Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:1
https://doi.org/10.28919/jmcs/6822
ISSN: 1927-5307

CONTINUOUS CONTROLLED K-FRAME FOR HILBERT C^*-MODULES

HAMID FARAJ[1,*], SAMIR KABBAJ[1], HATIM LABRIGUI[1], ABDESLAM TOURI[1], MOHAMED ROSSAFI[2]

[1]Laboratory of Partial Differential Equations, Spectral Algebra and Geometry Department of Mathematics, Faculty of Sciences, University Ibn Tofail, Kenitra, Morocco

[2]LaSMA Laboratory Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, B. P. 1796 Fes Atlas, Morocco

Abstract. In this paper, we introduce and study the concept of continuous controlled K-frame for Hilbert C^*-modules which is a generalization of discrete controlled K-frame.

Keywords: controlled frame; controlled K-frame; continuous controlled K-frame; C^*-algebra; Hilbert $\mathscr{A}$-modules.

2010 AMS Subject Classification: 41A58, 42C15.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [11]. Frames have been used in signal processing, image processing, data compression and sampling theory. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a

*Corresponding author
E-mail address: farajham19@gmail.com
Received September 25, 2021

Radon measure was proposed by G. Kaiser [14] and independently by Ali, Antoine and Gazeau [5]. These frames are known as continuous frames. Gabardo and Han in [10] called these frames associated with measurable spaces, Askari-Hemmat, Dehghan and Radjabalipour in [3] called them generalized frames and in mathematical physics they are referred to as coherent states [5]. In 2012, L. Gavruta [12] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Controlled frames in Hilbert spaces have been introduced by P. Balazs [4] to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Rahimi [17] defined the concept of controlled K-frames in Hilbert spaces and showed that controlled K-frames are equivalent to K-frames due to which the controlled operator C can be used as preconditions in applications. Controlled frames in C^*-modules were introduced by Rashidi and Rahimi [15], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. We extended the results of frames in Hilbert spaces to Hilbert C^*-modules (see [13], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29])

Motivated by the above literature, we introduce the notion of a continuous controlled K-frame in Hilbert C^*-modules.

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathscr{A}$-modules. Our references for C^*-algebras as [8, 6]. For a C^*-algebra $\mathscr{A}$ if $a \in \mathscr{A}$ is positive we write $a \geq 0$ and $\mathscr{A}^+$ denotes the set of positive elements of $\mathscr{A}$.

Definition 1.1. [18] Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{H}$ be a left $\mathscr{A}$-module, such that the linear structures of $\mathscr{A}$ and $\mathscr{H}$ are compatible. $\mathscr{H}$ is a pre-Hilbert $\mathscr{A}$-module if $\mathscr{H}$ is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle_{\mathscr{A}} : \mathscr{H} \times \mathscr{H} \to \mathscr{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x,x \rangle_{\mathscr{A}} \geq 0$ for all $x \in \mathscr{H}$ and $\langle x,x \rangle_{\mathscr{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax+y,z \rangle_{\mathscr{A}} = a\langle x,z \rangle_{\mathscr{A}} + \langle y,z \rangle_{\mathscr{A}}$ for all $a \in \mathscr{A}$ and $x,y,z \in \mathscr{H}$.

(iii) $\langle x,y \rangle_{\mathscr{A}} = \langle y,x \rangle_{\mathscr{A}}^*$ for all $x,y \in \mathscr{H}$.

For $x \in \mathscr{H}$, we define $||x|| = ||\langle x,x \rangle_{\mathscr{A}}||^{\frac{1}{2}}$. If $\mathscr{H}$ is complete with $||.||$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in C^*-algebra $\mathscr{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathscr{A}$-valued norm on $\mathscr{H}$ is defined by $|x| = \langle x,x \rangle_{\mathscr{A}}^{\frac{1}{2}}$ for $x \in \mathscr{H}$.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules, A map $T : \mathscr{H} \to \mathscr{K}$ is said to be adjointable if there exists a map $T^* : \mathscr{K} \to \mathscr{H}$ such that $\langle Tx, y\rangle_{\mathscr{A}} = \langle x, T^*y\rangle_{\mathscr{A}}$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

We reserve the notation $End^*_{\mathscr{A}}(\mathscr{H},\mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and $End^*_{\mathscr{A}}(\mathscr{H},\mathscr{H})$ is abbreviated to $End^*_{\mathscr{A}}(\mathscr{H})$.

Lemma 1.2. [2]. *Let $\mathscr{H}$ and $\mathscr{K}$ two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(\mathscr{H})$. Then the following statements are equivalente:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e, there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathscr{K}$.*

(iii) *T^* is bounded below with respect to the inner product, i.e, there is $m' > 0$ such that,*

$$\langle T^*x, T^*x\rangle_{\mathscr{A}} \geq m'\langle x,x\rangle_{\mathscr{A}}, x \in \mathscr{K}$$

Lemma 1.3. [18] *Let $\mathscr{H}$ and $\mathscr{K}$ two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(\mathscr{H})$. Then the following statements are equivalente,*

(i) *The operator T is bounded and $\mathscr{A}$-linear.*

(ii) *There exist $0 \leq k$ such that*

$$\langle Tx, Tx\rangle_{\mathscr{A}} \leq k\langle x,x\rangle_{\mathscr{A}} \qquad x \in \mathscr{H}.$$

For the following theorem, R(T) denote the range of the operator T.

