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# THE DYNAMICICS OF A PREY- PREDATOR MODEL WITH THE EXISTENCE OF DISEASE AND POLLUTION

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**Abstract:** An eco-epidemiological model consisting of a prey-predator system involving disease and pollution has been proposed and studied. It is assumed that the disease transmitted between the individual of prey species by contact with nonlinear incidence rate, however the predator preys upon the prey according to Holling type-II functional response. The existence, uniqueness and boundedness of the solution of the system are studied. The existence of all possible equilibrium points are discussed. The local stability of for each equilibrium point is investigated. The global stability of the positive equilibrium point is studied with the help of Lyapunov function. Finally further investigations for the global dynamics of the proposed system are carried out with the help of numerical simulations. It is observed that the system has a Hopf bifurcation near the positive equilibrium.

**Keywords:** Prey-predator model, Disease, Stability, Toxicant.

**2000 AMS Subject Classification:** 47H17; 47H05; 47H09

## 1. Introduction

Various kinds of pollutants like oxides of sulphur or oxides of carbon enter into both aquatic and terrestrial environment. These pollutants may be emitted into the environment from different sources (e.g. vehicles, thermal power plant, industries, refineries, etc.) as well as by incessant use of natural resources

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without recharging and cleaning them.

In recent decades, several investigators have been proposed and analyzed mathematical models to study the effects of toxicants on biological species [1-4]. In particular, Hallam et al. [5,6] have proposed and analyzed mathematical models to study the effects of toxicants on biological species when these are emitted into the environment from external sources. Hauping and Zhien [7] have been proposed a mathematical model to study the effect of a toxicant on natural stable two species communities. In these investigations the effects of a toxicant simultaneously on growth rate and carrying capacity of the species have not been considered. However, Freedman and Shukla [4] Proposed models to study the effects of toxicant on single-species and predator-prey system by assuming that the intrinsic growth rate of species decreases as the uptake concentration of the toxicant increases, while its carrying capacity decreases with the environmental concentration of the toxicant. Shukla et al [8, 9, 10] have been studied the survival of two competing species in a polluted environment using similar assumptions and showed that the usual competitive outcomes may be altered in the presence of a toxicant. Agarwal and Devi [11] proposed and analyzed a mathematical model to study the survival of resource-dependent competing species. They assumed that competing species and its resource are affected simultaneously by a toxicant emitted into the environment from external sources as well as formed by precursors of competing species. Sinha et al [12] have proposed a mathematical model to study the simultaneous effect of toxicant and disease on Lotka-Volterra prey-predator system..

In this paper however, an eco-epidemiological model consisting of diseased prey-predator involving nonlinear incidence rate and Holling type-II functional response has been proposed and analyzed. The dynamical behavior of a proposed model under the effect of toxicant has been investigated analytically as well as numerically.

## 2. The mathematical model

Consider the eco-epidemic model consisting of susceptible prey denoted by $S(t)$, infected prey denoted by $I(t)$ and a predator that denoted by $Y(t)$ in which the following assumptions are adopted:

1. The susceptible prey reproduces logistically while the infected prey does not grow, recover and reproduce, and do not compete for resources, and this is due to the fact that the disease makes

the infected prey individuals weak so when they compete with individuals of their own species (susceptible prey) they always failure.

2. The disease transmitted from infected prey to susceptible prey by contact, according to the following nonlinear incidence rate of the form $\frac{\lambda SI}{1+I}$ used originally by Capasso and Serio 1978 [13] in their modeling of cholera, $\lambda SI$ measures the infection force of the disease and $\frac{1}{1+I}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals [14].
3. The predator individuals feed on infected prey and susceptible prey with different rates. Indeed they consume the prey individuals ($S$ and $I$) according to $\frac{\alpha_1 S}{\beta+S+mI}$ and $\frac{\alpha_2 I}{\beta+S+mI}$, which are known as modified Holling type-II functional response.

Consequently, the dynamics of this eco-epidemic model can be written as follows

$$\begin{aligned}
\frac{dS}{dt} &= rS\left(1-\frac{S}{K}\right)-\frac{\lambda I}{1+I}S-\frac{\alpha_1 SY}{\beta+S+mI} \\
\frac{dI}{dt} &= \frac{\lambda I}{1+I}S-\frac{\alpha_2 IY}{\beta+S+mI}-\mu_1 I \\
\frac{dY}{dt} &= \frac{\theta_1 S+\theta_2 I}{\beta+S+mI}Y-\mu_2 Y
\end{aligned} \tag{1}$$

where $S(t), I(t)$ and $Y(t)$ represent the population density of the susceptible prey, infected prey and predator at time $t$ respectively. However the parameters in the above system are assumed to be positive values and can be described as follows: $r$ represents the intrinsic growth rate of susceptible prey; $K$ is the carrying capacity of the prey; $\lambda$ represents the infected rate; $\alpha_1$ and $\alpha_2$ represent the predation rate of $S$ and $I$ respectively; $\beta$ is the half saturation constant; $m$ represents the predator's favorite rate between $S$ and $I$. $\theta_1$ and $\theta_2$ are the conversion rates of $S$ and $I$ respectively; $\mu_1$ and $\mu_2$ are the natural death rates of $I$ and $Y$ respectively.

In addition to the above if we assume that, there are toxicants (pollutants) in the environment affect negatively on the growth of prey population (susceptible as well as infected) but not the predator population. Therefore, if we assume that, $W(t)$ be the toxicant concentration in the prey population (i.e. $S+I$) at time $t$; $Z(t)$ is the environment concentration of toxicant at time $t$. Consequently, the dynamics of the above eco-epidemic model in a polluted environment can be described by the following set of equations:

$$\begin{aligned}
\frac{dS}{dt} &= rS\left(1-\frac{S}{K}\right)-\frac{\lambda I}{1+I}S-\frac{\alpha_1 SY}{\beta+S+mI}-\sigma_1 SW = f_1(S,I,Y,Z,W)\\
\frac{dI}{dt} &= \frac{\lambda I}{1+I}S-\frac{\alpha_2 IY}{\beta+S+mI}-\mu_1 I-\sigma_2 IW = f_2(S,I,Y,Z,W)\\
\frac{dY}{dt} &= \frac{\theta_1 S+\theta_2 I}{\beta+S+mI}Y-\mu_2 Y = f_3(S,I,Y,Z,W)\\
\frac{dZ}{dt} &= Q-\mu_3 Z-\sigma_3 Z(S+I) = f_4(S,I,Y,Z,W)\\
\frac{dW}{dt} &= \sigma_3 Z(S+I)-\mu_4 W = f_5(S,I,Y,Z,W)
\end{aligned} \tag{2}$$

Here the new parameters can described as follows: $Q>0$ is the exogenous input rate of the toxicant in the environment; $\mu_3>0$ is the natural depletion rate of the environmental toxicant; $\mu_4>0$ is the natural washout rate of the toxicant from the organism; $\sigma_1$ and $\sigma_2$ are the rates at which susceptible and infected are decreasing due to toxicant; $\sigma_3$ is uptake rate of toxicant by organism. In addition, since the density of population cannot be negative then the state space of the system (2) is $R_+^5=\{(S,I,Y,Z,W)\in R^5 : S\geq 0,\ \ I\geq 0, Y\geq 0, Z\geq 0, W\geq 0\}$.

Obviously the interaction functions $f_1, f_2, f_3, f_4$ and $f_5$ of the system (2) are continuous and have continuous partial derivatives on the state space $R_+^5$, therefore these functions are Lipschizian on $R_+^5$ and then the solution of the system (2) with non negative initial condition exists and is unique. In addition all the solutions of the system (2) which initiate in the above state space are uniformly bounded as shown in the following theorem.

**Theorem 2.1** All the solutions of system (2) that initiate in the state space $R_+^5$ are uniformly bounded.

**Proof.** Let $(S(t), I(t), Y(t), Z(t), W(t))$ be any solution of the system (2) with the non-negative initial conditions. From the first equation we have $\frac{dS}{dt}\leq rS\left(1-\frac{S}{K}\right)$

Then by solving the above differential inequality, we obtain $\operatorname*{Lim\,Sup}_{t\to\infty} S(t)\leq K$. Let $R(t)=S(t)+I(t)+Y(t)+Z(t)+W(t)$, then from the system (2) we get

$$\frac{dR}{dt} = rS\left(1-\frac{S}{K}\right) - \frac{\alpha_1 SY}{\beta+S+mI} - \sigma_1 SW - \frac{\alpha_2 IY}{\beta+S+mI} - \mu_1 I - \sigma_2 IW + \frac{\theta_1 S+\theta_2 I}{\beta+S+mI}Y - \mu_2 Y + Q - \mu_3 Z - \mu_4 W$$

Now, since the conversion rate constant from prey population to predator population can not be exceeding the maximum predation rate constant of predator population to prey population. Hence from biological point of view, we have always $\theta_i \le \alpha_i$; $i = 1,2$. Hence we obtain that

$$\frac{dR}{dt} \le Q + (r+\mu)K - \mu R$$

here $\mu = \min\{\mu_1, \mu_2, \mu_3, \mu_4\}$. So again by solving the above linear differential inequality we get that $\lim_{t\to\infty} R(t) \le \frac{Q+(r+\mu)K}{\mu}$. Hence all solutions are uniformly bounded and the proof is complete.

## 3. Existence of equilibrium points

In this section, the existence of all possible equilibrium points of system (2) has been discussed. The system (2) may have five nonnegative equilibriums namely $E_1 = (0,0,0,\frac{Q}{\mu_3},0)$, $E_2 = (\bar{S},0,0,\bar{Z},\bar{W})$, $E_3 = (\hat{S},0,\hat{Y},\hat{Z},\hat{W})$, $E_4 = (\tilde{S},\tilde{I},0,\tilde{Z},\tilde{W})$ and $E_5 = (\widehat{S},\widehat{I},\widehat{Y},\widehat{Z},\widehat{W})$. The existence of $E_1 = (0,0,0,\frac{Q}{\mu_3},0)$ is obvious, however the existence of the other four equilibrium points is established as follows:

The equilibrium point $E_2 = (\bar{S},0,0,\bar{Z},\bar{W})$ exists uniquely in the positive region of $SZW-$space provided that there is a positive solution to the following set of equations

$$\begin{aligned}
& r\left(1-\frac{S}{K}\right) - \sigma_1 W = 0 \\
& Q - \mu_3 Z - \sigma_3 ZS = 0 \\
& \sigma_3 ZS - \mu_4 W = 0
\end{aligned} \tag{3a}$$

Straightforward computation shows that system (3a) has always the following unique positive solution.

$$\begin{aligned}
\bar{S} &= \frac{r\mu_4 K}{r\mu_4 + \sigma_1\sigma_3 K\bar{Z}} > 0,\ \bar{W} = \frac{r\sigma_3 K\bar{Z}}{\sigma_1\sigma_3 K\bar{Z} + r\mu_4} > 0 \\
\bar{Z} &= \frac{-[\sigma_3(r\mu_4 K - \sigma_1 QK) + r\mu_3\mu_4]}{2\sigma_1\sigma_3\mu_3 K} \\
&\quad + \frac{\sqrt{(\sigma_3(r\mu_4 K - \sigma_1 QK) + r\mu_3\mu_4)^2 + 4rQ\sigma_1\sigma_3\mu_3\mu_4 K}}{2\sigma_1\sigma_3\mu_3 K} > 0
\end{aligned} \tag{3b}$$

Therefore the equilibrium point $E_2$ always exists in the positive region of $SZW-$space.

