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SOME CONSTRUCTIONS OF GABOR K-FRAMES

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Abstract. It is known that a necessary condition for Gabor systems $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to yield a frame is that $ab \leqslant 1$ (where a,b are the lattice parameters). In this paper we will characterize and construct Gabor (Weyl-Heisenberg) K-frames when $ab > 1$, and windows are in the Weiner algebra. We will show that an oversampling $\{E_{m\frac{b}{q}}T_{n\frac{a}{p}}g\}_{m,n\in\mathbb{Z}}$ of a K-frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is also a K-frame. Moreover, we will give a concrete example, in the case when $g = \chi_{[0,1)}$, $a = 2$, $b = 1$, and we will derive some results.

Keywords: K-frame; pseudo-inverse; Weyl-Heisenberg K-frame.

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1. Introduction

The notion of a frame for Hilbert spaces was introduced by Duffin and Schaeffer [5]. This was done while probing into some questions in non-harmonic Fourier series. This idea seemed to have been unnoticed outside of this area until Daubechies, Grossmann and Meyer [3] brought it into light in 1986. the latter's showed that Duffin and Schaeffer's definition was an abstraction of the concept introduced by Gabor [6] in 1946 for doing signal analysis. Recently, the frames

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that have been introduced by Gabor are referred to as Gabor frames or Weyl-Heisenberg frames, and they play a vital role in signal analysis.

Frames are more general than bases; a frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are called the frame coefficients of the vector. But unlike an orthonormal basis each vector may have infinitely many different representations in terms of its frame coefficients.

The aim of Gabor analysis is to represent every signal f as a superposition of elementary basic functions of the form $e^{2\pi ima}g(x-nb)$, $m,n\in\mathbb{Z}$; g is a fixed function and a,b are fixed numbers in $\mathbb{R}^+$. If this is the case, the system $\{e^{2\pi ima}g(x-nb)\}_{m,n\in\mathbb{Z}}$, is called a Gabor frame.

In this work, we will be interested in when $\{e^{2\pi ima}g(x-nb)\}_{m,n\in\mathbb{Z}}$ is not a frame.

Then, we will be dealing with Gabor K-frame.

K-frames, as a new generalization of frames, have important applications. They help us to reconstruct elements from a range of a bounded linear operator K in a separable Hilbert space. The notion of K-frames has been introduced by L.*Găvruta* in order to study the atomic systems with respect to a bounded linear operator K in a separable Hilbert space H [7]. K-frames are more general than ordinary frames in the way that the lower frame bound only holds for the elements of the range of K. Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames. For instance, the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic. Our paper will discus Gabor (Weyl-Heisenberg) K-frames and will be organized as follows: Sections 2 , 3 , 4 and 5 will be devoted to displaying the required concepts for our focal point of study. In section 6, we will show that:

(i) For a given $g\in W$; $a,b>0$ such that $ab>1$ and $K\in B\left(L^2(\mathbb{R})\right)$
$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K-fram if and only if $\mathscr{R}(K)\subset\mathscr{R}(D_{g,a,b})$.

(ii) for a given window g in the wiener space and real $a,b>0$ with $ab>1$ and for every $h\in W$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a $K_{h,g}$-frame, where $K_{h,g}$ is the mixed frame operator[2].

iii Every oversampling in the form $\{E_{m\frac{b}{q}}T_{n\frac{a}{p}}g\}_{m,n\in\mathbb{Z}}$ of a Gabor K-frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is also a Gabor K-frame and we will establish some results when $g = \chi_{[0,1)}$.

Throughout this paper, we will also adopt the following notations: H is a separable Hilbert space ; $B(H)$ the space of all bounded linear operators on H; $l^2(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}/\|(a_n)\|_{l^2(\mathbb{N})} = \left(\sum_{n\in\mathbb{N}} |a_n|^2\right)^{1/2} < \infty\}$; $L^2(\mathbb{R})$ is the space of all square-integrable functions on the real line $\mathbb{R}$ with the inner product and norm on $L^2(\mathbb{R})$ denoted by $\langle .,. \rangle$ and $\|.\|_2$ respectively ; The characteristic function of a set $\mathbb{E} \subset \mathbb{R}$ is : $\chi_{\mathbb{E}}(x) = \begin{cases} 1 & : x \in \mathbb{E} \\ 0 & : x \notin \mathbb{E} \end{cases}$.