Theorem 1.4. [30] *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a C^*-algebra $\mathscr{A}$ and let T,S two operators for $End^*_{\mathscr{A}}(\mathscr{H})$. If $R(S)$ is closed, then the following statements are equivalent:*

(i) *$R(T) \subset R(S)$.*

(ii) *$TT^* \leq \lambda^2 SS^*$ for some $\lambda \geq 0$.*

(iii) *There exists $Q \in End^*_{\mathscr{A}}(\mathscr{H})$ such that $T = SQ$.*

2. Continuous Controlled K-Frame for Hilbert C^*-Modules

Let X be a Banach space, (Ω,μ) a measure space, and $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every

C^*-algebra and Hilbert C^*-module is a Banach space thus we can use this integral and its properties.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert C^*-modules, $\{\mathscr{K}_w : w \in \Omega\}$ is a family of subspaces of $\mathscr{K}$, and $End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K}_w)$ is the collection of all adjointable $\mathscr{A}$-linear maps from $\mathscr{H}$ into $\mathscr{K}_w$. We define

$$\oplus_{w\in\Omega}\mathscr{K}_w = \{x = \{x_w\}_{w\in\Omega} : x_w \in \mathscr{K}_w, \int_\Omega \|x_w\|^2 d\mu(w) < \infty\}.$$

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathscr{A}$-valued inner product is defined by $\langle x, y\rangle_{\mathscr{A}} = \int_\Omega \langle x_w, y_w\rangle_{\mathscr{A}} d\mu(w)$, the norm is defined by $\|x\| = \|\langle x, x\rangle_{\mathscr{A}}\|^{\frac{1}{2}}$. Therefore, $\oplus_{w\in\Omega}\mathscr{K}_w$ is a Hilbert C^*-module(see [14]).

Let $\mathscr{A}$ be a C^*-algebra, $l^2(\mathscr{A})$ is defined by,

$$l^2(\mathscr{A}) = \{\{a_\omega\}_{w\in\Omega} \subseteq \mathscr{A} : \|\int_\Omega a_\omega a^*_\omega d\mu(\omega)\| < \infty\}.$$

$l^2(\mathscr{A})$ is a Hilbert C^*-module (Hilbert $\mathscr{A} - module$) with pointwise operations and the inner product defined as,

$$\langle \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega}\rangle_{\mathscr{A}} = \int_\Omega a_\omega b^*_\omega d\mu(\omega), \{a_\omega\}_{w\in\Omega}, \{b_\omega\}_{w\in\Omega} \in l^2(\mathscr{A}),$$

and,

$$\|\{a_\omega\}_{w\in\Omega}\| = (\int_\Omega a_\omega a^*_\omega d\mu(\omega))^{\frac{1}{2}}.$$

Definition 2.1. Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a unital C^*-algebra, and $K \in End^*_{\mathscr{A}}(\mathscr{H})$. A mapping F: $\Omega \to \mathscr{H}$ is called a continuous K-Frame for $\mathscr{H}$ if :

- F is weakly-measurable, ie, for any $f \in \mathscr{H}$, the map
  $w \to \langle f, F(w)\rangle_{\mathscr{A}}$ is measurable on Ω.

- There exist two strictly positive constants A and B such that

$$A\langle K^*f, K^*f\rangle_{\mathscr{A}} \leq \int_\Omega \langle f, F(w)\rangle_{\mathscr{A}} \langle F(w), f\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}, f \in \mathscr{H}. \tag{2.1}$$

The elements A and B are called continuous K-frame bounds.

If $A = B$ we call this Continuous K-Frame a continuous tight K-Frame, and if $A = B = 1$ it is called a continuous Parseval K-Frame. If only the right-hand inequality of (2.1) is satisfied, we call F a continuous bessel mapping with Bessel bound B.

Let F be a continuous bessel mapping for Hilbert C^*- module $\mathscr{H}$ over $\mathscr{A}$.

The operator T :$\mathscr{H} \to l^2(\mathscr{A})$ defined by,

$$Tf = \{\langle f, F(\omega)\rangle_{\mathscr{A}}\}_{\omega\in\Omega},$$

is called the analysis operator.

There adjoint operator $T^* : l^2(\mathscr{A}) \to \mathscr{H}$ given by,

$$T^*(\{a_\omega\}_{\omega\in\Omega}) = \int_\Omega a_\omega F(\omega)d\mu(\omega),$$

is called the synthesis operator.