The disease free equilibrium point $E_3 = (\hat{S}, 0, \hat{Y}, \hat{Z}, \hat{W})$ exists uniquely in the positive region of $SYZW-$space provided that there is a positive solution to the following set of equations

$$\begin{aligned}
&r\left(1-\frac{S}{K}\right)-\frac{\alpha_1 Y}{\beta+S}-\sigma_1 W=0\\
&\frac{\theta_1 S}{\beta+S}=\mu_2\\
&\mu_3 Z+\sigma_3 ZS=Q\\
&\sigma_3 ZS-\mu_4 W=0
\end{aligned} \tag{4a}$$

Again straightforward computation shows that system (4a) has always the following unique positive solution.

$$\begin{aligned}
&\hat{S}=\frac{\beta\mu_2}{\theta_1-\mu_2}\ ,\ \hat{Z}=\frac{Q(\theta_1-\mu_2)}{\mu_3(\theta_1-\mu_2)+\sigma_3\beta\mu_2},\\
&\hat{W}=\frac{\sigma_3}{\mu_4}\left(\frac{Q\beta\mu_2}{\mu_3(\theta_1-\mu_2)+\sigma_3\beta\mu_2}\right),\ \hat{Y}=\frac{\beta+\hat{S}}{\alpha_1 K}\left[r(K-\hat{S})-\sigma_1 K\hat{W}\right]
\end{aligned} \tag{4b}$$

provided that

$$\theta_1 > \mu_2 \tag{5a}$$

$$\hat{W} < \frac{r}{\sigma_1 K}(K-\hat{S}) \tag{5b}$$

Therefore the disease free equilibrium point $E_3$ exists uniquely in the positive region of $SYZW-$space if and only if conditions (5a)-(5b) hold.

The predator free equilibrium point $E_4 = (\tilde{S}, \tilde{I}, 0, \tilde{Z}, \tilde{W})$ exists uniquely in the positive region of $SIZW-$space provided that there is a positive solution to the following set of equations

$$\begin{aligned}
&r\left(1-\frac{S}{K}\right)-\frac{\lambda I}{1+I}-\sigma_1 W=0\\
&\frac{\lambda}{1+I}S-\mu_1-\sigma_2 W=0\\
&Q-\mu_3 Z-\sigma_3 Z(S+I)=0\\
&\sigma_3 Z(S+I)-\mu_4 W=0
\end{aligned} \tag{6}$$

Straightforward computation gives that

$$\tilde{I}=\frac{-\left(\sigma_1\lambda+\frac{r\sigma_2}{K}\right)\tilde{S}+\sigma_1\mu_1+r\sigma_2}{\frac{r\sigma_2}{K}\tilde{S}+\sigma_2\lambda-(\sigma_1\mu_1+r\sigma_2)}=h_1(\tilde{S}) \tag{7a}$$

$$\tilde{Z} = \frac{Q}{\mu_3 + \sigma_3 (\tilde{S} + f_1(\tilde{S}))} = h_2(\tilde{S}) \tag{7b}$$

$$\tilde{W} = \frac{1}{\sigma_2}\left(\frac{\lambda\tilde{S}}{1 + f_1(\tilde{S})} - \mu_1\right) = h_3(\tilde{S}) \tag{7c}$$

While $\tilde{S} \in (0, K)$ represents a positive root of the following equation

$$H(S) = \sigma_3 h_2(S)(S + h_1(S)) - \mu_4 h_3(S) = 0 \tag{7d}$$

Obviously $h_i(S);\, i = 1,2,3$ are positive for all the value of $S \in (0, K]$ provided that the following conditions are satisfied:

$$\lambda > \frac{\sigma_1\mu_1 + r\sigma_2}{\sigma_2} \tag{8a}$$

$$S < \frac{\sigma_1\mu_1 + r\sigma_2}{\sigma_1\lambda + \frac{r\sigma_2}{K}} \tag{8b}$$

$$\mu_1 < \frac{\lambda S}{1 + h_1(S)} \tag{8c}$$

Moreover, by using intermediate value theorem, Eq. (7d) has a unique positive root namely $\tilde{S} \in (0, K)$, if $H(S) : [0, K] \to R$ is a continuous function with $H(0) > 0$ (or $H(0) < 0$); $H(K) < 0$ (or $H(K) > 0$) and $\frac{dH}{dS} = H'(S) < 0$ for all $S \in [0, K]$.

Now, since

$$H(0) = \frac{\sigma_3 Q(\sigma_1\mu_1 + r\sigma_2)}{\mu_3(\lambda\sigma_2 - \sigma_1\mu_1 - r\sigma_2) + \sigma_3(\sigma_1\mu_1 + r\sigma_2)} + \frac{\mu_1\mu_4}{\sigma_2}$$

which positive due to condition (8a). Also we have

$$H(K) = Q\frac{\sigma_3(K + h_1(K))}{\mu_3 + \sigma_3(K + h_1(K))} - \mu_4 h_3(K)$$

Clearly $H(K) < 0$ provided that the following condition holds

$$Q < \mu_4 h_3(K) \tag{8d}$$

Further, we have that

$$\frac{dH}{dS} = \frac{\sigma_3 Q(1+h_1'(S))}{\mu_3 + \sigma_3(S+h_1(S))}\left[1 - \frac{\sigma_3(S+h_1(S))}{\mu_3+\sigma_3(S+h_1(S))}\right] + \frac{\mu_4\lambda}{\sigma_2(1+h_1'(S))}\left[\frac{Sh_1'(S)}{1+h_1(S)} - 1\right]$$

Note that $\frac{dH}{dS} = H'(S) < 0$ for all $S \in [0,K]$ provided that the following condition holds

$$1 + h_1'(S) < 0, \ \forall S \in [0,K] \tag{8e}$$

Consequently, according to the intermediate value theorem, Eq. (7d) has a unique positive root namely $\tilde{S} \in (0,K)$ and hence the predator free equilibrium point $E_4$ exists uniquely in the positive region of $SIZW-$space provided that conditions (8a)-(8e) are satisfied.

Finally the positive equilibrium point $E_5 = (\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W})$ exists uniquely in the $Int.R_+^5$ if and only if there is a positive solution to the following set of equations

$$\hat{I} = \frac{\mu_2\beta + (\mu_2 - \theta_1)\hat{S}}{\theta_2 - m\mu_2} = \hat{h}_1(\hat{S}) \tag{9a}$$

$$\hat{Z} = \frac{Q}{\mu_3 + \sigma_3(\hat{S} + \hat{h}_1(\hat{S}))} = \hat{h}_2(\breve{S}) \tag{9b}$$

$$\hat{W} = \frac{\sigma_3}{\mu_4}\hat{h}_2(S)(S + \hat{h}_1(S)) = \hat{h}_3(S) \tag{9c}$$

$$\hat{Y} = \frac{\beta + \hat{S} + m\hat{h}_1(\hat{S})}{\alpha_2}\left(\frac{\lambda\hat{S}}{1+\hat{h}_1(\hat{S})} - \sigma_2\hat{h}_3(\hat{S}) - \mu_1\right) = \hat{h}_4(\hat{S}) \tag{9d}$$

While $\hat{S} \in (0,K)$ represents a positive root of the following equation

$$\hat{H}(S) = r\left(1 - \frac{S}{K}\right) - \frac{\lambda\left(\frac{\alpha_1}{\alpha_2}S + \hat{h}_1(S)\right)}{1+\hat{h}_1(S)} + \left(\frac{\alpha_1}{\alpha_2}\sigma_2 - \sigma_1\right)\hat{h}_3(S) + \frac{\alpha_1}{\alpha_2}\mu_1 = 0 \tag{9e}$$

Note that it is easy to verify that $\hat{h}_i(S);\ i = 1,2,3,4$ are positive for all values of $S \in (0,K]$ under the following conditions.

$$\frac{\theta_2}{m} > \mu_2 > \theta_1 \tag{10a}$$

$$\frac{\lambda\hat{S}}{1+\hat{h}_1(\hat{S})} > \mu_1 + \sigma_2\hat{h}_3(\hat{S}) \tag{10b}$$

Note that we have that

$$\widehat{H}(0) = r - \frac{\lambda\mu_2\beta}{(\theta_2 - m\mu_2) + \mu_2\beta} - \left(\sigma_1 - \frac{\alpha_1}{\alpha_2}\sigma_2\right)\frac{Q\sigma_3\mu_2\beta}{\mu_4[\mu_3(\theta_2 - m\mu_2) + \sigma_3\mu_2\beta]} + \frac{\alpha_1\mu_1}{\alpha_2}$$

Clearly, $\widehat{H}(0) > 0$ under the following conditions

$$r > \lambda \tag{10c}$$

$$\frac{\alpha_1\mu_1}{\alpha_2} \geq \frac{Q}{\mu_4}\left(\sigma_1 - \frac{\alpha_1}{\alpha_2}\sigma_2\right) > 0 \tag{10d}$$

Also, it is easy to verify that

$$\widehat{H}(K) = -\frac{\alpha_1}{\alpha_2}\left(\frac{\lambda K}{1+\widehat{h}_1(K)} - \sigma_2\widehat{h}_3(K) - \mu_1\right) - \frac{\lambda\widehat{h}_1(K)}{1+\widehat{h}_1(K)} - \sigma_1\widehat{h}_3(K)$$

Here $\widehat{H}(K) < 0$ due to condition (10b). Moreover we have

$$\frac{d\widehat{H}}{ds} = -\frac{r}{K} - \frac{\lambda}{\left(1+\widehat{h}_1(S)\right)^2}\left(\frac{\alpha_1}{\alpha_2} + \widehat{h}_1'(S) + \frac{\alpha_1}{\alpha_2}\left(\widehat{h}_1(S) - S\widehat{h}_1'(S)\right)\right) + \left(\frac{\alpha_1}{\alpha_2}\sigma_2 - \sigma_1\right)\widehat{h}_3'(S)$$

where

$$\widehat{h}_1'(S) = \frac{\mu_2 - \theta_1}{\theta_2 - m\mu_2}$$

$$\widehat{h}_3'(S) = \frac{\sigma_3 Q\mu_3(1+\widehat{h}_1'(S))}{\mu_4[\mu_3 + \sigma_3(S + \widehat{h}_1(S))]^2} > 0$$

Obviously $\widehat{h}_1'(S) > 0$ due to conditions (10a) and from the above we have $\widehat{h}_1(S) > S\widehat{h}_1'(S)$ always true and then $\frac{d\widehat{H}}{dS} < 0$ for all the value of $S \in [0, K]$ due to condition (10d). Therefore, from the intermediate value theorem, Eq. (9e) has a unique positive root namely $\widehat{S} \in (0, K)$ and hence the positive equilibrium point $E_5$ exists uniquely in the $Int.R_+^5$ if and only if conditions (10a)-(10d) are satisfied.

## 4. Stability analysis

In this section, the stability analysis of each possible equilibrium point of system (2) is carried out by using Linearization method with the help of Routh-Huritiz criterion or Lyapunov function.

The Jacobian matrix for the system (2) at the point $E_1$ is written as

$$V(E_1)=\begin{pmatrix} r & 0 & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 & 0 \\ 0 & 0 & -\mu_2 & 0 & 0 \\ -\sigma_3\frac{Q}{\mu_3} & -\sigma_3\frac{Q}{\mu_3} & 0 & -\mu_3 & 0 \\ \sigma_3\frac{Q}{\mu_3} & \sigma_3\frac{Q}{\mu_3} & 0 & 0 & -\mu_4 \end{pmatrix}$$

Therefore, $V(E_1)$ has the following eigenvalues $r>0,-\mu_1<0,-\mu_2<0,-\mu_3<0$ and $-\mu_4<0$.