The essential supremum of a function f is :

$$\|f\|_\infty = ess\,sup_{x\in\mathbb{R}}|f(x)| = inf\{\alpha \in \mathbb{R} / f(x) < \alpha \quad a.e\}.$$

$\mathscr{R}_K$ and $\mathscr{N}_K$ are the range and the kernel of a bounded operator $K : H \longrightarrow H$, $K^\dagger$ is the pseudo-inverse of K ; $C_c(\mathbb{R})$ is the space of continuous compactly supported functions on $\mathbb{R}$.

Theorem 1.1. [4] *Let $L_1 \in B(H_1,H)$, $L_2 \in B(H_2,H)$ be two bounded operators. The following statements are equivalent:*

(i) $\mathscr{R}(L_1) \subset \mathscr{R}(L_1)$
(ii) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ *for some* $\lambda \geq 0$ *and*
(iii) *there exists a bounded operator* $X \in B(H_1,H_2)$ *so that* $L_1 = L_2X$.

Definition 1.2. A sequence $\{f_k\}_{k\in\mathbb{N}}$ of elements in a separable Hilbert space H is a frame for H if there exist constants $A,B > 0$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

If $\{f_k\}_{k\in\mathbb{N}}$ is a frame only for $\overline{span}\{f_k\}_{k\in\mathbb{N}}$, it is called is a frame sequence. The operator

$$T : l^2(\mathbb{N}) \longrightarrow H, \quad T\{c_k\}_{k\in\mathbb{N}} = \sum_{k\in\mathbb{N}} c_k f_k$$

(called the synthesis operator) is well defined and bounded iff $\{f_k\}_{k\in\mathbb{N}}$ is a Bessel sequence.

It is well known that (see[1] Thm 4.1) if $\{f_k\}_{k\in\mathbb{N}}$ is a frame then T is bounded; linear and onto (surjective), and $\overline{span}\{f_k\}_{k\in\mathbb{N}} = H$

If $\{f_k\}_{k\in\mathbb{N}}$ is a frame sequence then

$$\mathscr{R}(T) = \overline{span}\{f_k\}_{k\in\mathbb{N}}.$$

2. K-FRAMES

We first recall the concepts of K-frames, the atomic system of K

Definition 2.1. [7]. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called a K-frame for H, if there exist constants $A, B > 0$ such that:

$$A\|K^*f\|^2 \leqslant \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \forall f \in H. \tag{2.1}$$

We call A, B the lower frame bound and the upper frame bound for K-frame $\{f_n\}_{n\in\mathbb{N}}$, respectively. If only the right inequality of (2.1) holds, $\{f_n\}_{n\in\mathbb{N}}$ is called a Bessel sequence

Remark 2.2. If $K = I_H$, then K-frames are just the ordinary frames. Hence, K-frames arise naturally as a generalization of ordinary frames

Example 2.3. Let $H = \mathbb{C}^n$ and $\{e_1, e_2, \ldots\ldots, e_n\}$ be an orthonormal basis of H. Define $K : H \longrightarrow H$ by : $f_i = Ke_i = e_i$ for $1 \leqslant i \leqslant n-1$, and $f_n = Ke_n = e_{n-1}$

$\{f_i\}_{i=1}^n$ is a K-frame for H.

Example 2.4. Suppose that $H = l^2(\mathbb{N})$, let $\{e_n\}_{n\in\mathbb{N}}$ be the standard ortonormal basis of H , $\{e_n\}_{n\in\mathbb{N}}$ is an ordinary frame for $l^2(\mathbb{N})$. Define $K : H \longrightarrow H$ by : $g_i = Ke_i = e_{i+2}$ for every $i \in \mathbb{N}$.

Clearly, $\{g_i\}_{i\in\mathbb{I}}$ is a K-frame for H.