By composing T and T^*, we obtain the continuous K-frame operator, $S : \mathscr{H} \to \mathscr{H}$ defined by

$$Sf = \int_\Omega \langle f, F(\omega)\rangle_{\mathscr{A}} F(\omega)d\mu(\omega).$$

It's clear to see that S is positive, bounded and selfadjoint (see [5]).

For the following definition we need to introduce, $GL^+(\mathscr{H})$ be the set of all positive bounded linear invertible operators on $\mathscr{H}$ with bounded inverse.

Definition 2.2. Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a unital C^*-algebra and $K \in End^*_{\mathscr{A}}(\mathscr{H})$, $C \in GL^+(\mathscr{H})$. A mapping F :$\Omega \to \mathscr{H}$ is called a continuous C-controlled K-Frame in $\mathscr{H}$ if :

- F is weakly-measurable, ie, for any $f \in \mathscr{H}$, the map
  $w \to \langle f, F(w)\rangle_{\mathscr{A}}$ is measurable on Ω.
- There exists two strictly positive constants A and B such that

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \int_\Omega \langle f, F(w)\rangle_{\mathscr{A}} \langle CF(w), f\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}, f \in \mathscr{H}. \tag{2.2}$$

The elements A and B are called continuous C-controlled K-frame bounds.

If $A = B$ we call this continuous C-controlled K-Frame a continuous tight C-Controlled K-Frame, and if $A = B = 1$ it is called a continuous Parseval C-Controlled K-Frame. If only the right-hand inequality of (2.2) is satisfied, we call F a continuous C-controlled bessel mapping with Bessel bound B.

Example 2.3.

$$\begin{aligned} H = \mathscr{A} &= l^2(\mathbb{C}) \\ &= \left\{ \{a_n\}_{n=1}^{\infty} \subset \mathbb{C} \ / \ \sum_{n=1}^{\infty} |a_n|^2 < +\infty \right\}. \end{aligned}$$

$\mathscr{A}$ is recognized as a Hilbert $\mathscr{A}$-Module with the $\mathscr{A}$-inner product

$$< \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} >_{\mathscr{A}} = \left\{ a_n \overline{b_n} \right\}_{n=1}^{\infty}.$$

Consider now the borned linear operator

$$\begin{array}{rccc} C: & H & \to & H \\ & \{a_n\}_{n=1}^{\infty} & \longmapsto & \{\alpha a_n\}_{n=1}^{\infty} \end{array}$$

where $\alpha \in \mathbb{R}_+^*$. Then C is positive invertible and

$$C^{-1}(\{a_n\}_{n=1}^{\infty}) = \left\{ \alpha^{-1} a_n \right\}_{n=1}^{\infty}.$$

Let (Ω, μ) the measure space where $\Omega = [0,1]$ and μ is the lebesgue measure and let

$$\begin{array}{rccc} F: & \Omega & \to & H \\ & w & \longmapsto & F_w = \left\{ \frac{w}{n} \right\}_{n=1}^{\infty} \end{array}.$$

In the author hand, consider the projection

$$\begin{array}{rccc} K: & H & \to & H \\ & \{a_n\}_{n=1}^{\infty} & \longmapsto & (a_1, .., a_r, 0, ...) \end{array}$$

where r is an integer $(r \geq 2)$.

It's clair that $K^* = K$ and for each $f = \{a_n\}_{n=1}^{\infty} \in H = l^2(\mathbb{C})$, one has

$$\begin{aligned} \int_{\Omega} &< f, F_w >_{\mathscr{A}} < CF_w, f >_{\mathscr{A}} d\mu(w) = \int_{[0,1]} \left\{ \frac{w}{n} a_n \right\}_{n=1}^{\infty} . \left\{ \alpha \frac{w}{n} \overline{a_n} \right\}_{n=1}^{\infty} d\mu(w) \\ &= \int_{[0,1]} \left\{ \alpha \frac{w^2}{n^2} |a_n|^2 \right\}_{n=1}^{\infty} d\mu(w) \\ &= \frac{\alpha}{3} \left\{ \frac{|a_n|^2}{n^2} \right\}_{n=1}^{\infty}. \end{aligned}$$

Hence

$$\int_{\Omega} < f, F_w >_{\mathscr{A}} < CF_w, f >_{\mathscr{A}} d\mu(w) \leq \frac{\alpha \pi^2}{18} < \{a_n\}_{n=1}^{\infty}, \{a_n\}_{n=1}^{\infty} >_{\mathscr{A}}.$$

Furthermore,

$$\begin{aligned} &< CK^*f, K^*f >_{\mathscr{A}} = < (\alpha a_1, .., \alpha a_r, 0, ...), (a_1, .., a_r, 0, ...) >_{\mathscr{A}} \\ &= (\alpha |a_1|^2, .., \alpha |a_r|^2, 0, ...). \end{aligned}$$