Accordingly the equilibrium point $E_1$ is a saddle point with unstable manifold in the $S-$direction and stable manifold in the other directions.

Now the local stability conditions for $E_2,E_3,E_4$ and $E_5$ of system (2) are established in the following theorems.

**Theorem4.1.** The equilibrium point $E_2=(\bar{S},0,0,\bar{Z},\bar{W})$ of system (2) is locally asymptotically stable provided that the following two conditions are satisfied:

$$\lambda\bar{S}<\mu_1+\sigma_1\bar{W} \tag{11a}$$

$$\frac{\theta_1\bar{S}}{\beta+\bar{S}}<\mu_2 \tag{11b}$$

**Proof.** It is easy to verify that the Jacobian matrix of the system (2) at the equilibrium point $E_2=(\bar{S},0,0,\bar{Z},\bar{W})$ is given by

$$V(E_2)=\begin{pmatrix} -\frac{r}{K}\bar{S} & -\lambda\bar{S} & -\frac{\alpha_1\bar{S}}{\beta+\bar{S}} & 0 & -\sigma_1\bar{S} \\ 0 & \lambda\bar{S}-\mu_1-\sigma_2\bar{W} & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta_1\bar{S}}{\beta+\bar{S}}-\mu_2 & 0 & 0 \\ -\sigma_3\bar{Z} & -\sigma_3\bar{Z} & 0 & -(\mu_3+\sigma_3\bar{S}) & 0 \\ \sigma_3\bar{Z} & \sigma_3\bar{Z} & 0 & \sigma_3\bar{S} & -\mu_4 \end{pmatrix}$$

Then the characteristic equation of $V(E_2)$ can be written as:

$$\left(\lambda\bar{S}-\mu_1-\sigma_2\overline{W}-\bar{\gamma}\right)\left(\frac{\theta_1\bar{S}}{\beta+\bar{S}}-\mu_2-\bar{\gamma}\right)(\bar{\gamma}^3+A_1\bar{\gamma}^2+A_2\bar{\gamma}+A_3)=0$$

Hence, either $\bar{\gamma}_I=\lambda\bar{S}-\mu_1-\sigma_2\overline{W}$ or $\bar{\gamma}_Y=\frac{\theta_1\bar{S}}{\beta+\bar{S}}-\mu_2$ or $\bar{\gamma}^3+A_1\bar{\gamma}^2+A_2\bar{\gamma}+A_3=0$, here $\bar{\gamma}_I$ and $\bar{\gamma}_Y$ represent the eigenvalues of $V(E_2)$ in the $I$ and $Y$ direction respectively and and the coefficients of the above third order polynomial can be written as:

$$A_1=\frac{r}{K}\bar{S}+\mu_4+\mu_3+\sigma_3\bar{S}>0$$

$$A_2=\left(\frac{r}{K}\bar{S}+\mu_4\right)\left(\mu_3+\sigma_3\bar{S}\right)+\frac{r}{K}\mu_4\bar{S}+\sigma_1\sigma_3\bar{S}\bar{Z}>0$$

$$A_3=\frac{r}{K}\mu_4\bar{S}\left(\mu_3+\sigma_3\bar{S}\right)+\sigma_1\sigma_3\mu_3\bar{S}\bar{Z}>0$$

Also

$$\Delta=A_1A_2-A_3=\left(\frac{r}{K}\bar{S}+\mu_4\right)\left[\frac{r}{K}\mu_4+\sigma_2\sigma_3\bar{Z}\right]\bar{S}+\sigma_1\sigma_3{}^2\bar{S}^2\bar{Z}$$
$$+\left(\frac{r}{K}\bar{S}+\mu_4+\mu_3+\sigma_3\bar{S}\right)\left(\frac{r}{K}\bar{S}+\mu_4\right)\left(\mu_3+\sigma_3\bar{S}\right)>0$$

Therefore by using Routh-Huritiz criterion all the roots of the above third order polynomial (the eigenvalues of $V(E_2)$ in the $S-$direction, $Z-$direction and $W-$direction) have negative real parts. However the eigenvalues $\bar{\gamma}_I$ and $\bar{\gamma}_Y$ are negative provided that conditions (11a) and (11b) are satisfied. Consequently, the equilibrium point $E_2$ is locally asymptotically stable under the given conditions and then the proof is complete. ■

**Theorem4.2.** Assume that the disease free equilibrium point $E_3=(\hat{S},0,\hat{Y},\hat{Z},\hat{W})$ of system (2) exists. Then it is locally asymptotically stable provided that

$$\frac{\lambda\hat{S}-(\mu_1+\sigma_2\hat{W})}{\alpha_2}<\frac{\hat{Y}}{\beta+\hat{S}}<\frac{r}{\alpha_1K} \tag{12}$$

**Proof.** It is easy to verify that the Jacobian matrix for the system (2) at $E_3=(\hat{S},0,\hat{Y},\hat{Z},\hat{W})$ is given by

$$V(E_3)=\begin{pmatrix} \frac{\alpha_1\hat{S}\hat{Y}}{(\beta+\hat{S})}-\frac{r\hat{S}}{K} & \hat{S}\left[-\lambda+\frac{\alpha_1 m\hat{Y}}{(\beta+\hat{S})^2}\right] & -\frac{\alpha_1\hat{S}}{(\beta+\hat{S})} & 0 & -\sigma_1\hat{S} \\ 0 & \lambda\hat{S}-\frac{\alpha_2\hat{Y}}{(\beta+\hat{S})}-\mu_1-\sigma_2\hat{W} & 0 & 0 & 0 \\ \frac{\theta_1\beta\hat{Y}}{(\beta+\hat{S})^2} & \hat{Y}\left[\frac{\theta_2\beta+\hat{S}(\theta_2-m\theta_1)}{(\beta+\hat{S})^2}\right] & 0 & 0 & 0 \\ -\sigma_3\hat{Z} & -\sigma_3\hat{Z} & 0 & -(\mu_3+\sigma_3\hat{S}) & 0 \\ \sigma_3\hat{Z} & \sigma_3\hat{Z} & 0 & \sigma_3\hat{S} & -\mu_4 \end{pmatrix}$$

Then the characteristic equation of $V(E_3)$ can be written as:

$$\left(\lambda\hat{S}-\frac{\alpha_2\hat{Y}}{\beta+\hat{S}}-\mu_1-\sigma_2\hat{W}-\hat{\gamma}\right)(\hat{\gamma}^4+\hat{D}_1\hat{\gamma}^3+\hat{D}_2\hat{\gamma}^2+\hat{D}_3\hat{\gamma}+\hat{D}_4)=0$$

So either $\hat{\gamma}_I=\lambda\hat{S}-\frac{\alpha_2\hat{Y}}{\beta+\hat{S}}-\mu_1-\sigma_2\hat{W}$ or $(\hat{\gamma}^4+\hat{D}_1\hat{\gamma}^3+\hat{D}_2\hat{\gamma}^2+\hat{D}_3\hat{\gamma}+\hat{D}_4)=0$, where $\hat{\gamma}_I$ represents the eigenvalue of $V(E_3)$ in the $I-$direction and the coefficients of the above fourth order polynomial can be written as:

$$\hat{D}_1=-c_{11}+\hat{R}_1,$$

$$\hat{D}_2=\hat{R}_2-c_{11}\hat{R}_1+\hat{R}_3+\hat{R}_4,$$

$$\hat{D}_3=\hat{R}_1\hat{R}_2-c_{11}\hat{R}_4+\mu_3\hat{R}_3$$

$$\hat{D}_4=\hat{R}_2\hat{R}_4$$

with

$$\hat{R}_1=\mu_3+\mu_4+\sigma_3\hat{S}>0,$$

$$\hat{R}_2=\frac{\alpha_1\theta_1\beta\hat{S}\hat{Y}}{(\beta+\hat{S})^3}>0$$

$$\hat{R}_3=\sigma_1\sigma_3\hat{S}\hat{Z}>0,$$

$$\hat{R}_4=\mu_3\mu_4+\sigma_3\mu_4\hat{S}>0$$

Clearly we have $\hat{\gamma}_I<0$ and $c_{11}=\frac{\alpha_1\hat{S}\hat{Y}}{\beta+\hat{S}}-\frac{r\hat{S}}{K}<0$ under condition (12). Consequently, we obtain that $\hat{D}_i>0$ for all values of $i=1,2,3,4$ . Further more since the value $\hat{\Delta}=(\hat{D}_1\hat{D}_2-\hat{D}_3)\hat{D}_3-\hat{D}_1{}^2\hat{D}_4$ is determined as below

$$\begin{aligned}\hat{\Delta}=-c_{11}\hat{R}_2[\hat{R}_1\hat{R}_2+\mu_3\hat{R}_3+2\hat{R}_1\hat{R}_4]+\hat{R}_1\hat{R}_4[\mu_3\hat{R}_3-c_{11}\hat{R}_4]\\+\hat{D}_3[-c_{11}(\hat{R}_3+\hat{R}_1\hat{D}_1)+\hat{R}_3(\hat{R}_1-\mu_3)]>0\end{aligned}$$

Thus by Routh-Hurwitz criterion all the roots of the above fourth order polynomial (eigenvalues of $V(E_3)$ in the $S-$direction, $Y-$direction, $Z-$direction and $W-$direction) have negative real parts, and hence $E_3=(\hat{S},0,\hat{Y},\hat{Z},\hat{W})$ is locally asymptotically stable. ■

**Theorem4.3.** Assume that the predator free equilibrium point $E_4=(\tilde{S},\tilde{I},0,\tilde{Z},\tilde{W})$ of system (2) exists. Then it is locally asymptotically stable provided that

$$\sigma_1\tilde{S}>\sigma_2 \tag{13a}$$

$$(\tilde{R}_7-\tilde{R}_1\tilde{R}_2)\tilde{D}_3<(\tilde{R}_1+\tilde{R}_2)(\tilde{R}_4+\tilde{R}_5)\tilde{R}_6 \tag{13b}$$

$$(\tilde{R}_2-\tilde{R}_1)\left(\tilde{R}_3-\mu_4[\mu_3+\sigma_3(\tilde{S}+\tilde{I})]\right)>\sigma_3\tilde{Z}(\tilde{R}_4+\tilde{R}_5)+(\sigma_1\tilde{S}+\sigma_2\tilde{I})\tilde{R}_6 \tag{13c}$$

$$\tilde{\theta}_1\tilde{S}+\tilde{\theta}_2\tilde{I}<\mu_2\tilde{B} \tag{13d}$$

Where $\tilde{R}_i; i=1,2,\cdots,7$, $\tilde{D}_j; j=1,2,3,4$ and $\tilde{B}$ are stated in the proof.

