infty^2\langle K^*Q^{-1}\xi, K^*Q^{-1}\xi\rangle, \xi \in \mathscr{X}.$$

Hence

$$\begin{aligned} C\|Q\|_\infty^{-2}\langle K^*\xi, K^*\xi\rangle &\leq C\langle K^*Q^{-1}\xi, K^*Q^{-1}\xi\rangle \\ &\leq \sum_{i\in I}\langle T_iQQ^{-1}\xi, T_iQQ^{-1}\xi\rangle (= \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle) \\ &\leq D\|Q^{-1}\|_\infty^2\langle \xi,\xi\rangle. \end{aligned}$$

Since A and B are the best bounds of K-operator frame $\{T_i\}_{i\in I}$, we have

$$C\|Q\|_\infty^{-2} \leq A \text{ and } B \leq D\|Q^{-1}\|_\infty^2 \tag{3.4}$$

Hence the inequality (3.2) follows from (3.3) and (3.4). □

Theorem 3.9. *A sequence $\{T_i\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$ is a K-operator frame for $\mathscr{X}$ if and only if $Ran(K) \subset Ran(R^*)$, where R is the analysis operator of K-operator frame.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. Then there exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{T_i\}_{i\in I}$.

Since $S = RR^*$ then $R^*(R^*)^* \geqslant AKK^*$.Therefore by Lemma 2.7 $\text{Ran}(K) \subseteq \text{Ran}(R^*)$.

Conversely, suppose that $\text{Ran}(K) \subseteq \text{Ran}(R^*)$. Then $KK^* \leq \lambda^2 R^*(R^*)^*$.Thus $KK^* \leq \lambda^2 S$. Therefore by Theorem 3.4 $\{T_i\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ □

Theorem 3.10. *Let $K \in Hom^*_{\mathscr{A}}(\mathscr{X})$ and $\{T_i\}_{i\in I}$ be K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. If $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$ is bounded surjective operator with $Q = Q^*$ and $QK = KQ$, then $\{T_iQ\}_{i\in I}$ is K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.*

Proof. We have

$$A\langle K^*Q^*\xi, K^*Q^*\xi\rangle = A\langle Q^*K^*\xi, Q^*K^*\xi\rangle$$

Suppose that Q is surjective. Then by Proposition **??** there are $m, M > 0$ such that

$$mA\langle K^*\xi, K^*\xi\rangle \leq A\langle Q^*K^*\xi, Q^*K^*\xi\rangle \leq \sum_{i\in I}\langle T_iQ^*\xi, T_iQ^*\xi\rangle, \xi \in \mathscr{X}.$$

and

$$\begin{aligned}\sum_{i\in I}\langle T_iQ^*\xi, T_iQ^*\xi\rangle &\leq B\langle Q^*\xi, Q^*\xi\rangle \\ &= B\langle Q\xi, Q\xi\rangle \\ &\leq BM\langle \xi, \xi\rangle\end{aligned}$$

Therefore, we obtain

$$mA\langle K^*\xi, K^*\xi\rangle \leq \sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle_{\leq} BM\langle \xi, \xi\rangle$$

Hence, $\{T_iQ\}_{i\in I}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.11. *Let $K \in Hom^*_{\mathscr{A}}(\mathscr{X})$ and $\{T_i\}_{i\in I}$ be K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. If $Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$ be an isometry with $K^*Q = QK^*$, then $\{T_iQ\}$ is K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.*

Proof. Suppose $\{T_i\}_{i\in I}$ is K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. Then, for each $\xi \in \mathscr{X}$, we have

$$\begin{aligned}\sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle &\geq A\langle K^*Q\xi, K^*Q\xi\rangle \\ &= A\langle QK^*\xi, QK^*\xi\rangle \\ &= A\langle K^*\xi, K^*\xi\rangle\end{aligned}$$

Also,

$$\sum_{i\in I}\langle T_iQ\xi, T_iQ\xi\rangle \leq B\|Q\|^2_\infty\langle\xi,\xi\rangle$$