Then for $A = \frac{1}{3r^2}$, one obtain

$$\frac{\alpha}{3r^2}(|a_1|^2, .., |a_r|^2, 0, ...) \leq \left\{ \frac{\alpha}{3} \frac{|a_n|^2}{n^2} \right\}_{n=1}^{\infty}.$$

The conclusion is

$$\frac{1}{3r^2} < C^{1/2}K^*f, C^{1/2}K^*f >_{\mathscr{A}} \leq \int_{\Omega} < f, F_w >_{\mathscr{A}} < CF_w, f >_{\mathscr{A}} d\mu(w) \leq \frac{\alpha\pi^2}{18} < f, f >_{\mathscr{A}}$$

Let F be a continuous C-controlled bessel mapping for Hilbert C^*- module $\mathscr{H}$ over $\mathscr{A}$. We define the operator frame

$S_C : \mathscr{H} \to \mathscr{H}$ by,

$$S_C f = \int_{\Omega} \langle f, F(\omega) \rangle_{\mathscr{A}} CF(\omega) d\mu(\omega).$$

Remark 2.4. From definition of S and S_C, we have, $S_C = CS$.

Using [16] , S_C is $\mathscr{A}$-linear and bounded. Thus, it is adjointable.

Since $\langle S_C x, x \rangle_{\mathscr{A}} \geq 0$, for any $x \in \mathscr{H}$, it result, again from [16], that S_C is positive and selfadjoint.

Theorem 2.5. *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module, $K \in End^*_{\mathscr{A}}(\mathscr{H})$, and $C \in GL^+(\mathscr{H})$. Let $F : \Omega \to \mathscr{H}$ a map. Suppose that $CK = KC$, $R(C^{\frac{1}{2}}) \subset R(K^*C^{\frac{1}{2}})$ with $R(K^*C^{\frac{1}{2}})$ is closed. Then F is a continuous C-controlled K-frame for $\mathscr{H}$ if and only if there exist two constants $0 < A, B < \infty$ such that :*

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\int_{\Omega} \langle f, F(w) \rangle_{\mathscr{A}} \langle CF(w), f \rangle_{\mathscr{A}} d\mu(w)\| \leq B\|f\|^2, f \in \mathscr{H}. \tag{2.3}$$

Proof. ($\Longrightarrow$) obvious.

For the converse, we suppose that $0 < A, B < \infty$ such that :

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\int_{\Omega} \langle f, F(w) \rangle_{\mathscr{A}} \langle CF(w), f \rangle_{\mathscr{A}} d\mu(w)\| \leq B\|f\|^2, f \in \mathscr{H}.$$

We have,

$$\begin{aligned}\|\int_{\Omega}\langle f,F(w)\rangle_{\mathscr{A}}\langle CF(w),f\rangle_{\mathscr{A}}d\mu(w)\| &= \|\langle S_C f,f\rangle_{\mathscr{A}}\|\\ &= \|\langle CSf,f\rangle_{\mathscr{A}}\|\\ &= \|\langle (CS)^{\frac{1}{2}}f,(CS)^{\frac{1}{2}}f\rangle_{\mathscr{A}}\|\\ &= \|(CS)^{\frac{1}{2}}f\|^2.\end{aligned}$$

Since, $R(C^{\frac{1}{2}})\subset R(K^*C^{\frac{1}{2}})$ with $R(K^*C^{\frac{1}{2}})$ is closed, then by theorem 1.4, there exists $0\leq m$ such that,

$$(C^{\frac{1}{2}})(C^{\frac{1}{2}})^*\leq m(K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*.$$

Thus,

$$\langle (C^{\frac{1}{2}})(C^{\frac{1}{2}})^*f,f\rangle_{\mathscr{A}}\leq m\langle (K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*f,f\rangle_{\mathscr{A}}.$$

Consequently,

$$\|C^{\frac{1}{2}}f\|^2\leq m\|K^*C^{\frac{1}{2}}f\|^2.$$

Then,

$$A\|C^{\frac{1}{2}}f\|^2\leq Am\|K^*C^{\frac{1}{2}}f\|^2\leq m\|(CS)^{\frac{1}{2}}f\|^2.$$

Hence,

$$\frac{A}{m}\|C^{\frac{1}{2}}f\|^2\leq \|(CS)^{\frac{1}{2}}f\|^2.$$

So,

$$\sqrt{\frac{A}{m}}\|C^{\frac{1}{2}}f\|\leq \|(CS)^{\frac{1}{2}}f\|. \tag{2.4}$$

From lemma1.2, we have,

$$\sqrt{\frac{A}{m}}\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle_{\mathscr{A}}\leq \langle C^{\frac{1}{2}}S^{\frac{1}{2}}f,C^{\frac{1}{2}}S^{\frac{1}{2}}f\rangle_{\mathscr{A}}.$$

Then,

$$\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle_{\mathscr{A}}\leq \sqrt{\frac{m}{A}}\langle CSf,f\rangle_{\mathscr{A}}.$$