Definition 2.5. [9] A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called an atomic system for K, if the following conditions are satisfied

(i) $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence ;

(ii) For any $x \in H$, there exists $a_x = \{a_n^x\}_n \in l^2(\mathbb{N})$ such that:

$$Kx = \sum_{n\in\mathbb{N}} a_n f_n.$$

Where $\|a_x\|_{l^2(\mathbb{N})} \leq C\|x\|$, C is a positive constant .

Example 2.6. Let H be a separable Hilbert space, $K \in B(H)$, and $\{h_i\}_{i\in I}$ defined by: $h_i = Ke_i$ where $\{e_i\}_{i\in I}$ is an orthonormal basis of H.

We can see that: $\{h_n\}_{i\in I}$ is an atomic system for K by letting:

$$a_h := \{\langle h, e_i\rangle\}_{i\in I} \quad \forall h \in H.$$

The proof of the following proposition can be founded in [7] (Theorem 3).

Proposition 2.7. *Let $\{f_n\}_{n\in\mathbb{N}} \subset H$. Then the following assertions are equivalent*

(i) $\{f_n\}_{n\in\mathbb{N}}$ *is an atomic system for K ;*

(ii) $\{f_n\}_{n\in\mathbb{N}}$ *is a K-frame for H .*

3. Gabor Frames

Gabor analysis in $L^2(\mathbb{R})$ is based on two classes of operators on $L^2(\mathbb{R})$, namely: Translation by $a \in \mathbb{R}$

$$T_a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), (T_a f)(x) = f(x-a); \tag{3.1}$$

Modulation by $b \in \mathbb{R}$

$$E_b : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), (E_b f)(x) = e^{2\pi i b x} f(x). \tag{3.2}$$

We begin this section with the following definition.

Definition 3.1. A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ where $a, b > 0$ and $g \in L^2(\mathbb{R})$ is a fixed function. Frames of this type are also called Weyl-Heisenberg frames. The function g is called the window function or the generator.

Example 3.2. Consider the system $\{E_m T_{na}\chi_{[0,c)}\}_{m,n\in\mathbb{Z}}$.
It is a Gabor frame when $a \leq c \leq 1$.

Theorem 3.3. [2] *(Theorem 11.3.1). Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, such that $ab > 1$. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$.*

Remark 3.4. We can assume that either the translation parameter or the modulation parameter in a Gabor frame is equal to 1. This can be obtained by a scaling of g, (in an arbitrary Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$) i.e., by replacing g with a function of the type

$$D_c g(x) = \frac{1}{\sqrt{c}} g\left(\frac{x}{c}\right). \tag{3.3}$$

Proposition 3.5. [2] *Let $g \in L^2(\mathbb{R})$ and $a,b,c > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame (Bessel sequence respectively). Then, with $g_c := D_c g$, the Gabor family $\left\{E_{mb/c}T_{nac}g_c\right\}_{m,n\in\mathbb{Z}}$ is a frame (Bessel sequence respectively) with the same frame bounds as $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.*

4. Pseudo-Inverse

The Pseudo-inverse Operator exist in the literature as a type of generalized inverses.

Lemma 4.1. [2]*(Lemma 2.5.1). Let H, K be Hilbert spaces, and suppose that $U : K \longrightarrow H$ is a bounded operator with closed range $\mathscr{R}_U$. Then there exists a bounded operator $U^\dagger : H \longrightarrow K$ for which*

$$UU^\dagger x = x, \forall x \in \mathscr{R}_U. \tag{4.1}$$

Remark 4.2. The operator $U^\dagger$ is called the pseudo-inverse of U. And it is the unique operator satisfying that:

$$\mathscr{N}_{U^\dagger} = \mathscr{R}_U^\perp, \mathscr{R}_{U^\dagger} = \mathscr{N}_U^\perp, and, UU^\dagger x = x, \forall x \in \mathscr{R}_U. \tag{4.2}$$

Example 4.3. Let K be as in example 2.4.
It is easy to see that : $\mathscr{N}_K = span\{e_0, e_1\}$; $\mathscr{R}_K = span\{e_i\}_{i\geq 2}$, and $K^\dagger$ is defined by : $K^\dagger(e_i) = \begin{cases} e_{i-2} & : i \geq 2 \\ 0 & : i = 0,1 \end{cases}$.