Hence $\{T_iQ\}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.12. *Let $\{T_i\}_{i\in I}$ and $\{R_i\}_{i\in I}$ be K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame operator S_T and S_R respectively. Then $K = PS_T^{1/2} + QS_R^{1/2}$ for some $P, Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$*

Proof. Let $\{T_i\}_{i\in I}$ and $\{R_i\}_{i\in I}$ be K-operator frames for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with frame operator S_T and S_R respectively. Then by Lemma 2.7, there exist $A_1, A_2 > 0$ such that $S_T \geq A_1KK^*$ and $S_R \geq A_2KK^*$. Therefore, by Douglas Theorem, we get $Ran(K) \subset Ran\left(S_T^{1/2}\right)$ and $Ran(K) \subset Ran\left(S_R^{1/2}\right)$. Hence $Ran(K) \subset Ran\left(S_T^{1/2}\right) + Ran\left(S_R^{1/2}\right)$. Thus, we obtain $K = PS_T^{1/2} + QS_R^{1/2}$ for some $P, Q \in Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.13. *Let $\{T_i\}_{i\in I}$ be a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with the frame operator S and let P be a positive operator such that $SP^* = P^*S$. Then $\{T_i + T_iP\}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. Moreover, for any natural number n, $\{T_i + T_iP^n\}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$ with the frame operator S. Then, there exist $\lambda > 0$ such that $S \geq \lambda KK^*$. The frame operator for $\{T_i + T_iP\}$ is given by

$$\begin{aligned}\sum_{i\in I}(T_i + T_iP)^*(T_i + T_iP)(\xi) &= \sum_{i\in I}T_i^*(T_i(\xi) + T_iP(\xi)) + P^*T_i^*(T_i(\xi) + T_iP(\xi)) \\ &= S(I+P)^*(I+P)(\xi)\end{aligned}$$

Since $S(I+P^*)(I+P) \geq S \geq \lambda KK^*$, $\{T_i + T_iP\}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$.

Similarly, for any natural number n, $\{T_i + T_iP^n\}$ is a K-operator frame for $Hom^*_{\mathscr{A}}(\mathscr{X})$. □

Theorem 3.14. *Let $(\mathscr{X},\mathscr{A},\langle.,.\rangle_{\mathscr{A}})$ and $(\mathscr{X},\mathscr{B},\langle.,.\rangle_{\mathscr{B}})$ be two Hilbert $\mathscr{C}^*$-modules and let $\varphi:\mathscr{A}\longrightarrow\mathscr{B}$ be a $*$-homomorphism and θ be a map on $\mathscr{X}$ such that $\langle\theta\xi,\theta\eta\rangle_{\mathscr{B}}=\varphi(\langle\xi,\eta\rangle_{\mathscr{A}})$ for all $\xi,\eta\in\mathscr{X}$. Also, suppose that $\{T_i\}_{i\in I}\subset Hom^*_{\mathscr{A}}(\mathscr{X})$ is a K-operator frame for $(\mathscr{X},\mathscr{A},\langle.,.\rangle_{\mathscr{A}})$ with frame operator $S_{\mathscr{A}}$ and lower and upper operator frame bounds A, B respectively. If θ is surjective, $\theta K^*=K^*\theta$, $\theta T_i=T_i\theta$ and $\theta T_i^*=T_i^*\theta$ for each i in I, then $\{T_i\}_{i\in I}$ is a K-operator frame for $(\mathscr{X},\mathscr{B},\langle.,.\rangle_{\mathscr{B}})$ with frame operator $S_{\mathscr{B}}$ and lower and upper operator frame bounds A, B respectively, and $\langle S_{\mathscr{B}}\theta\xi,\theta\eta\rangle_{\mathscr{B}}=\varphi(\langle S_{\mathscr{A}}\xi,\eta\rangle_{\mathscr{A}})$.*

Proof. Let $\eta\in\mathscr{X}$ then there exists $\xi\in\mathscr{X}$ such that $\theta\xi=\eta$ (θ is surjective). By the definition of K-operator frames we have

$$A\langle K^*\xi,K^*\xi\rangle_{\mathscr{A}}\leq\sum_{i\in I}\langle T_i\xi,T_i\xi\rangle_{\mathscr{A}}\leq B\langle\xi,\xi\rangle_{\mathscr{A}}.$$