**Proof.** It is easy to verify that the Jacobian matrix for the system (2) at the point $E_4$ is given by $V(E_4)=(c_{ij})_{5\times5}$ and $i,j=1,2,...,5$; where

$$c_{11}=-\frac{r\tilde{S}}{K},\ c_{12}=-\frac{\lambda\tilde{S}}{\tilde{A}^2},\ c_{13}=-\frac{\alpha_1\tilde{S}}{\tilde{B}},\ c_{14}=0,\ c_{15}=-\sigma_1\tilde{S},\ c_{21}=\frac{\lambda\tilde{I}}{\tilde{A}},$$

$$c_{22}=-\frac{\lambda\tilde{S}\tilde{I}}{\tilde{A}^2},\ c_{23}=-\frac{\alpha_2\tilde{I}}{\tilde{B}},\ c_{24}=0,\ c_{25}=-\sigma_2\tilde{I},\ c_{31}=0,\ c_{32}=0,$$

$$c_{33}=\frac{\tilde{S}(\theta_1-\mu_2)+\tilde{I}(\theta_2-m\mu_2)-\beta\mu_2}{\tilde{B}},\ c_{34}=0,\ c_{35}=0,\ c_{41}=-\sigma_3\tilde{Z},$$

$$c_{42}=-\sigma_3\tilde{Z},\ c_{43}=0,\ c_{44}=-(\mu_3+\sigma_3(\tilde{S}+\tilde{I})),\ c_{45}=0,\ c_{51}=\sigma_3\tilde{Z},$$

$$c_{52}=\sigma_3\tilde{Z},\ c_{53}=0,\ c_{54}=\sigma_3(\tilde{S}+\tilde{I}),\ c_{55}=-\mu_4.$$

here $\tilde{A}=1+\tilde{I}$ and $\tilde{B}=\beta+\tilde{S}+m\tilde{I}$. Therefore the characteristic equation of the $V(E_4)$ is given by:

$$(c_{33}-\tilde{\gamma})(\tilde{\gamma}^4+D_1\tilde{\gamma}^3+D_2\tilde{\gamma}^2+D_3\tilde{\gamma}+D_4)=0$$

Then, either $\tilde{\gamma}_Y=c_{33}$ or $\tilde{\gamma}^4+\tilde{D}_1\tilde{\gamma}^3+\tilde{D}_2\tilde{\gamma}^2+\tilde{D}_3\tilde{\gamma}+\tilde{D}_4=0$, where $\tilde{\gamma}_Y$ denotes to the eigenvalue of $V(E_4)$ in the $Y-$direction, while the coefficients of the above fourth order polynomial are determined as follows:

$$\tilde{D}_1 = -(\tilde{R}_1 + \tilde{R}_2)$$

$$\tilde{D}_2 = \tilde{R}_1\tilde{R}_2 + \tilde{R}_3 - \tilde{R}_7 + \mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})]$$

$$\tilde{D}_3 = -\tilde{R}_2\tilde{R}_3 - \mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})]\tilde{R}_1 + \sigma_3\tilde{Z}(\tilde{R}_4 + \tilde{R}_5) + (\sigma_1\tilde{S} + \sigma_2\tilde{I})\tilde{R}_6$$

$$\tilde{D}_4 = \mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})]\tilde{R}_3 + (\tilde{R}_4 + \tilde{R}_5)\tilde{R}_6$$

with

$$\tilde{R}_1 = c_{11} + c_{22} = -\tilde{S}\left(\frac{r}{K} + \frac{\lambda\tilde{I}}{(1+\tilde{I})^2}\right) < 0$$

$$\tilde{R}_2 = c_{44} + c_{55} = -\left(\mu_4 + \mu_3 + \sigma_3(\tilde{S} + \tilde{I})\right) < 0$$

$$\tilde{R}_3 = c_{11}c_{22} - c_{12}c_{21} = \frac{r\lambda\tilde{S}^2\tilde{I}}{K(1+\tilde{I})^2} + \frac{\lambda^2\tilde{S}\tilde{I}}{(1+\tilde{I})^3} > 0$$

$$\tilde{R}_4 = c_{15}c_{22} - c_{12}c_{25} = \frac{\lambda\tilde{S}\tilde{I}}{(1+\tilde{I})^2}(\sigma_1\tilde{S} - \sigma_2)$$

$$\tilde{R}_5 = c_{11}c_{25} - c_{15}c_{21} = \frac{r\sigma_2\tilde{S}\tilde{I}}{K} + \frac{\lambda\sigma_1\tilde{S}\tilde{I}}{1+\tilde{I}} > 0$$

$$\tilde{R}_6 = c_{41}c_{54} - c_{44}c_{51} = c_{42}c_{54} - c_{44}c_{52} = \mu_3\sigma_3\tilde{Z} > 0$$

$$\tilde{R}_7 = c_{15}c_{51} + c_{25}c_{52} = -\sigma_3\tilde{Z}(\sigma_1\tilde{S} + \sigma_2\tilde{I}) < 0$$

Clearly we have $\tilde{R}_4 > 0$ due to conditions (13a), hence $\tilde{D}_i > 0$ for all values of $i = 1,2,3,4$.

Moreover, $\tilde{\Delta} = (\tilde{D}_1\tilde{D}_2 - \tilde{D}_3)\tilde{D}_3 - \tilde{D}_1{}^2\tilde{D}_4$ can be computed as follows:

$$\begin{aligned}\tilde{\Delta} = (\tilde{R}_1 + \tilde{R}_2)&\left[(\tilde{R}_7 - \tilde{R}_1\tilde{R}_2)\tilde{D}_3 - (\tilde{R}_1 + \tilde{R}_2)(\tilde{R}_4 + \tilde{R}_5)\tilde{R}_6\right] \\ &+ \left[\mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})](\tilde{R}_4 + \tilde{R}_5) + (\sigma_1\tilde{S} + \sigma_2\tilde{I})\tilde{R}_6\right] \\ &\quad \times\left[(\tilde{R}_2 - \tilde{R}_1)\left(\tilde{R}_3 - \mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})]\right)\right. \\ &\qquad\qquad \left. - \sigma_3\tilde{Z}(\tilde{R}_4 + \tilde{R}_5) - (\sigma_1\tilde{S} + \sigma_2\tilde{I})\tilde{R}_6\right] \\ &\qquad + \tilde{R}_1\tilde{R}_2\left[\tilde{R}_3 - \mu_4[\mu_3 + \sigma_3(\tilde{S} + \tilde{I})]\right]^2\end{aligned}$$

Now it is easy to verify that the first term of $\tilde{\Delta}$ is positive under conditions (13a) and (13b), while the second term is positive under conditions (13a) and (13c). Consequently $\tilde{\Delta} > 0$ under the conditions (13a)-(13c) and hence by Routh-Hurwitz criterion all the roots of the above fourth order polynomial (eigenvalues of $V(E_4)$ in the $S-$direction, $I-$direction, $Z-$direction and $W-$direction) have negative real parts. In addition to the above, it is clear that condition (13d) guarantees the negativity of the eigenvalue $\tilde{\gamma}_Y$. Therefore the predator free equilibrium point $E_4=(\tilde{S},\tilde{I},0,\tilde{Z},\tilde{W})$ is locally asymptotically stable and then the proof is complete. ■

**Theorem4.4.** Assume that the positive equilibrium point $E_5=(\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W})$ of the system (2) exists and let the following inequalities hold:

$$\hat{S}<\frac{\beta\theta_2}{m\theta_1-\theta_2} \quad \text{or} \quad \hat{I}<\frac{\beta\theta_1}{\theta_2-m\theta_1} \tag{14a}$$

$$\frac{r}{K}>\frac{\alpha_1\hat{Y}}{\hat{B}^2} \tag{14b}$$

$$\frac{\lambda\hat{S}}{\hat{A}^2}>\frac{m\alpha_2\hat{Y}}{\hat{B}^2} \tag{14c}$$

$$(d_{12})^2<\frac{4}{9}d_{11}d_{22} \tag{14d}$$

$$(d_{15})^2<\frac{4}{9}d_{11}d_{55} \tag{14e}$$

$$(d_{14})^2=(d_{24})^2<\min\left\{\frac{4}{9}d_{11}d_{44},\frac{4}{9}d_{22}d_{44}\right\} \tag{14f}$$

$$(d_{25})^2<\frac{4}{9}d_{22}d_{55} \tag{14g}$$

$$(d_{45})^2<\frac{4}{9}d_{44}d_{55} \tag{14h}$$

where $d_{ij},\ \forall i,j=1,2,\cdots,5$ are determined in then proof . Then $E_5$ is locally asymptotically stable.

**Proof.** It is easy to verify that, the linearized system of system (2) can be written as

$$\frac{dX}{dt}=\frac{dU}{dt}=V(E_5)\,U$$

here $X=(S,I,Y,Z,W)^t$ and $U=(u_1,u_2,u_3,u_4,u_5)^t$ with $u_1=S-\hat{S}$ , $u_2=I-\hat{I}$ , $u_3=Y-\hat{Y}$ , $u_4=Z-\hat{Z}$ and $u_5=W-\hat{W}$ . Moreover, $V(E_5)=(\hat{c}_{ij})_{5\times5}$ ; $i,j=1,2,\ldots,5$ represents the Jacobian matrix of system (2) at the positive equilibrium point $E_5$ and has the following elements:

$$\hat{c}_{11}=-r\frac{\hat{S}}{K}+\frac{\alpha_1\hat{S}\hat{Y}}{\hat{B}^2},\ \hat{c}_{12}=\frac{\alpha_1m\hat{S}\hat{Y}}{\hat{B}^2}-\frac{\lambda\hat{S}}{\hat{A}^2},\ \hat{c}_{13}=-\frac{\alpha_1\hat{S}}{\hat{B}},\ \hat{c}_{14}=0,\ \hat{c}_{15}=-\sigma_1\hat{S},$$

$$\widehat{c}_{21}=\frac{\lambda\widehat{I}}{\widehat{A}}+\frac{\alpha_2\widehat{I}\widehat{Y}}{\widehat{B}^2},\ \widehat{c}_{22}=\frac{-\lambda\widehat{S}\widehat{I}}{\widehat{A}^2}+\frac{m\alpha_2\widehat{I}\widehat{Y}}{\widehat{B}^2},\ \widehat{c}_{23}=-\frac{\alpha_2\widehat{I}}{\widehat{B}},\ \widehat{c}_{24}=0,\ \widehat{c}_{25}=-\sigma_2\widehat{I},$$

$$\widehat{c}_{31}=\frac{\beta\theta_1\widehat{Y}+(m\theta_1-\theta_2)\widehat{I}\widehat{Y}}{\widehat{B}^2},\ \widehat{c}_{32}=\frac{\beta\theta_2\widehat{Y}+(\theta_2-m\theta_1)\widehat{S}\widehat{Y}}{\widehat{B}^2},\ \widehat{c}_{33}=0,\ \widehat{c}_{34}=0,\ \widehat{c}_{35}=0,$$

$$\widehat{c}_{41}=-\sigma_3\widehat{Z},\ \widehat{c}_{42}=-\sigma_3\widehat{Z},,\ \widehat{c}_{43}=0,\ \widehat{c}_{44}=-\mu_3-\sigma_3(\widehat{S}+\widehat{I}),\ \widehat{c}_{45}=0,$$

$$\widehat{c}_{51}=\sigma_3\widehat{Z},\ \widehat{c}_{52}=\sigma_3\widehat{Z},\ \widehat{c}_{53}=0,\ \widehat{c}_{54}=\sigma_3\left(\widehat{S}+\widehat{I}\right)\ \text{and}\ \ \widehat{c}_{55}=-\mu_4.$$

here $\widehat{A}=1+\widehat{I}$ and $\widehat{B}=\beta+\widehat{S}+m\widehat{I}$. Now, consider the following function

$$V=a_1\frac{u_1^2}{2\widehat{S}}+a_2\frac{u_2^2}{2\widehat{I}}+a_3\frac{u_3^2}{2\widehat{Y}}+a_4\frac{u_4^2}{2}+a_5\frac{u_5^2}{2}$$

where $a_i,\ i=1,2,3,4,5$ are positive constants to be chosen appropriately. It is clearly that $V:\Re_+^5\to\Re$ and is a continuously differentiable function with $V(0,0,0,0,0)=0$ and $V(S,I,Y,Z,W)>0$ for all $(S,I,Y,Z,W)\in\Re_+^5$ and $(S,I,Y,Z,W)\neq(0,0,0,0,0)$. Hence it is a positive definite function. Now, by differentiating $V$ with respect to time $t$, we obtain

$$\frac{dV}{dt}=a_1\frac{u_1}{\widehat{S}}\frac{du_1}{dt}+a_2\frac{u_2}{\widehat{I}}\frac{du_2}{dt}+a_3\frac{u_3}{\widehat{Y}}\frac{du_3}{dt}+a_4u_4\frac{du_4}{dt}+a_5u_5\frac{du_5}{dt}$$