5. The Wiener Space W

Our reference for this section is [2] .

Definition 5.1. Given $a > 0$, the Wiener space is defined by:

$$W := \{g : \mathbb{R} \longrightarrow \mathbb{C} / g \, is \, mesurable \, and, \sum_{k\in\mathbb{Z}} \|g\chi_{[k,k+1)}\|_\infty < \infty\}. \tag{5.1}$$

the space W is also called a Wiener amalgam space and is often denoted by $W(L^\infty, l^1)$.

W is a Banach space with respect to the norm

$$\|g\|_{W,a} = \sum_{k\in\mathbb{Z}} \|g\chi_{[ka,(k+1)a)}\|_\infty .$$

The space W is independent of the choice of a, and different choices give equivalent norms.
$\mathscr{S} \subset \mathscr{S}_0 \subset W$, where $\mathscr{S}$ is the Schwartz space and $\mathscr{S}_0$ is *Feichtinger's* algebra.
$W \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and W is a dense subspace of $L^2(\mathbb{R})$.

Proposition 5.2. [2] *(Prop 11.5.2) If $g \in W$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for any choice of $a,b>0$.*

6. Main Results

Let us define the analysis operator $C_g : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ and the synthesis operator D_g : $l^2(\mathbb{Z}^2) \longrightarrow L^2(\mathbb{R})$ associated with $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ by :

$$C_g(f) = \{\langle f, E_{mb}T_{na}g\rangle\}_{m,n\in\mathbb{Z}} \tag{6.1}$$

and

$$D_g(c) = \sum_{m,n\in\mathbb{Z}} c_{m,n}E_{mb}T_{na}g. \tag{6.2}$$

Let $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ be an orthonormal basis of $l^2(\mathbb{Z}^2)$. One can see that

$$D_g(e_{m,n}) = E_{mb}T_{na}g.$$

The operators D_g and C_g are bounded iff $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence, and D_g is the adjoint of C_g. Moreover:

$$\|C_g(f)\|^2_{l^2(\mathbb{Z}^2)} = \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2, f \in L^2(\mathbb{R}). \tag{6.3}$$

Definition 6.1. The operator defined by composing analysis and synthesis operators from different systems will be noted $K_{h,g}$:

$$K_{h,g}(f) = D_g \circ C_h(f) = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g. \tag{6.4}$$

Theorem 6.2. *Let $g \in W$ and $a,b>0$ such that $ab>1$, let $K \in B\left(L^2(\mathbb{R})\right)$. Then the following holds:*

(i) *If* $\mathscr{R}(K) \subseteq \mathscr{R}(D_g)$ *then* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a* K *frame for* $L^2(\mathbb{R})$.

(ii) *If* $\mathscr{R}(D_g) \subsetneq \mathscr{R}(K)$ *then* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is not a* K *frame for* $L^2(\mathbb{R})$.

(iii) *If* $\mathscr{R}(K) = \mathscr{R}(D_g)$ *then* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a Parseval* K *frame for* $L^2(\mathbb{R})$

Proof. Let $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ be an orthonormal basis of $l^2(\mathbb{Z}^2)$. For $f \in L^2(\mathbb{R})$ and $(m,n) \in \mathbb{Z}^2$

$$\langle D_g^* f, e_{mn}\rangle = \langle f, D_g e_{mn}\rangle = \langle f, E_{mb}T_{na}g\rangle$$

then

$$D_g^* f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle e_{mn}$$

hence

$$\|D_g^* f\|^2 = \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \quad \forall f \in L^2(\mathbb{R}) \tag{6.5}$$

If $\mathscr{R}(K) \subseteq \mathscr{R}(D_g)$, then by Douglas majoration theorem [4] we get

$$AKK^* \leq D_g D_g^*$$

this means that

$$A\|K^* f\|^2 \leq \|D_g^* f\|^2 \quad \forall f \in L^2(\mathbb{R})$$

and by (6.5) we claim that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K frame for $L^2(\mathbb{R})$.