By lemma 2.8 we have

$$\varphi(A\langle K^*\xi,K^*\xi\rangle_{\mathscr{A}})\leq\varphi(\sum_{i\in I}\langle T_i\xi,T_i\xi\rangle_{\mathscr{A}})\leq\varphi(B\langle\xi,\xi\rangle_{\mathscr{A}}).$$

By the definition of $*$-homomorphism we have

$$A\varphi(\langle K^*\xi,K^*\xi\rangle_{\mathscr{A}})\leq\sum_{i\in I}\varphi(\langle T_i\xi,T_i\xi\rangle_{\mathscr{A}})\leq B\varphi(\langle\xi,\xi\rangle_{\mathscr{A}}).$$

By the relation betwen θ and φ we get

$$A\langle\theta K^*\xi,\theta K^*\xi\rangle_{\mathscr{B}}\leq\sum_{i\in I}\langle\theta T_i\xi,\theta T_i\xi\rangle_{\mathscr{B}}\leq B\langle\theta\xi,\theta\xi\rangle_{\mathscr{B}}.$$

By the relation betwen θ, K^* and T_i we have

$$A\langle K^*\theta\xi,K^*\theta\xi\rangle_{\mathscr{B}}\leq\sum_{i\in I}\langle T_i\theta\xi,T_i\theta\xi\rangle_{\mathscr{B}}\leq B\langle\theta\xi,\theta\xi\rangle_{\mathscr{B}}.$$

Then

$$A\langle K^*\eta,K^*\eta\rangle_{\mathscr{B}}\leq\sum_{i\in I}\langle T_i\eta,T_i\eta\rangle_{\mathscr{B}}\leq B\langle\eta,\eta\rangle_{\mathscr{B}},\forall\eta\in\mathscr{X}.$$

On the other hand we have

$$\begin{aligned}
\varphi(\langle S_{\mathscr{A}}\xi,\eta\rangle_{\mathscr{A}}) &= \varphi(\langle\sum_{i\in I} T_i^*T_i\xi,\eta\rangle_{\mathscr{A}}) \\
&= \sum_{i\in I}\varphi(\langle T_i\xi,T_i\eta\rangle_{\mathscr{A}}) \\
&= \sum_{i\in I}\langle\theta T_i\xi,\theta T_i\eta\rangle_{\mathscr{B}} \\
&= \sum_{i\in I}\langle T_i\theta\xi,T_i\theta\eta\rangle_{\mathscr{B}} \\
&= \langle\sum_{i\in I} T_i^*T_i\theta\xi,\theta\eta\rangle_{\mathscr{B}} \\
&= \langle S_{\mathscr{B}}\theta\xi,\theta\eta\rangle_{\mathscr{B}}.
\end{aligned}$$

Which completes the proof. □

4. TENSOR PRODUCT

The minimal or injective tensor product of the pro-C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A}\otimes\mathscr{B}$, is the completion of the algebraic tensor product $\mathscr{A}\otimes_{\text{alg}}\mathscr{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathscr{X}$ is a Hilbert module over a pro-C^*-algebra $\mathscr{A}$ and $\mathscr{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathscr{B}$. The algebraic tensor product $\mathscr{X}\otimes_{\text{alg}}\mathscr{Y}$ of $\mathscr{X}$ and $\mathscr{Y}$ is a pre-Hilbert $\mathscr{A}\otimes\mathscr{B}$-module with the action of $\mathscr{A}\otimes\mathscr{B}$ on $\mathscr{X}\otimes_{\text{alg}}\mathscr{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b)=\xi a\otimes\eta b \text{ for all } \xi\in\mathscr{X},\eta\in\mathscr{Y},a\in\mathscr{A} \text{ and } b\in\mathscr{B}$$

and the inner product

$$\langle\cdot,\cdot\rangle:\left(\mathscr{X}\otimes_{\text{alg}}\mathscr{Y}\right)\times\left(\mathscr{X}\otimes_{\text{alg}}\mathscr{Y}\right)\to\mathscr{A}\otimes_{\text{alg}}\mathscr{B}. \text{ defined by}$$

$$\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle=\langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$$