Substituting the values of $\frac{du_1}{dt},\frac{du_2}{dt},\frac{du_3}{dt},\frac{du_4}{dt}$, and $\frac{du_5}{dt}$ in the above equation, and after doing some algebraic manipulation; we get that:

$$\begin{aligned}\frac{dV}{dt}&=a_1\frac{\widehat{c}_{11}}{\widehat{S}}u_1^{\ 2}+a_2\frac{\widehat{c}_{22}}{\widehat{I}}u_2^{\ 2}+a_4\widehat{c}_{44}u_4^{\ 2}+a_5\widehat{c}_{55}u_5^{\ 2}\\&+u_1u_2\left(a_1\frac{\widehat{c}_{12}}{\widehat{S}}+a_2\frac{\widehat{c}_{21}}{\widehat{I}}\right)+u_1u_3\left(a_1\frac{\widehat{c}_{13}}{\widehat{S}}+a_3\frac{\widehat{c}_{31}}{\widehat{Y}}\right)+u_1u_5\left(a_1\frac{\widehat{c}_{15}}{\widehat{S}}+a_2\widehat{c}_{51}\right)\\&+u_2u_3\left(a_2\frac{\widehat{c}_{23}}{\widehat{I}}+a_3\frac{\widehat{c}_{32}}{\widehat{Y}}\right)+u_2u_5\left(a_2\frac{\widehat{c}_{25}}{\widehat{I}}+a_5\widehat{c}_{52}\right)+a_4\widehat{c}_{41}u_1u_4\\&+a_4\widehat{c}_{41}u_2u_4+a_5\widehat{c}_{54}u_4u_5\end{aligned}$$

Obviously, condition (14a) guarantees that $\widehat{c}_{31}>0$ and $\widehat{c}_{32}>0$, condition (14b) guarantees that $\widehat{c}_{11}<0$, while condition (14c) guarantees that $\widehat{c}_{22}<0$. Now by choosing the constants as $a_3=a_4=a_5=1$, $a_1=\frac{\widehat{c}_{31}\widehat{B}}{\alpha_1\widehat{Y}}$ and $a_2=\frac{\widehat{c}_{32}\widehat{B}}{\alpha_2\widehat{Y}}$. Therefore, by substituting the values of $a_i,\ i=1,2,3,4,5$ in $\frac{dV}{dt}$ and then rearrange the resulting terms, we get

$$\begin{aligned}\frac{dV}{dt} &= -\frac{d_{11}}{3}u_1{}^2 + d_{12}u_1u_2 - \frac{d_{22}}{3}u_2{}^2 - \frac{d_{11}}{3}u_1{}^2 + d_{15}u_1u_5 - \frac{d_{55}}{3}u_5{}^2 \\ &\quad - \frac{d_{11}}{3}u_1{}^2 + d_{14}u_1u_4 - \frac{d_{44}}{3}u_4{}^2 - \frac{d_{22}}{3}u_2{}^2 + d_{25}u_2u_5 - \frac{d_{55}}{3}u_5{}^2 \\ &\quad - \frac{d_{22}}{3}u_2{}^2 + d_{24}u_2u_4 - \frac{d_{44}}{3}u_4{}^2 - \frac{d_{44}}{3}u_4{}^2 + d_{45}u_4u_5 - \frac{d_{55}}{3}u_5{}^2\end{aligned}$$

Where

$$d_{11} = \frac{\hat{c}_{31}\hat{B}}{\alpha_1\hat{Y}}\left(\frac{r}{K} - \frac{\alpha_1\hat{Y}}{\hat{B}^2}\right),\ d_{12} = \frac{\hat{c}_{31}\hat{B}}{\alpha_1\hat{Y}}\left(\frac{\alpha_1 m\hat{Y}}{\hat{B}^2} - \frac{\lambda}{\hat{A}^2}\right) + \frac{\hat{c}_{32}\hat{B}}{\alpha_2\hat{Y}}\left(\frac{\lambda}{\hat{A}} + \frac{\alpha_2\hat{Y}}{\hat{B}^2}\right),\ d_{14} = -\sigma_3\hat{Z}$$

$$d_{15} = -\frac{\hat{c}_{31}\hat{B}\sigma_1}{\alpha_1\hat{Y}} + \sigma_3\hat{Z},\ d_{22} = \frac{\hat{c}_{32}\hat{B}}{\alpha_2\hat{Y}}\left(\frac{\lambda\hat{S}}{\hat{A}^2} - \frac{\alpha_2 m\hat{Y}}{\hat{B}^2}\right),\ d_{24} = -\sigma_3\hat{Z},$$

$$d_{25} = -\frac{\hat{c}_{32}\hat{B}\sigma_2}{\alpha_2\hat{Y}} + \sigma_3\hat{Z},\ d_{44} = \mu_3 + \sigma_3(\hat{S} + \hat{I}),\ d_{45} = \sigma_3(\hat{S} + \hat{I}),\ d_{55} = \mu_4$$

Obviously, conditions (14a) and (14b) guarantee that $d_{11} > 0$, while conditions (14a) with (14c) guarantee that $d_{22} > 0$. Hence due to the given conditions (14d)-(14h), then $\frac{dV}{dt}$ will be negative . Consequently $\frac{dV}{dt} < 0$, according to the Lyapunov stability theorem the origin and hence $E_5 = (\hat{S}, \hat{I}, \hat{Y}, \hat{Z}, \hat{W})$ is locally asymptotically stable point. ■

**Theorem4.5.** Assume that the positive equilibrium point $E_5 = (\hat{S}, \hat{I}, \hat{Y}, \hat{Z}, \hat{W})$ is locally asymptotically stable. Then it is a globally asymptotically stable in the sub region $\Omega$ of $Int.R_+^5$, that satisfy the following conditions.

$$\hat{Y} < \min\left\{\frac{r P_2(S,I)}{\alpha_1 K}, \frac{\lambda\hat{S}P_2(S,I)}{\alpha_2 m P_1(I)}\right\} \tag{15a}$$

$$(q_{12})^2 < \frac{4}{9}q_{11}q_{22} \tag{15b}$$

$$(q_{14})^2 = (q_{24})^2 < \min\left\{\frac{4}{9}q_{11}q_{44}, \frac{4}{9}q_{22}q_{44}\right\} \tag{15c}$$

$$(q_{15})^2 < \frac{4}{9}q_{11}q_{55} \tag{15d}$$

$$(q_{25})^2 < \frac{4}{9}q_{22}q_{55} \tag{15e}$$

$$(q_{45})^2 < \frac{4}{9}q_{44}q_{55} \tag{15f}$$

where $P_1(I) = (1+I)(1+\hat{I}) = A\hat{A}$ , $P_2(S,I) = (\beta + S + mI)(\beta + \hat{S} + m\hat{I}) = B\hat{B}$ and

$q_{ij};\ \forall\, i,\, j = 1,2,\cdots,5$ are given in the proof.

**Proof.** Consider the following function:

$$V(S,I,Y,Z,W) = \hat{C}_1\left(S - \hat{S} - \hat{S}\ln\left(\frac{S}{\hat{S}}\right)\right) + \hat{C}_2\left(I - \hat{I} - \hat{I}\ln\left(\frac{I}{\hat{I}}\right)\right) + \hat{C}_3\left(Y - \hat{Y} - \hat{Y}\ln\left(\frac{Y}{\hat{Y}}\right)\right) + \hat{C}_4\frac{\left(Z-\hat{Z}\right)^2}{2} + \hat{C}_5\frac{\left(W-\hat{W}\right)^2}{2}$$

where $\hat{C}_i, \forall i = 1,2,\cdots 5$ are positive constants to be determined. It is easy to see that $V(S,I,Y,Z,W) \in C^1(R^5, R)$ and $V(\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W}) = 0$, while $V(S,I,Y,Z,W) > 0$ for all $(S,I,Y,Z,W) \in R_+^5$ with $(S,I,Y,Z,W) \neq (\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W})$, then

$$\begin{aligned}\frac{dV}{dt} &= \hat{C}_1(S-\hat{S})\left[r\left(1-\frac{S}{K}\right) - \frac{\lambda I}{A} - \frac{\alpha_1 Y}{B} - \sigma_1 W\right] \\ &\quad + \hat{C}_2(I-\hat{I})\left[\frac{\lambda S}{A} - \frac{\alpha_2 Y}{B} - \mu_1 - \sigma_2 W\right] + \hat{C}_3(Y-\hat{Y})\left[\frac{\theta_1 S + \theta_2 I}{B} - \mu_2\right] \\ &\quad + \hat{C}_4(Z-\hat{Z})\left[Q - \mu_3 Z - \sigma_3 Z(S+I)\right] - \hat{C}_5(W-\hat{W})\left[\sigma_3 Z(S+I) - \mu_4 W\right]\end{aligned}$$

Then after doing some algebraic manipulations, we get

$$\begin{aligned}\frac{dV}{dt} &= -\hat{C}_1\left[\frac{r}{K} - \frac{\alpha_1\hat{Y}}{P_2(S,I)}\right](S-\hat{S})^2 - \hat{C}_2\left[\frac{\lambda\hat{S}}{P_1(I)} - \frac{m\alpha_2\hat{Y}}{P_2(S,I)}\right](I-\hat{I})^2 \\ &\quad - \hat{C}_4\left[\mu_3 + \sigma_3(\hat{S}+\hat{I})\right](Z-\hat{Z})^2 - \hat{C}_5\mu_4(W-\hat{W})^2 \\ &\quad + \left[\hat{C}_1\left(\frac{\alpha_1 m\hat{Y}}{P_2(S,I)} - \frac{\lambda}{P_1(I)}\right) + \hat{C}_2\left(\frac{\lambda\hat{A}}{P_1(I)} + \frac{\alpha_2\hat{Y}}{P_2(S,I)}\right)\right](S-\hat{S})(I-\hat{I}) \\ &\quad + \left[\frac{\hat{C}_3}{P_2(S,I)}\left[\theta_1\beta + (\theta_1 m - \theta_2)\hat{I}\right] - \frac{\hat{C}_1\alpha_1\hat{B}}{P_2(S,I)}\right](S-\hat{S})(Y-\hat{Y}) \\ &\quad - \hat{C}_4\sigma_3 Z(S-\hat{S})(Z-\hat{Z}) + \left[\hat{C}_5\sigma_3 Z - \sigma_1\hat{C}_1\right](S-\hat{S})(W-\hat{W}) \\ &\quad + \left[\frac{\hat{C}_3}{P_2(S,I)}\left[\theta_2\beta + (\theta_2 - \theta_1 m)\hat{S}\right] - \frac{\hat{C}_2\alpha_2\hat{B}}{P_2(S,I)}\right](I-\hat{I})(Y-\hat{Y}) \\ &\quad - \hat{C}_4\sigma_3 Z(I-\hat{I})(Z-\hat{Z}) + \left[\hat{C}_5\sigma_3 Z - \sigma_2\hat{C}_2\right](I-\hat{I})(W-\hat{W}) \\ &\quad + \hat{C}_5\sigma_3(\hat{S}+\hat{I})(Z-\hat{Z})(W-\hat{W})\end{aligned}$$