So,

$$\langle C^{\frac{1}{2}}f,C^{\frac{1}{2}}f\rangle_{\mathscr{A}}\leq \sqrt{\frac{m}{A}}\langle S_Cf,f\rangle_{\mathscr{A}}.$$

One the deduce

$$\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \|K^*\|^2\langle C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq \|K^*\|^2\sqrt{\frac{m}{A}}\langle S_C f, f\rangle_{\mathscr{A}}.$$

Hence,

$$\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{m}}\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \langle S_C f, f\rangle_{\mathscr{A}}. \tag{2.5}$$

Since S_C is positive, selfadjoint and bounded $\mathscr{A}$-linear map, we can write

$$\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle_{\mathscr{A}} = \langle S_C f, f\rangle_{\mathscr{A}} = \int_{\omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}}d\mu(w).$$

From lemma 1.3, there exists $D > 0$ such that,

$$\langle S_C^{\frac{1}{2}}f, S_C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq D\langle f, f\rangle_{\mathscr{A}},$$

hence,

$$\langle S_C f, f\rangle_{\mathscr{A}} \leq D\langle f, f\rangle_{\mathscr{A}}. \tag{2.6}$$

Therfore by (2.5) and (2.6), we conclude that F is a continuous C-controlled K-frame in Hilbert C^*-module $\mathscr{H}$ with frame bounds $\frac{1}{\|K^*\|^2}\sqrt{\frac{A}{m}}$ and D. □

Lemma 2.6. *Let $C \in GL^+(\mathscr{H})$. Suppose $CS_C = S_C C$ and $R(S_C^{\frac{1}{2}}) \subset R((CS_C)^{\frac{1}{2}})$ with $R((CS_C)^{\frac{1}{2}})$ is closed. Then $\|S_C^{\frac{1}{2}}f\|^2 \leq \lambda\|(CS_C)^{\frac{1}{2}}f\|^2$ for some $\lambda \geq 0$.*

Proof. By theorem1.4, there exists some $\lambda > 0$ such that,

$$(S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda(CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^*.$$

Hence,

$$\langle(S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^*f, f\rangle_{\mathscr{A}} \leq \lambda\langle(CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^*f, f\rangle_{\mathscr{A}}.$$

So,

$$\|S_C^{\frac{1}{2}}f\|^2 \leq \lambda\|(CS_C^{\frac{1}{2}})f\|^2, f \in \mathscr{H}.$$

□

Theorem 2.7. *Let $F:\Omega\to\mathscr{H}$ a map and $C\in GL^{+}(\mathscr{H})$. Suppose $CS_C=S_CC$ and $R(S_C^{\frac{1}{2}})\subset R((CS_C)^{\frac{1}{2}})$ with $R((CS_C)^{\frac{1}{2}})$ is closed. Then F is a continuous C-controlled Bessel mapping with bound B if and only if $U:l^2(\mathscr{A})\to\mathscr{H}$ defined by $U(\{a_w\}_{w\in\Omega})=\int_\Omega a_wCF(w)d\mu(w)$ is well defined bounded with $\|U\|\leq\sqrt{B}\|C^{\frac{1}{2}}\|$.*

Proof. Assume that F is a continuous C-controlled Bessel with bound B. Hence ,

$$\|\int_\Omega\langle f,F(w)\rangle_{\mathscr{A}}\langle CF(w),f\rangle_{\mathscr{A}}d\mu(w)\|\leq B\|f\|^2, f\in\mathscr{H}.$$

So,

$$\|\langle S_Cf,f\rangle_{\mathscr{A}}\|\leq B\|f\|^2.$$

In the begining, we show that U is well defined .

For each $\{a_w\}_{w\in\Omega}\in l^2(\mathscr{A})$,

$$\begin{aligned}
\|U(\{a_w\}_{\omega\in\Omega})\|^2 &= \sup_{f\in\mathscr{H},\|f\|=1}\|\langle U(\{a_w\}_{\omega\in\Omega}),f\rangle_{\mathscr{A}}\|^2\\
&= \sup_{f\in\mathscr{H},\|f\|=1}\|\langle\int_\Omega a_wCF(w)d\mu(w),f\rangle_{\mathscr{A}}\|^2\\
&= \sup_{f\in\mathscr{H},\|f\|=1}\|\int_\Omega a_w\langle CF(w),f\rangle_{\mathscr{A}}d\mu(w)\|^2\\
&\leq \sup_{f\in\mathscr{H},\|f\|=1}\|\int_\Omega\langle f,CF(w)\rangle_{\mathscr{A}}\langle CF(w),f\rangle_{\mathscr{A}}d\mu(w)\|.\|\int_\Omega a_wa_w^*d\mu(w)\|\\
&= \sup_{f\in\mathscr{H},\|f\|=1}\|\langle\int_\Omega\langle f,CF(w)\rangle_{\mathscr{A}}CF(w)d\mu(w),f\rangle_{\mathscr{A}}\|.\|\int_\Omega a_wa_w^*d\mu(w)\|\\
&= \sup_{f\in\mathscr{H},\|f\|=1}\|\langle CS_Cf,f\rangle_{\mathscr{A}}\|.\|\int_\Omega a_wa_w^*d\mu(w)\|\\
&= \sup_{f\in\mathscr{H},\|f\|=1}\|\langle(CS_C)^{\frac{1}{2}}f,(CS_C)^{\frac{1}{2}}f\rangle_{\mathscr{A}}\|.\|\{a_w\}_{\omega\in\Omega}\|^2\\
&\leq \sup_{f\in\mathscr{H},\|f\|=1}\|(C)^{\frac{1}{2}}\|^2\|(S_Cf)^{\frac{1}{2}}\|^2\|\{a_w\}_{\omega\in\Omega}\|^2\\
&\leq B\|(C)^{\frac{1}{2}}\|^2\|\{a_w\}_{\omega\in\Omega}\|^2.
\end{aligned}$$