Suppose $\mathscr{R}(D_g) \subsetneq \mathscr{R}(K)$, let $f \in \mathscr{R}(K) \setminus \mathscr{R}(D_g)$ such that $f \neq 0$, since

$$\mathscr{R}(D_g) = \overline{span}(E_{mb}T_{na}g)$$

then

$$\langle D_g(C_{mn}), f\rangle_{L^2(\mathbb{R})} = 0 \quad \forall (C_{mn}) \in l^2(\mathbb{Z}^2),$$

hence

$$\langle (C_{mn}), D_g^* f\rangle_{l^2(\mathbb{Z}^2)} = 0 \quad \forall (C_{mn}) \in l^2(\mathbb{Z}^2)$$

then

$$0 = \|D_g^* f\|^2_{l^2(\mathbb{Z})} = \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2$$

but $\|K^* f\| \neq 0$, so $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is not a K frame for $L^2(\mathbb{R})$.

If $\mathscr{R}(K) = \mathscr{R}(D_g)$ then there exist $A > 0$ such that

$$AKK^* = D_g D_g^*$$

So

$$A\|K^* f\|^2 = \|D_g^* f\|^2 \quad \forall f \in L^2(\mathbb{R})$$

ie

$$A\|K^* f\|^2 = \| \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \quad \forall f \in L^2(\mathbb{R}).$$

□

Theorem 6.3. *Let $g \in W$ and $a, b > 0$ such that $ab > 1$. Then for every $h \in W$: $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a $K_{h,g}$ - frame , i.e there exist $A, B > 0$ such that :*

$$A\|K_{h,g}^* f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \leq B\|f\|^2, \forall f \in L^2(\mathbb{R}).$$

Proof. It is easy to see that the right hand side of the inequality holds because $\{\langle f, E_{mb}T_{na}h\rangle\}_{m,n\in\mathbb{Z}} \in l^2\left(\mathbb{Z}^2\right)$.
On the other hand;

$$\|K_{h,g}^* f\| = sup_{\|u\|=1}|\langle K_{h,g}^* f, u\rangle| = sup_{\|u\|=1}|\langle f, K_{h,g}u\rangle|$$

and by definition of $K_{h,g}$:

$$K_{h,g}u = \sum_{m,n\in\mathbb{Z}} \langle u, E_{mb}T_{na}h\rangle E_{mb}T_{na}g.$$

So

$$\begin{aligned}
\|K_{h,g}^* f\| &= \sup_{\|u\|=1} |\sum_{m,n\in\mathbb{Z}} \langle E_{mb}T_{na}h, u\rangle \langle f, E_{mb}T_{na}g\rangle| \\
&\leq \sup_{\|u\|=1} \left(\sum_{m,n\in\mathbb{Z}} |\langle E_{mb}T_{na}h, u\rangle|^2\right)^{1/2} . \left(\sum_{m,n} |\langle f, E_{mb}T_{na}g\rangle|^2\right)^{1/2} \\
&\leq \sup_{\|u\|=1} B(h,a,b)\|u\|^2 \left(\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2\right)^{1/2} .
\end{aligned}$$

Let

$$A = \frac{1}{B(h,a,b)}$$

It follows that

$$A\|K^*f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2$$

this complete the proof. $\square$

Remark 6.4. Let K be an operator such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K - frame, is K in the form $K_{h,g}$ for some h and g in W?

Corollary 6.5. *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K-frame, then it is a T-frame for every $T \in B(L^2(\mathbb{R}))$ such that $\mathscr{R}(T) \subset \mathscr{R}(K)$*

The following Theorem gives a partial answer to the problem:

Problem 6.6. Let $K \in B(L^2(H))$ and $a > 0$, $b > 0$ such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K-frame. The system $\{E_{m\frac{bp}{q}}T_{n\frac{ar}{s}}g\}_{m,n\in\mathbb{Z}}(p,p,r,s$ are in $\mathbb{N}^*$ and $p \leq p$, $r \leq s$) is it a Gabor K-frame?.