So by choosing the constants $\hat{C}_i, \forall i = 1,2,\cdots 5$ as follow

$$\hat{C}_1 = 1, \hat{C}_2 = \frac{\alpha_1\left(\theta_2\beta + (\theta_2 - \theta_1 m)\tilde{S}\right)}{\alpha_2\left(\theta_1\beta + (\theta_1 m - \theta_2)\tilde{I}\right)},\ \hat{C}_3 = \frac{\alpha_1(\beta + \tilde{S} + m\tilde{I})}{\theta_1\beta + (\theta_1 m - \theta_2)\tilde{I}} \text{ and } \hat{C}_4 = \hat{C}_5 = \frac{1}{\sigma_3}$$

which are positive due to the local stability condition (14a). Then we get that

$$\frac{dV}{dt} < -\frac{q_{11}}{3}(S-\hat{S})^2 + q_{12}(S-\hat{S})(I-\hat{I}) - \frac{q_{22}}{3}(I-\hat{I})^2$$
$$-\frac{q_{11}}{3}(S-\hat{S})^2 + q_{14}(S-\hat{S})(Z-\hat{Z}) - \frac{q_{44}}{3}(Z-\hat{Z})^2$$
$$-\frac{q_{11}}{3}(S-\hat{S})^2 + q_{15}(S-\hat{S})(W-\hat{W}) - \frac{q_{55}}{3}(W-\hat{W})^2$$
$$-\frac{q_{22}}{3}(I-\hat{I})^2 + q_{24}(I-\hat{I})(Z-\hat{Z}) - \frac{q_{44}}{3}(Z-\hat{Z})^2$$
$$-\frac{q_{22}}{3}(I-\hat{I})^2 + q_{25}(I-\hat{I})(W-\hat{W}) - \frac{q_{55}}{3}(W-\hat{W})^2$$
$$-\frac{q_{44}}{3}(Z-\hat{Z})^2 + q_{45}(Z-\hat{Z})(W-\hat{W}) - \frac{q_{55}}{3}(W-\hat{W})^2$$

where

$$q_{11} = \left[\frac{r}{K} - \frac{\alpha_1\hat{Y}}{P_2(S,I)}\right], \quad q_{22} = \frac{\alpha_1(\theta_2\beta + (\theta_2-\theta_1 m)\tilde{S})}{\alpha_2(\theta_1\beta + (\theta_1 m - \theta_2)\tilde{I})}\left[\frac{\lambda\hat{S}}{P_1(I)} - \frac{m\alpha_2\hat{Y}}{P_2(S,I)}\right],$$

$$q_{44} = \frac{1}{\sigma_3}\left[\mu_3 + \sigma_3(\hat{S}+\hat{I})\right], \quad q_{55} = \frac{1}{\sigma_3}\mu_4,$$

$$q_{12} = \left(\frac{\alpha_1 m\hat{Y}}{P_2(S,I)} - \frac{\lambda}{P_1(I)}\right) + \frac{\alpha_1(\theta_2\beta + (\theta_2-\theta_1 m)\tilde{S})}{\alpha_2(\theta_1\beta + (\theta_1 m - \theta_2)\tilde{I})}\left(\frac{\lambda\hat{A}}{P_1(I)} + \frac{\alpha_2\hat{Y}}{P_2(S,I)}\right), \quad q_{14} = q_{24} = -Z,$$

$$q_{15} = Z - \sigma_1, \quad q_{25} = Z - \sigma_2 \quad \text{and} \quad q_{45} = \hat{S} + \hat{I}$$

Clearly $q_{11}$ and $q_{22}$ are positive provided that condition (15a) holds. Consequently, due to conditions (15b)-(15f),we obtain that $\frac{dV}{dt} < 0$ is negative definite and hence $V$ is Lyapunov function with respect to $E_5 = (\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W})$. So $E_5 = (\hat{S},\hat{I},\hat{Y},\hat{Z},\hat{W})$ is a globally asymptotically stable in $\Omega \subset Int.R_+^5$ that satisfy the given conditions. ■

## 5. Numerical Simulation

In this section the global dynamics of system (2) is investigated numerically. The objectives are confirm our analytical results and discuss the role of the existence of disease and toxicant on the dynamical behaviour of the system. For the following set of hypothetical, biologically feasible, set of parameters, definitely different set of hypothetical parameters can be chosen also, system (2) is solved numerically starting at different initial points as illustrated in Fig. (1a)-(1e).

$$\begin{gathered} r=1, K=500, \lambda=1, \alpha_1=1, \alpha_2=1, \ \beta=20, m=1, \ \theta_1=0.5, \ \theta_2=0.5, \\ \mu_1=0.05, \ \mu_2=0.1, \ \sigma_1=0.001, \ \sigma_2=0.01, Q=5, \ \mu_3=0.2, \mu_4=0.2, \sigma_3=0.05 \end{gathered} \tag{16}$$

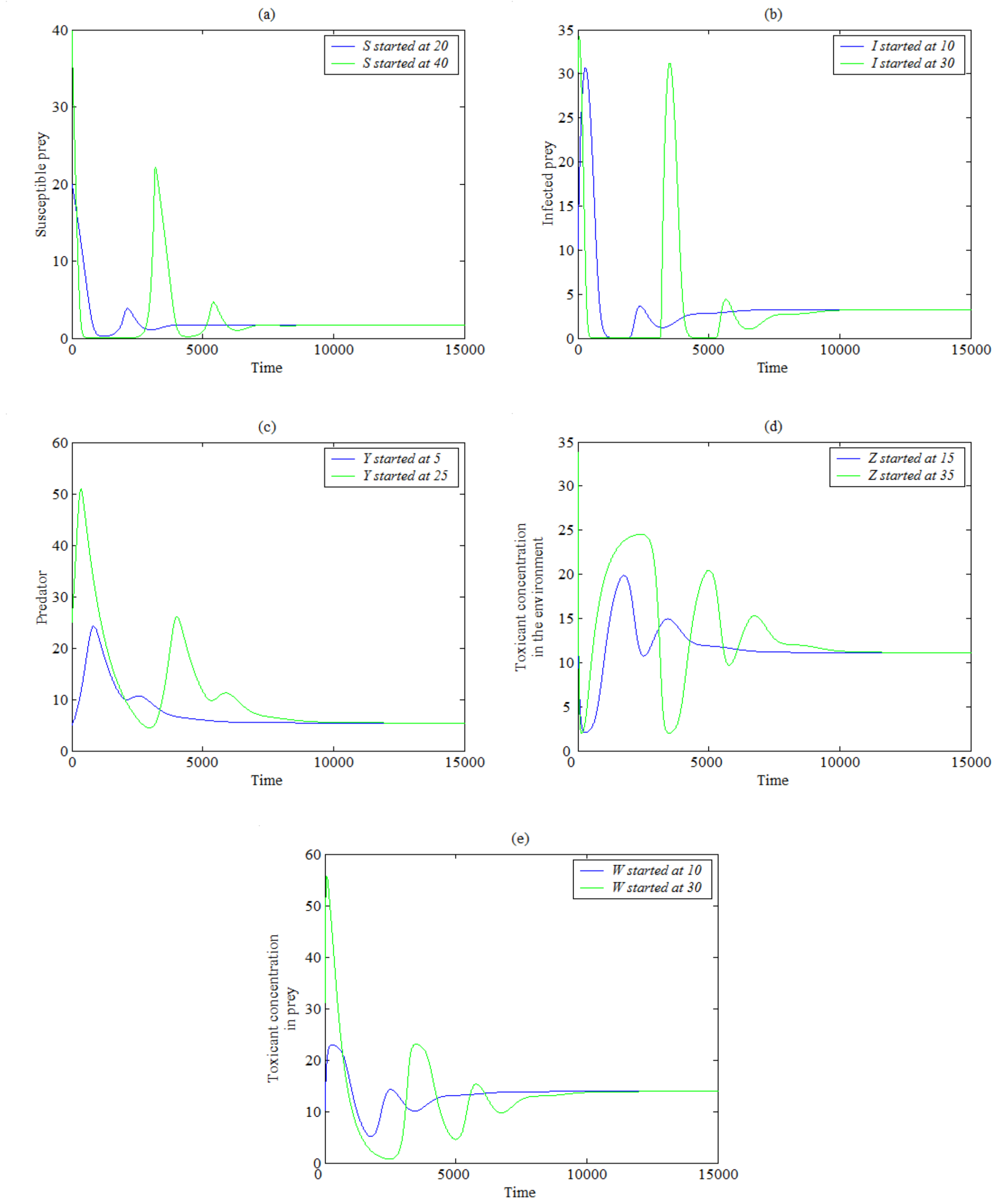


Fig. 1: The solution of system (2) approaches asymptotically to the positive equilibrium point $E_5 = (1.73, 3.26, 5.42, 11.11, 13.88)$ starting from different initial points. (a) Trajectories of $S$. (b) Trajectories of $I$. (c) Trajectories of $Y$. (d) Trajectories of $Z$. (e) Trajectories of $W$.

It is clear from above figures that, system (2) has a globally asymptotically stable point for the above set of data. However, for the above set of data with the intrinsic growth rate of the susceptible prey

$r = 1.25$, system (2) has a periodic dynamics in the $Int.R_+^5$ as illustrated in Fig. (2).

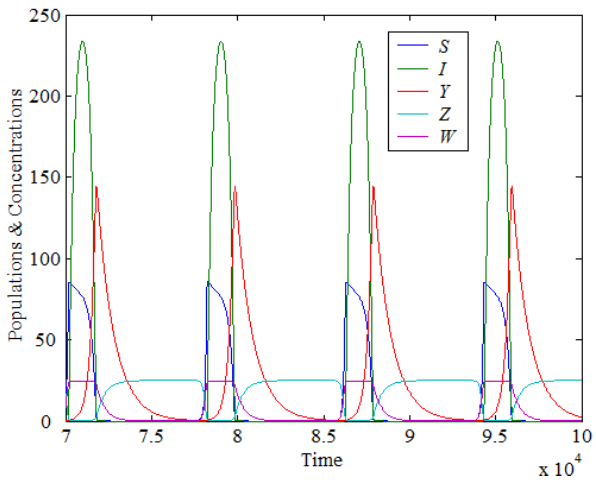


Fig. 2: The solution of system (2) approaches to periodic dynamics in the $Int.R_+^5$ for the parameter values in Eq. (16) with $r = 1.25$.

Further investigation has been down by varying the intrinsic growth rate of the susceptible prey keeping the rest of parameters as in Eq. (16), it is observed that, system (2) has a globally stable positive equilibrium point for the range $r \leq 1.2$, while it has a periodic dynamics for the range $r > 1.2$.