Then,

$$\|U\|\leq\sqrt{B}\|(C)^{\frac{1}{2}}\|.$$

Hence U is well defined and bounded.

Now, suppose that U is well defined, and

$$\|U\| \leq \sqrt{B}\|(C)^{\frac{1}{2}}\|.$$

For any $f \in \mathscr{H}$ and $\{a_w\}_{\omega\in\Omega} \in l^2(\mathscr{A})$, we have,

$$\begin{aligned}\langle f, U(\{a_w\}_{\omega\in\Omega})\rangle_{\mathscr{A}} &= \langle f, \int_\Omega a_w CF(w)d\mu(w)\rangle_{\mathscr{A}} \\ &= \int_\Omega \langle a_w^* Cf, F(w)\rangle_{\mathscr{A}} d\mu(w) \\ &= \int_\Omega \langle Cf, F(w)\rangle_{\mathscr{A}} a_w^* d\mu(w) \\ &= \langle \{\langle Cf, F(w)\rangle_{\mathscr{A}}\}_{\omega\in\Omega}, \{a_w\}_{\omega\in\Omega}\rangle_{\mathscr{A}}.\end{aligned}$$

Then, U has an adjoint, and

$$U^* f = \{\langle Cf, F(w)\rangle_{\mathscr{A}}\}_{\omega\in\Omega}.$$

Also,

$$\begin{aligned}\|U\|^2 &= \sup_{\|(\{a_w\}_{\omega\in\Omega})\|=1} \|U(\{a_w\}_{\omega\in\Omega})\|^2 \\ &= \sup_{\|(\{a_w\}_{\omega\in\Omega})\|=1, \|f\|=1} \|\langle U(\{a_w\}_{\omega\in\Omega}), f\rangle_{\mathscr{A}}\|^2 \\ &= \sup_{\|(\{a_w\}_{\omega\in\Omega})\|=1, \|f\|=1} \|\langle \{a_w\}_{\omega\in\Omega}, U^* f\rangle_{\mathscr{A}}\|^2 \\ &= \sup_{\|f\|=1} \|U^* f\|^2 \\ &= \|U^*\|^2\end{aligned}$$

So,

$$\|U^* f\|^2 = \|\langle U^* f, U^* f\rangle_{\mathscr{A}}\| = \|\langle UU^* f, f\rangle_{\mathscr{A}}\| = \|\langle CS_C f, f\rangle_{\mathscr{A}}\|.$$

Then,

$$\|U^* f\|^2 = \|(CS_C)^{\frac{1}{2}} f\|^2 \leq B\|(C)^{\frac{1}{2}}\|^2 \|f\|^2. \tag{2.7}$$

From lemma 2.6, we have,

$$\|(S_C)^{\frac{1}{2}} f\|^2 \leq \lambda \|(CS_C)^{\frac{1}{2}} f\|^2,$$

for some $\lambda > 0$.

Using (2.7) we get,

$$\begin{aligned} \|(S_C)^{\frac{1}{2}} f\|^2 &\leq \lambda \|(CS_C)^{\frac{1}{2}} f\|^2 \\ &\leq \lambda B \|C^{\frac{1}{2}}\|^2 \|f\|^2. \end{aligned}$$

Hence F is a continuous C-controlled Bessel mapping with Bessel bound $\lambda B\|C^{\frac{1}{2}}\|^2$. □

Proposition 2.8. *Let F be a continuous C-controlled K-frame for $\mathscr{H}$ with bounds A and B. Then :*

$$ACKK^*I \leq S_C \leq B.I.$$

Proof. Suppose F is a continuous C-controlled K-frame with bounds A and B. Then,

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \int_{\Omega} \langle f, F(w)\rangle_{\mathscr{A}} \langle CF(w), f\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}.$$

Hence,

$$A\langle CKK^*f, f\rangle_{\mathscr{A}} \leq \langle S_C f, f\rangle_{\mathscr{A}} \leq B\langle f, f\rangle_{\mathscr{A}}.$$

So,

$$ACKK^*I \leq S_C \leq B.I.$$

□

Proposition 2.9. *Let F be a continuous C-controlled Bessel mapping for $\mathscr{H}$, and $C \in GL^+(\mathscr{H})$. Then F is a continuous C-controlled K-frame for $\mathscr{H}$ if and only if there exists $A > 0$ such that:*

$$ACKK^* \leq CS.$$

Proof. ($\Longrightarrow$) obvious.