We also know that for $z=\sum_{i=1}^{n}\xi_i\otimes\eta_i$ in $\mathscr{X}\otimes_{alg}\mathscr{Y}$ we have $\langle z,z\rangle_{\mathscr{A}\otimes\mathscr{B}}=\sum_{i,j}\langle\xi_i,\xi_j\rangle_{\mathscr{A}}\otimes\langle\eta_i,\eta_j\rangle_{\mathscr{B}}\geq 0$ and $\langle z,z\rangle_{\mathscr{A}\otimes\mathscr{B}}=0$ iff $z=0$.

The external tensor product of $\mathscr{X}$ and $\mathscr{Y}$ is the Hilbert module $\mathscr{X}\otimes\mathscr{Y}$ over $\mathscr{A}\otimes\mathscr{B}$ obtained by the completion of the pre-Hilbert $\mathscr{A}\otimes\mathscr{B}$-module $\mathscr{X}\otimes_{\text{alg}}\mathscr{Y}$.

If $P \in M(\mathscr{X})$ and $Q \in M(\mathscr{Y})$ then there is a unique adjointable module morphism $P \otimes Q$: $\mathscr{A} \otimes \mathscr{B} \to \mathscr{X} \otimes \mathscr{Y}$ such that $(P\otimes Q)(a\otimes b) = P(a) \otimes Q(b)$ and $(P\otimes Q)^*(a\otimes b) = P^*(a) \otimes Q^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, [8]).

Let I and J be countable index sets.

Theorem 4.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two Hilbert pro-C*-modules over unital pro-C*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $\{T_i\}_{i\in I} \subset Hom^*_{\mathscr{A}}(\mathscr{X})$ be a K_1-operator frame for $\mathscr{X}$ and $\{R_j\}_{j\in J} \subset Hom^*_{\mathscr{B}}(\mathscr{Y})$ be a K_2-operator frame for $\mathscr{Y}$ with frame operators S_T and S_R and operator frame bounds (A,B) and (C,D) respectively. Then $\{T_i \otimes R_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hibert $\mathscr{A} \otimes \mathscr{B}$-module $\mathscr{X} \otimes \mathscr{Y}$ with frame operator $S_T \otimes S_R$ and lower and upper operator frame bounds AC and BD, respectively.*

Proof. By the definition of K_1-operator frame $\{T_i\}_{i\in I}$ and K_2-operator frame $\{R_j\}_{j\in J}$ we have

$$A\langle K_1^*\xi, K_1^*\xi\rangle_{\mathscr{A}} \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle_{\mathscr{A}} \leq B\langle \xi,\xi\rangle_{\mathscr{A}}, \forall \xi \in \mathscr{X}.$$

$$C\langle K_2^*\eta, K_2^*\eta\rangle_{\mathscr{B}} \leq \sum_{j\in J}\langle R_j\eta, R_j\eta\rangle_{\mathscr{B}} \leq D\langle \eta,\eta\rangle_{\mathscr{B}}, \forall \eta \in \mathscr{K}.$$

Therefore

$$\begin{aligned}&(A\langle K_1^*\xi, K_1^*\xi\rangle_{\mathscr{A}}) \otimes (C\langle K_2^*\eta, K_2^*\eta\rangle_{\mathscr{B}})\\ &\leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle_{\mathscr{A}} \otimes \sum_{j\in J}\langle R_j\eta, R_j\eta\rangle_{\mathscr{B}}\\ &\leq (B\langle \xi,\xi\rangle_{\mathscr{A}}) \otimes (D\langle \eta,\eta\rangle_{\mathscr{B}}), \forall \xi \in \mathscr{X}, \forall \eta \in \mathscr{Y}.\end{aligned}$$