The effect of varying the infected rate on the dynamics of system (2) is studied. For the parameter values given in Eq. (16) with $\lambda \leq 0.8$, $0.8 < \lambda \leq 1.21$ and $\lambda \geq 1.22$ the solution of system (2) approaches to periodic attractor, positive equilibrium point $E_5$ and predator free equilibrium point $E_4$ respectively, as illustrated in the following two figures.

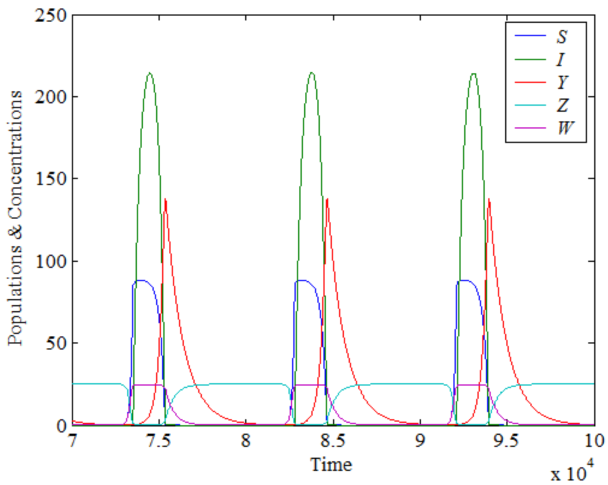


Fig. 3: The solution of system (2) approaches to periodic dynamics in the $Int.R_+^5$ for the parameter values in Eq. (16) with $\lambda = 0.8$.

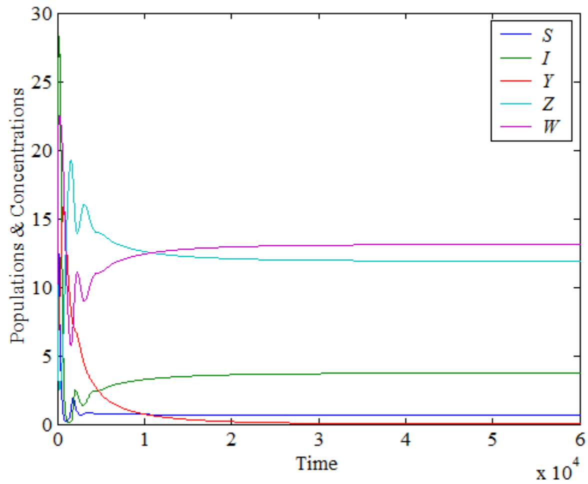


Fig. 4: The solution of system (2) approaches to predator free equilibrium point $E_4 = (0.68, 3.72, 0, 11.88, 13.11)$ for the parameter values in Eq. (16) with $\lambda = 1.25$.

The effect of varying the predation rates $\alpha_1$ and $\alpha_2$ on the dynamics of system (2) is also studied by solving the system numerically for the parameters values used in Fig. (2), that is mean Eq. (16) with $r = 1.25$, with $\alpha_1 = \alpha_2 = 2$ and the trajectories of system (2) are drawn in Fig. (5).

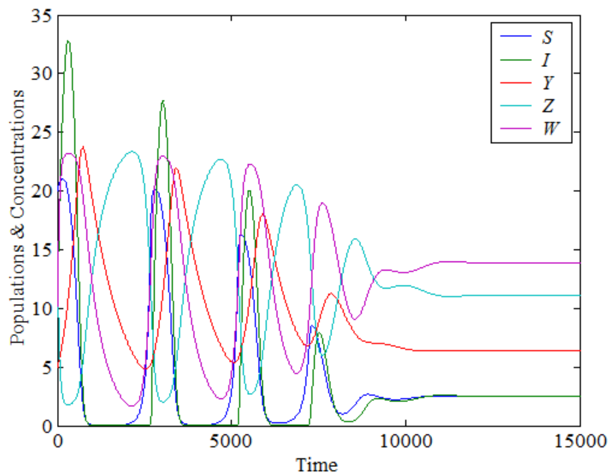


Fig. 5: The solution of system (2) approaches asymptotically to positive equilibrium point $E_5 = (2.47, 2.52, 6.42, 11.11, 13.88)$ in the $Int.R_{+}^{5}$ for the parameter values in Eq. (16) with $r = 1.25$ and $\alpha_1 = \alpha_2 = 2$.

According to the above figure, the solution of system (2) transfer from periodic as in Fig. (2) to positive equilibrium point when the predation rates $\alpha_1$ and $\alpha_2$ increase simultaneously to $\alpha_1 = \alpha_2 = 2$.

Now the effect of varying the half saturation constant $\beta$ on the dynamics of system (2) is discussed. it is observed that, for the data given by Eq. (16) with $\beta \leq 10$, $10 < \beta < 130$ and $\beta \geq 130$ the system approaches to periodic attractor, positive equilibrium point $E_5$ and predator free equilibrium point $E_4$ respectively, as illustrated in the following two figures.

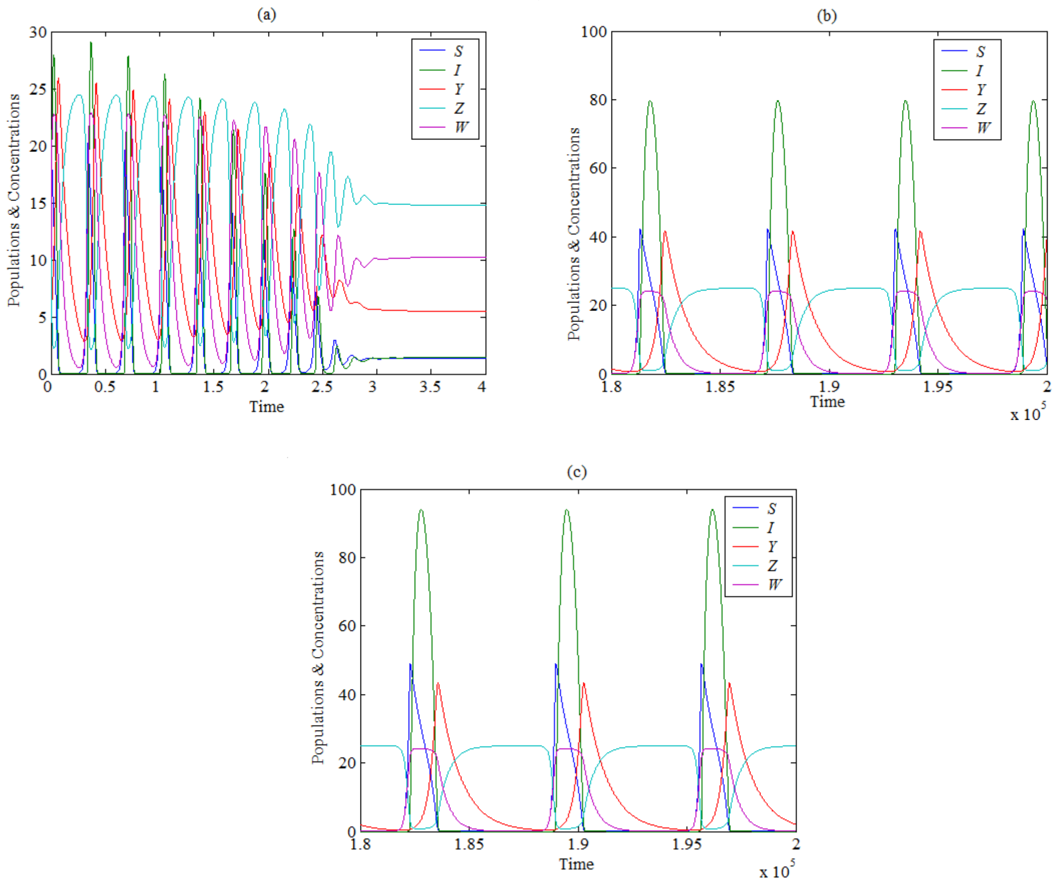


Fig. 6: The time series of system (2) for the data given by Eq. (16) with different values of the half saturation constant $\beta$. (a) System (2) approaches to positive equilibrium point for $\beta = 11$. (b) ) System (2) approaches to small periodic attractor for $\beta = 10$. (c) System (2) approaches to large periodic attractor for $\beta = 9$.

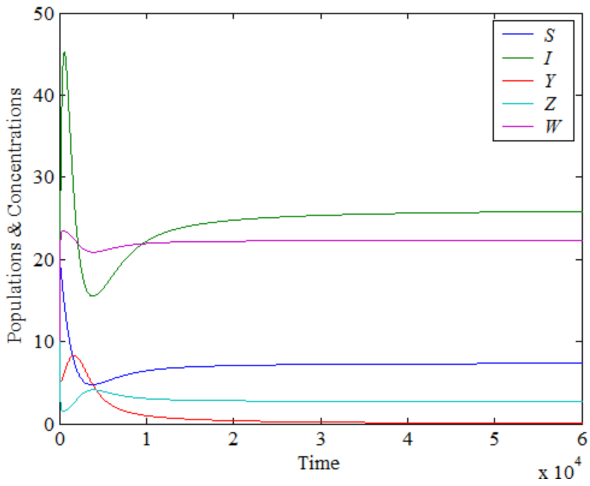


Fig. 7: The solution of system (2) approaches to predator free equilibrium point $E_4 = (7.36, 25.94, 0, 2.67, 22.32)$ for the parameter values in Eq. (16) with $\beta = 140$.

According to the Fig. (6), it is clear that system (2) undergo a Hopf bifurcation when the half saturation constant passes through the value $\beta = 10$. Note that, the effect of varying other parameters are also studied and the following results are observed. The solution of system (2) approaches asymptotically to the positive equilibrium point $E_5$ in case of increasing the death rate of the infected species $\mu_1$, however it approaches to predator free equilibrium point $E_4$ in case of increasing the natural death rate of predator species $\mu_2$. More over, the solution of system (2) approaches asymptotically to the predator free equilibrium point $E_4$ in case of decreasing the conversion rates $\theta_1$ and $\theta_2$ simultaneously. Also, it is observed that the solution of system (2) approaches asymptotically to the predator free equilibrium point $E_4$ in case of increasing the reduction rate of the susceptible prey due to the existence of toxicant given by $\sigma_1$.

Now the effect of varying the reduction rate of the infected prey due to the existence of toxicant , that is $\sigma_2$, on the dynamics of system (2) is investigated by solving the system (2) numerically using the data given in Eq. (16). It is observed that for $\sigma_2 \leq 0.26$ the solution of system (2) still has a globally asymptotically stable positive equilibrium point however for $\sigma_2 > 0.26$ system (2) losses the stability and approaches to periodic dynamics in $Int.R_+^5$. Further, for the data given by Eq. (16) with $r = 1.25$, the dynamics of system (2) transfer from periodic to asymptotic stable at the positive equilibrium point for $\sigma_2 = 0.05$ and then transfer again to periodic dynamic for $\sigma_2 = 0.2$ see Fig. (8a)-(8c).

Also, the effect of varying the exogenous input rate of the toxicant in the environment $Q$ on the dynamics of system (2) is studied numerically. It is observed that increasing the parameter $Q$ with the rest of parameters given by Eq. (16) with $r = 1.25$ leads to stability of the system, see for example the typical figure given by Fig. (9).