($\Longleftarrow$) Assume that there exists $A > 0$ such that: $ACKK^* \leq CS$,

then,

$$A\langle CKK^*f, f\rangle_{\mathscr{A}} \leq \langle S_C f, f\rangle_{\mathscr{A}}.$$

Hence,

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \langle S_C f, f\rangle_{\mathscr{A}}.$$

Therefore,

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w).$$

Hence F is a continuous C-controlled K-frame.

□

Proposition 2.10. *Let $C \in GL^+(\mathscr{H})$, $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and F be a continuous C-controlled K-frame for $\mathscr{H}$ with lower and upper frames bounds A and B respectivelty. Suppose $KC = CK$ and $R(C^{\frac{1}{2}}) \subset R(K^*C^{\frac{1}{2}})$with $R(K^*C^{\frac{1}{2}})$ is closed. Then F is continuous K-frame for $\mathscr{H}$ with lower and upper frames bounds $A\|C^{\frac{-1}{2}}\|^{-2}\|(C)^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$ respectively.*

Proof. Assume that F is a continuous C-controlled K-frame with lower and upper frames bounds A and B. From theorem 2.5, we have:

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w)\| \leq B\|f\|^2, f \in \mathscr{H}.$$

Then,

$$\begin{aligned}
A\|K^*f\|^2 &= A\|C^{\frac{-1}{2}}C^{\frac{1}{2}}K^*f\|^2 \\
&\leq A\|C^{\frac{-1}{2}}\|^2\|C^{\frac{1}{2}}K^*f\|^2 \\
&\leq \|C^{\frac{-1}{2}}\|^2\|\int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w)\|.
\end{aligned}$$

So,

$$A\|K^*f\|^2 \leq \|C^{\frac{1}{2}}\|^2\|\langle S_C f, f\rangle_{\mathscr{A}}\|. \tag{2.8}$$

Moreover,

$$\begin{aligned}
\langle S_C f, f\rangle_{\mathscr{A}} &= \langle CSf, f\rangle_{\mathscr{A}} \\
&= \langle (CS)^{\frac{1}{2}}f, (CS)^{\frac{1}{2}}f\rangle_{\mathscr{A}} \\
&= \|(CS)^{\frac{1}{2}}f\|^2 \\
&\leq \|(C)^{\frac{1}{2}}\|^2.\|(S)^{\frac{1}{2}}f\|^2 \\
&= \|(C)^{\frac{1}{2}}\|^2.\langle (S)^{\frac{1}{2}}f, (S)^{\frac{1}{2}}f\rangle_{\mathscr{A}} \\
&= \|(C)^{\frac{1}{2}}\|^2.\langle Sf, f\rangle_{\mathscr{A}},
\end{aligned}$$

then,

$$\langle S_C f,f\rangle_{\mathscr{A}} \leq \|(C)^{\frac{1}{2}}\|^2.\langle Sf,f\rangle_{\mathscr{A}}. \tag{2.9}$$

From (2.8)and (2.9), we have,

$$\begin{aligned} A\|K^*f\|^2 &\leq \|C^{\frac{-1}{2}}\|^2\|(C)^{\frac{1}{2}}\|^2\langle Sf,f\rangle_{\mathscr{A}} \\ &= \|C^{\frac{-1}{2}}\|^2\|(C)^{\frac{1}{2}}\|^2\int_\Omega\langle f,F(w)\rangle_{\mathscr{A}}\langle F(w),f\rangle_{\mathscr{A}}d\mu(w). \end{aligned}$$

Hence,

$$\|C^{\frac{-1}{2}}\|^{-2}\|(C)^{\frac{1}{2}}\|^{-2}A\|K^*f\|^2 \leq \int_\Omega\langle f,F(w)\rangle_{\mathscr{A}}\langle F(w),f\rangle_{\mathscr{A}}d\mu(w).$$

Moreover,

$$\begin{aligned} \|\int_\Omega\langle f,F(w)\rangle_{\mathscr{A}}\langle F(w),f\rangle_{\mathscr{A}}d\mu(w)\| &= \|\langle Sf,f\rangle_{\mathscr{A}}\| \\ &= \|\langle C^{-1}CSf,f\rangle_{\mathscr{A}}\| \\ &= \|\langle (C^{-1}CS)^{\frac{-1}{2}}f,(C^{-1}CS)^{\frac{-1}{2}}f\rangle_{\mathscr{A}}\| \\ &= \|(C^{-1}CS)^{\frac{1}{2}}f\|^2 \\ &\leq \|C^{\frac{-1}{2}}\|^2\|(CS)^{\frac{1}{2}}f\|^2 \\ &= \|C^{\frac{-1}{2}}\|^2\langle (CS)^{\frac{1}{2}}f,(CS)^{\frac{1}{2}}f\rangle_{\mathscr{A}} \\ &= \|C^{\frac{-1}{2}}\|^2\langle CSf,f\rangle_{\mathscr{A}} \\ &\leq \|C^{\frac{-1}{2}}\|^2B\|f\|^2. \end{aligned}$$