Then

$$\begin{aligned}&AC(\langle K_1^*\xi, K_1^*\xi\rangle_{\mathscr{A}} \otimes \langle K_2^*\eta, K_2^*\eta\rangle_{\mathscr{B}})\\ &\leq \sum_{i\in I, j\in J}\langle T_i\xi, T_i\xi\rangle_{\mathscr{A}} \otimes \langle R_j\eta, R_j\eta\rangle_{\mathscr{B}}\\ &\leq BD(\langle \xi,\xi\rangle_{\mathscr{A}} \otimes \langle \eta,\eta\rangle_{\mathscr{B}}), \forall \xi \in \mathscr{X}, \forall \eta \in \mathscr{Y}.\end{aligned}$$

Consequently we have

$$\begin{aligned}&AC\langle K_1^*\xi \otimes K_2^*\eta, K_1^*\xi \otimes K_2^*\eta\rangle_{\mathscr{A}\otimes\mathscr{B}}\\ &\leq \sum_{i\in I, j\in J}\langle T_i\xi \otimes R_j\eta, T_i\xi \otimes R_j\eta\rangle_{\mathscr{A}\otimes\mathscr{B}}\\ &\leq BD\langle \xi\otimes\eta, \xi\otimes\eta\rangle_{\mathscr{A}\otimes\mathscr{B}}, \forall \xi \in \mathscr{X}, \forall \eta \in \mathscr{Y}.\end{aligned}$$

Then for all $\xi\otimes\eta$ in $\mathscr{X}\otimes\mathscr{Y}$ we have

$$\begin{aligned}
&AC\langle (K_1\otimes K_2)^*(\xi\otimes\eta),(K_1\otimes K_2)^*(\xi\otimes\eta)\rangle_{\mathscr{A}\otimes\mathscr{B}}\\
&\leq \sum_{i\in I,j\in J}\langle (T_i\otimes R_j)(\xi\otimes\eta),(T_i\otimes R_j)(\xi\otimes\eta)\rangle_{\mathscr{A}\otimes\mathscr{B}}\\
&\leq BD\langle \xi\otimes\eta,\xi\otimes\eta\rangle_{\mathscr{A}\otimes\mathscr{B}}.
\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathscr{X}\otimes_{alg}\mathscr{Y}$ and then it's satisfied for all $z\in\mathscr{X}\otimes\mathscr{Y}$. It shows that $\{T_i\otimes R_j\}_{i\in I,j\in J}$ is a $K_1\otimes K_2$-operator frame for Hilbert $\mathscr{A}\otimes\mathscr{B}$-module $\mathscr{X}\otimes\mathscr{Y}$ with lower and upper operator frame bounds AC and BD, respectively.

By the definition of frame operator S_T and S_R we have

$$S_T\xi=\sum_{i\in I}T_i^*T_i\xi,\forall\xi\in\mathscr{X}.$$

$$S_R\eta=\sum_{j\in J}R_j^*R_j\eta,\forall\eta\in\mathscr{K}.$$

Therefore

$$\begin{aligned}
(S_T\otimes S_R)(\xi\otimes\eta)&=S_T\xi\otimes S_R\eta\\
&=\sum_{i\in I}T_i^*T_i\xi\otimes\sum_{j\in J}R_j^*R_j\eta\\
&=\sum_{i\in I,j\in J}T_i^*T_i\xi\otimes R_j^*R_j\eta\\
&=\sum_{i\in I,j\in J}(T_i^*\otimes R_j^*)(T_i\xi\otimes R_j\eta)\\
&=\sum_{i\in I,j\in J}(T_i^*\otimes R_j^*)(T_i\otimes R_j)(\xi\otimes\eta)\\
&=\sum_{i\in I,j\in J}(T_i\otimes R_j)^*)(T_i\otimes R_j)(\xi\otimes\eta).
\end{aligned}$$

Now by the uniqueness of frame operator, the last expression is equal to $S_{T\otimes R}(\xi\otimes\eta)$. Consequently we have $(S_T\otimes S_R)(\xi\otimes\eta)=S_{T\otimes R}(\xi\otimes\eta)$. The last equality is satisfied for every finite sum of elements in $\mathscr{X}\otimes_{alg}\mathscr{Y}$ and then it's satisfied for all $z\in\mathscr{X}\otimes\mathscr{Y}$. It shows that $(S_T\otimes S_R)(z)=S_{T\otimes R}(z)$. So $S_{T\otimes R}=S_T\otimes S_R$. □