Theorem 6.7. *Let $K \in B(L^2(H))$ and $a > 0$, $b > 0$. If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K-frame, then $\{E_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ is a K-frame for every $\alpha > 0$, $\beta > 0$ such that $\alpha = \frac{a}{p}$ and $\beta = \frac{b}{q}$ ($p,q \in \mathbb{N}^*$).*

Proof. We can always reduce to case $\beta = b$ and $\alpha = \frac{a}{q}$ (see proposition 3.5).
Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a K-frame,then:

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{n\frac{a}{q}}g\rangle|^2 &= \sum_{l=0}^{q-1}\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{(nq+l)\frac{a}{q}}g\rangle|^2 \\
&= \sum_{l=0}^{q-1}\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{l\frac{a}{q}}T_{na}g\rangle|^2 \\
&= \sum_{l=0}^{q-1}\sum_{m,n\in\mathbb{Z}} |\langle T_{-l\frac{a}{q}}f, E_{mb}T_{na}g\rangle|^2 \\
&= \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 + \sum_{l=1}^{q-1}\sum_{m,n\in\mathbb{Z}} |\langle T_{-l\frac{a}{q}}f, E_{mb}T_{na}g\rangle|^2 \\
&\geq A\|K^*f\|^2
\end{aligned}$$

$\square$

Corollary 6.8. *Let $g \in W$, if $\alpha = \frac{a}{p}$ and $\beta = \frac{b}{q}$ $(p, q \in \mathbb{N}^*)$ then;*

$$\mathscr{R}(D_{g,a,b}) \subset \mathscr{R}(D_{g,\alpha,\beta})$$

Example 6.9. Let $h = g = \chi_{[0,1)}$ and $a = 2$, $b = 1$.

We denote $K_{h,g}$ in this example by $K_{g,g} := K$

Note that $K^* = (D_g \circ C_g)^* = C_g^* \circ D_g^* = D_g \circ C_g = K$, and K is the frame operator corresponding to $\{E_m T_{2n} \chi_{[0,1)}\}_{m,n\in\mathbb{Z}}$. Let us show that $\{E_m T_{2n} \chi_{[0,1)}\}_{m,n\in\mathbb{Z}}$ is a K-frame.

$$K^* f = K f = \sum_{m,n\in\mathbb{Z}} \langle f, E_m T_{2n} \chi_{[0,1)} \rangle E_m T_{2n} \chi_{[0,1)}$$

Let $f \in C_c(\mathbb{R})$:

$$\begin{aligned}
\langle Kf, Kf \rangle &= \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2 \langle E_m T_{2n} \chi_{[0,1)}, E_m T_{2n} \chi_{[0,1)} \rangle \\
&= \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2 \|E_m T_{2n} \chi_{[0,1)}\|^2_{L^2(\mathbb{R})} \\
&= \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2 \|\chi_{[0,1)}\|^2_{L^2(\mathbb{R})} \\
&= \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2.
\end{aligned}$$

By density of $C_c(\mathbb{R})$ in $L^2(\mathbb{R})$, the equality hold for every $f \in L^2(\mathbb{R})$.

Then:

$$\|K^* f\|^2 = \sum_{m,n\in\mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2$$

So, $\{E_n T_{2n} \chi_{[0,1)}\}_{m,n\in\mathbb{Z}}$ is a K-frame for $L^2(\mathbb{R})$.

Let $\mathbb{E} := \cup_{k\in\mathbb{Z}} ([0,1) + 2k)$ and $\mathbb{F} := \cup_{k\in\mathbb{Z}} ([1,2) + 2k)$.

$\mathbb{E}$ and $\mathbb{F}$ are $2\mathbb{Z}$- periodic sets in $\mathbb{R}$.

We see that $\mathbb{E} \cup \mathbb{F} = \mathbb{R}$, also $L^2(\mathbb{E})$ and $L^2(\mathbb{F})$ are two closed subspaces of $L^2(\mathbb{R})$.

Moreover:

$$L^2(\mathbb{E}) \oplus L^2(\mathbb{F}) = L^2(\mathbb{R}). \tag{6.6}$$

Where $L^2(\mathbb{E}) := \{f \in L^2(\mathbb{R}) : f = 0 \text{ on } \mathbb{R} \setminus \mathbb{E}\}$.