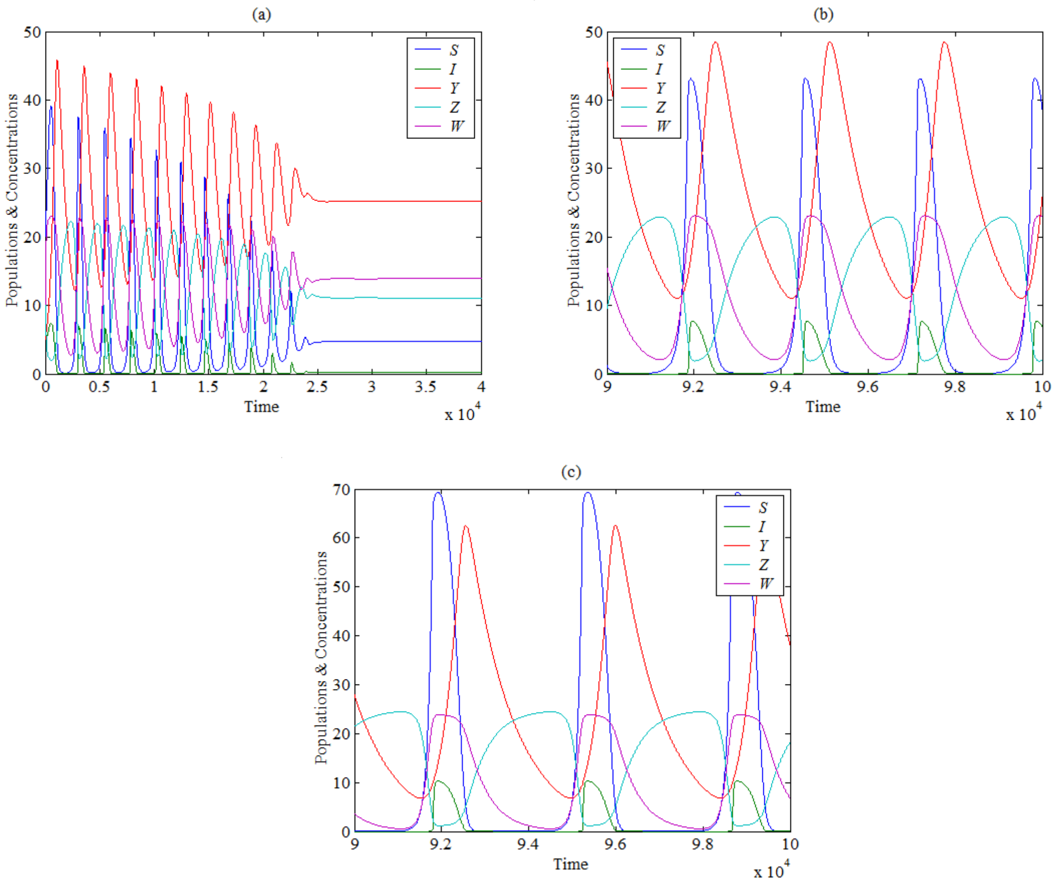


Fig. 8: The time series of system (2) for the data given by Eq. (16) with $r = 1.25$ for different values of the parameter $\sigma_2$. (a) System (2) approaches to positive equilibrium point for $\sigma_2 = 0.19$. (b) ) System (2) approaches to small periodic attractor for $\sigma_2 = 0.2$. (c) System (2) approaches to large periodic attractor for $\sigma_2 = 0.25$.

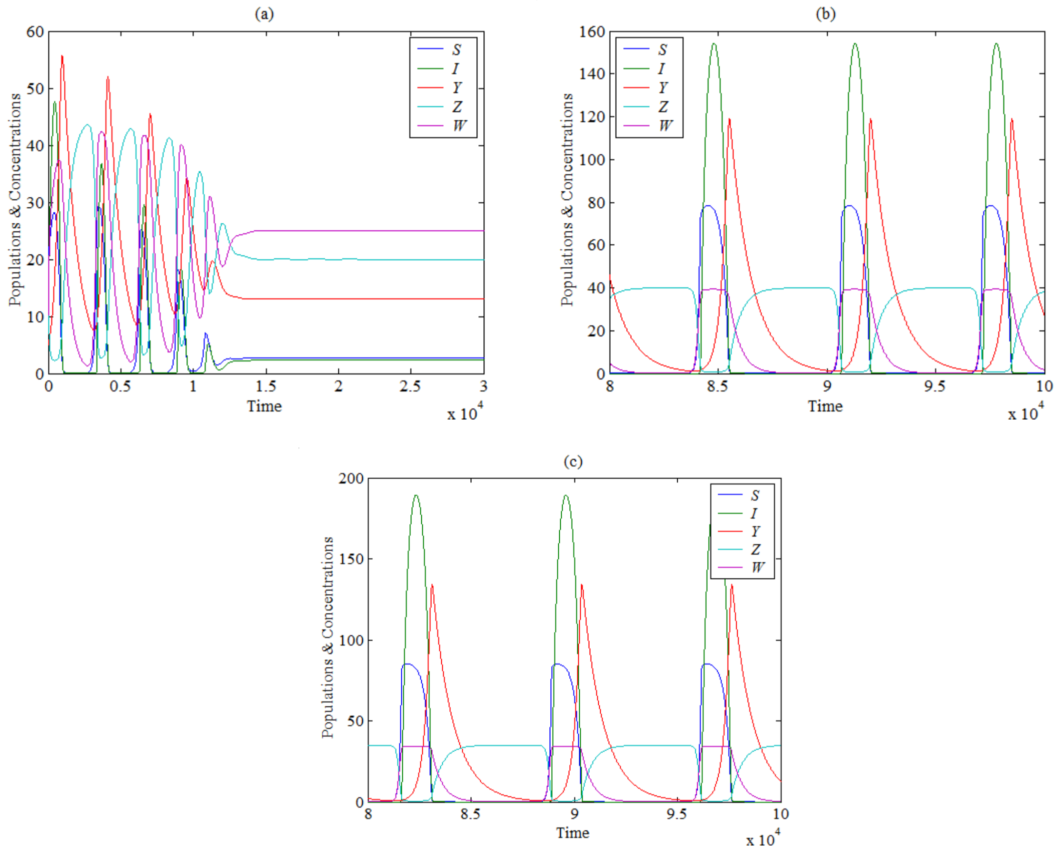


Fig. 9: The time series of system (2) for the data given by Eq. (16) with $r = 1.25$ for different values of the parameter $Q$. System (2) (a) approaches to positive equilibrium point for $Q = 9$. (b) approaches to small periodic attractor for $Q = 8$. (c) approaches to large periodic attractor for $Q = 7$.

Clearly, due to Fig. (8a)-(8c) and Fig. (9a)-(9c) system (2) have a Hopf bifurcation, which occurred when the parameters $\sigma_2$ and $Q$ pass through specific bifurcation values respectively. Finally, it is observed that, decreasing the natural depletion rates $\mu_3$ and $\mu_4$ keeping the rest of parameters fixed at the data given by Eq. (16) with $r = 1.25$ lead to transfer the dynamics of system (2) from periodic to asymptotic stability at the positive equilibrium point. However, varying the parameters $m$ and $\sigma_3$ do not effect on the pattern of the behaviour of system (2).

## 6. Discussions and Conclusions

In this paper, a prey-predator model with the existence of disease and pollution has been proposed and analyzed. The uniqueness and boundedness of solutions of the system are discussed. The local as well as global stability analysis for the proposed system are performed. Moreover, in order to confirm our analytical results and specified which combination of parameters control the dynamical behaviour of system (2) numerical simulations are used for biologically feasible set of hypothetical parameters that given by Eq. (16). For this set of data, it is observed that:

1. Increasing the intrinsic grow rate of the susceptible prey above a specific value say $\bar{r}$ ,the system loss the stability and approaches to periodic dynamics. Consequently, increasing this parameter destabilizing the system.
2. When the system has an asymptotically stable positive equilibrium point, then increasing the infective rate $\lambda$ above a specific value, say $\lambda_1$, leads to extinction in the predator species and hence system (2) losses the persistence and approaches asymptotically to predator free equilibrium point. however decreasing the infective rate $\lambda$ has destabilizing effect on the dynamics of system (2) and then system (2) approaches asymptotically to periodic dynamics in $Int.R_+^5$. On the other hand, when the system has periodic dynamics then increasing the parameter $\lambda$ slightly has stabilizing effect on the dynamics of system (2) however further increasing will causes extinction in predator species and the system will approaches to predator free equilibrium point.
3. Varying the half saturation parameter $\beta$ has the same effect on the dynamics of system (2) as that shown in case of varying the infected rate $\lambda$. In fact the occurrence of Hopf bifurcation is clearly shown in case of varying $\beta$.
4. Increasing the predation rates $\alpha_1$ and $\alpha_2$ simultaneously have stabilizing effect on the dynamical behavior of system (2).
5. Increasing each of the parameters, the death rate of the infected species $\mu_1$ or the exogenous input rate of the toxicant in the environment $Q$ , has stabilizing effect on the dynamics of system (2). Further more, the occurrence of Hopf bifurcation is clearly shown in case of decreasing the parameter $Q$.
6. When the system has an asymptotically stable positive equilibrium point, then increasing the natural death rate of predator species $\mu_2$ above a specific value, leads to extinction in the predator species and hence system (2) losses the persistence and approaches asymptotically to

predator free equilibrium point. however decreasing the natural death rate of predator species $\mu_2$ has destabilizing effect on the dynamics of system (2) and then system (2) approaches asymptotically to periodic dynamics in $Int.R_+^5$. On the other hand, when the system has periodic dynamics then increasing the natural death rate of predator species $\mu_2$ will causes extinction in predator species and the system will approaches to predator free equilibrium point.

7. Decreasing the conversion rates $\theta_1$ and $\theta_2$ simultaneously causes extinction in predator species and the system losses its persistence and approached asymptotically to predator free equilibrium point.
8. Varying the parameters, preference rate $m$ and the uptake rate of toxicant by organism $\sigma_3$, have no effect on the pattern of the dynamics of system (2).
9. When the system has an asymptotically stable positive equilibrium point, then varying the natural depletion rates $\mu_3$ and $\mu_4$ have no effect on the dynamical behaviour of the system. However, when the system has periodic dynamics then decreasing the natural depletion rates $\mu_3$ and $\mu_4$ have stabilizing effect on the dynamics of system (2) and the system approaches asymptotically to positive equilibrium point.
10. When the system has an asymptotically stable positive equilibrium point, then increasing the reduction rate of the susceptible prey due to the existence of toxicant given by $\sigma_1$ leads to extinction in predator species and the system will approaches to predator free equilibrium point. However, when the system has periodic dynamics then increasing slightly the parameter $\sigma_1$ stabilizing the dynamics and hence system (2) approaches asymptotically to positive equilibrium point. Moreover, increasing the value of parameter $\sigma_1$ further causes extinction in predator species and the system will approaches to predator free equilibrium point.
11. When the system has an asymptotically stable positive equilibrium point, then increasing the reduction rate of the infected prey due to the existence of toxicant given by $\sigma_2$ leads to destabilizing in the system and hence the solution will approaches to periodic dynamics. However, when the system has periodic dynamics then increasing slightly the parameter $\sigma_2$ stabilizing the dynamics and hence system (2) approaches asymptotically to positive equilibrium

point. Moreover, increasing the value of parameter $\sigma_2$ further causes destabilizing again in the system and then the solution will approaches asymptotically to periodic dynamics in the $Int.R_+^5$.

Accordingly, system (2) has rich dynamics in the domain $R_+^5$. In fact it is observed that the system is very sensitive to varying in the parameters values $\lambda$, $\beta$, $\sigma_1$, $\sigma_2$ and $Q$, while it has less sensitivity in case of varying the other parameters

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