5. DUAL OF K-OPERATOR FRAME

In the following we define the Dual K-operator frame and we give some properties

Definition 5.1. Let $K \in Hom^{*}_{\mathscr{A}}(\mathscr{X})$ and $\{T_i \in Hom^{*}_{\mathscr{A}}(\mathscr{X}), i \in I\}$ be a K-operator frame for the Hilbert $\mathscr{A}$-module $\mathscr{X}$. An operator Bessel sequences $\{R_i \in Hom^{*}_{\mathscr{A}}(\mathscr{X}), i \in I\}$ is called a K-dual operator frame for $\{T_i\}_{i\in I}$ if $K\xi = \sum_{i\in I} T_i^{*} R_i \xi$ for all $\xi \in \mathscr{X}$.

Example 5.2. Let $K \in Hom_{\mathscr{A}}{}^{*}(\mathscr{X})$ be a surjective operator and $\{T_i \in Hom^{*}_{\mathscr{A}}(\mathscr{X}), i \in I\}$ be a K-operator frame for $\mathscr{X}$ with frame operator S, then S is invertible.

For all $\xi \in \mathscr{X}$ we have :

$S\xi = \sum_{i\in I} T_i^{*} R_i \xi$.

So $K\xi = \sum_{i\in I} T_i^{*} R_i S^{-1} K\xi$.

Then the sequence $\{T_i S^{-1} K \in Hom^{*}_{\mathscr{A}}(\mathscr{X}), i \in I\}$ is a dual K-operator frame of $\{T_i \in Hom^{*}_{\mathscr{A}}(\mathscr{X}), i \in I\}$

Theorem 5.3. *Let $K \in Hom^{*}_{\mathscr{A}}(\mathscr{X})$ be an invertible element such that both are uniformly bounded and $Rang(K)$ is closed, and let $\{T_i\}_{i\in I}$ be K-operator frame for $Hom^{*}_{\mathscr{A}}(\mathscr{X})$ with frame operator S and frame bounds A and B respectively. Then $\{T_i \pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^{*} K\}$ is a K-dual of $\{T_i\}_{i\in I}$*

Proof. Let $\{T_i\}$ be a K-operator frame for $Hom^{*}_{\mathscr{A}}(\mathscr{X})$. Since $S : Rang(K) \to S(Rang(K))$ is invertible, we have

$$
\begin{aligned}
K\xi &= \left(S^{-1}_{|Rang(K)} S_{|Rang(K)}\right)^{*} K\xi \\
&= S_{|Rang(K)} \left(S^{-1}_{|Rang(K)}\right)^{*} K\xi \\
&= S\pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^{*} K\xi \\
&= \sum_{i\in I} T_i^{*} T_i \pi_{S(Rang(K))} \left(S^{-1}_{|Rang(K)}\right)^{*} K\xi, \text{ for all } \xi \in \mathscr{X}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\sum_{i\in I} \langle T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^{*} K\xi, T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^{*} K\xi \rangle &= \sum_{i\in I} \langle T_i^{*} T_i \pi_{S(Rang(K))} \left(S^{-1}\right)^{*} K\xi, \left(S^{-1}\right)^{*} K\xi \rangle \\
&= \left\langle S\left(S^{-1}\right)^{*} K\xi, \left(S^{-1}\right)^{*} K\xi \right\rangle \\
&= \left\langle K\xi, \left(S^{-1}\right)^{*} K\xi \right\rangle \\
&\leq A^{-1} \|K^{-1}\|^{2}_{\infty} \|K\|^{2}_{\infty} \langle \xi, \xi \rangle, \xi \in \mathscr{X}
\end{aligned}
$$

Hence $\left\{T_i\pi_{Rang(K)}\left(S^{-1}\right)^*K\right\}$ is a dual of the K-operator frame $\{T_i\}$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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