Every f in $L^2(\mathbb{R})$ can be written as $f = f_1 + f_2$, with $f_1 = f\chi_{\mathbb{E}}$ and $f_2 = f\chi_{\mathbb{F}}$.

It is easy to see from the definition of $\mathbb{E}$, that for every $f \in L^2(\mathbb{F})$ we have $Kf = 0$. Then $\{E_{mb}T_{na}\chi_{[0,1)}\}_{m,n\in\mathbb{Z}}$ cannot be complete , so it is not an ordinary frame for $L^2(\mathbb{R})$.

Corollary 6.10. *Let K, $\mathbb{E}$ and $\mathbb{F}$ as above. Then $\mathscr{N}_K = L^2(\mathbb{F})$, and $\mathscr{R}_K = L^2(\mathbb{E})$*

Proof. From [2](Theorem 12.2.1):

$$K_{h,g}f = \frac{1}{b}\sum_{k\in\mathbb{Z}} f(x-k/b)\sum_{n\in\mathbb{Z}} g(x-na)\overline{h(x-na-k/b)}$$

This equality is known as the Walnut representation (See [2], and[8] for more details). Then:

$$\begin{aligned}
Kf &= \sum_{k\in\mathbb{Z}} f(x-k)\sum_{n\in\mathbb{Z}} \chi_{[0,1)}(x-2n)\overline{\chi_{[0,1)}(x-2n-k)} \\
&= f(x)\sum_{n\in\mathbb{Z}} |\chi_{[0,1)}(x-2n)|^2 + \sum_{k\neq 0} f(x-k)\sum_{n\in\mathbb{Z}} \chi_{[0,1)}(x-2n)\overline{\chi_{[0,1)}(x-2n-k)}
\end{aligned}$$

If $x \in \mathbb{F}$, then: $x-2n \in \mathbb{F}$ for all $n \in \mathbb{Z}$, hence $Kf(x) = 0$.

If $x \in \mathbb{E}$, then:

$$\begin{aligned}
Kf &= \sum_{k\in\mathbb{Z}} f(x-k)\sum_{n\in\mathbb{Z}} \chi_{[0,1)}(x-2n)\overline{\chi_{[0,1)}(x-2n-k)} \\
&= f(x)\sum_{n\in\mathbb{Z}} |\chi_{[0,1)}(x-2n)|^2 + \sum_{k\neq 0} f(x-k)\sum_{n\in\mathbb{Z}} \chi_{[0,1)}(x-2n)\chi_{[0,1)}(x-2n-k) \\
&= f(x)\sum_{n\in\mathbb{Z}} |\chi_{[2n,2n+1)}(x)|^2 + \sum_{k\neq 0} f(x-k)\sum_{n\in\mathbb{Z}} \chi_{[2n,2n+1)}(x)\chi_{[2n+k,2n+k+1)}(x) \\
&= f(x)\sum_{n\in\mathbb{Z}} \chi_{[2n,2n+1)}(x) \\
&= f(x)\chi_{\mathbb{E}} = f_1(x).
\end{aligned}$$

□

Based on what has been said, we can deduce that $K = P_{\mathbb{E}}$, where $P_{\mathbb{E}}$ is the orthogonal projection on $\mathbb{E}$.

Corollary 6.11. *$\{E_mT_{2n}\chi_{[0,1)}\}_{m,n\in\mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{E})$.*

Proof. We have already seen that:

$$\|Kf\|^2 = \sum_{m,n\in\mathbb{Z}} |\langle f, E_mT_{2n}\chi_{[0,1)}\rangle|^2.$$

As $Kf = f_1$, and $\langle f, E_m T_{2n} \chi_{[0,1)} \rangle = \langle f_1, E_m T_{2n} \chi_{[0,1)} \rangle$.

We certainly have:

$$\|f\|^2 = \sum_{m,n \in \mathbb{Z}} |\langle f, E_m T_{2n} \chi_{[0,1)} \rangle|^2, \quad for every f \in L^2(\mathbb{E}).$$

$\square$

The value of this can be noticed in the ease with witch we are able to reconstruct signals that appears in proper subspaces of $L^2(\mathbb{R}))$ as the subspace $\mathbb{E}$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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