Then F is a continuous K-frame for $\mathscr{H}$ with lower and upper frames bounds $A\|C^{\frac{-1}{2}}\|^{-2}\|(C)^{\frac{1}{2}}\|^{-2}$ and $B\|C^{\frac{-1}{2}}\|^2$.

□

Proposition 2.11. *Let $C\in GL^+(\mathscr{H})$ and $K\in End^*_{\mathscr{A}}(\mathscr{H})$. We Suppose that $KC=CK$, $R(C^{\frac{1}{2}})\subset R(K^*C^{\frac{1}{2}})$ with $R(K^*C^{\frac{1}{2}})$ is closed and F is a continuous K-frame for $\mathscr{H}$ with lower and upper frames bounds A and B respectivlty.*

Then F is continuous C-controlled K-frame for $\mathscr{H}$ with lower and upper frames bounds A and $\|C\|\|S\|$.

Proof. Assume that F is a continuous K-frame for $\mathscr{H}$ with lower and upper frames bounds A and B. Then we have:

$$A\langle K^*f,K^*f\rangle_{\mathscr{A}} \leq \int_{\Omega}\langle f,F(w)\rangle_{\mathscr{A}}\langle F(w),f\rangle_{\mathscr{A}}d\mu(w) \leq B\langle f,f\rangle_{\mathscr{A}},$$

Since $\langle K^*f,K^*f\rangle_{\mathscr{A}} > 0$ and $\langle f,f\rangle_{\mathscr{A}} > 0$ then,

$$A\|K^*f\|^2 \leq \|\int_{\Omega}\langle f,F(w)\rangle_{\mathscr{A}}\langle F(w),f\rangle_{\mathscr{A}}d\mu(w)\| \leq B\|f\|^2. \tag{2.10}$$

Then for every $f \in \mathscr{H}$,

$$\begin{aligned}
A\|C^{\frac{1}{2}}K^*f\|^2 &= A\|K^*C^{\frac{1}{2}}f\|^2 \\
&\leq \|\int_{\Omega}\langle C^{\frac{1}{2}}f,F(w)\rangle_{\mathscr{A}}\langle F(w),C^{\frac{1}{2}}f\rangle_{\mathscr{A}}d\mu(w)\| \\
&= \|\langle\int_{\Omega}\langle C^{\frac{1}{2}}f,F(w)\rangle_{\mathscr{A}}F(w)d\mu(w),C^{\frac{1}{2}}f\rangle_{\mathscr{A}}\| \\
&= \|\langle C^{\frac{1}{2}}Sf,C^{\frac{1}{2}}f\rangle_{\mathscr{A}}\| \\
&= \|\langle CSf,f\rangle_{\mathscr{A}}\| \\
&= \|\langle Sf,Cf\rangle_{\mathscr{A}}\| \\
&\leq \|Sf\|.\|Cf\|,
\end{aligned}$$

then

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \|\langle S_Cf,f\rangle_{\mathscr{A}}\| \leq \|S\|.\|C\|\|f\|^2. \tag{2.11}$$

By (2.11) and theorem2.5, we conclude that F is continuous C-controlled K-frame for $\mathscr{H}$ with lower and upper frames bounds A and $\|C\|\|S\|$.

□

Theorem 2.12. *Let $C \in GL^+(\mathscr{H})$, and F be a continuous C-controlled K-frame for $\mathscr{H}$ with bounds A and B. Let $M,K \in End^*_{\mathscr{A}}(\mathscr{H})$ such that $R(M) \subset R(K)$, $R(K)$ is closed and C commutes with M^* and K^*. Then F is continuous C-controlled M-frame for $\mathscr{H}$.*

Proof. Assume that F be a continuous C-controlled K-frame for $\mathscr{H}$ with bounds A and B, then,

$$A\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \leq \int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}, f \in \mathscr{H}. \tag{2.12}$$

Since $R(M) \subseteq R(K)$, by theorem 1.4, there exists some $0 \leq \lambda$ such that

$$MM^* \leq \lambda KK^*.$$

Hence,

$$\langle MM^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq \lambda\langle KK^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle_{\mathscr{A}},$$

then,

$$\frac{A}{\lambda}\langle MM^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq A\langle KK^*C^{\frac{1}{2}}f, C^{\frac{1}{2}}f\rangle_{\mathscr{A}}.$$

By (2.12), we have,

$$\frac{A}{\lambda}\langle M^*C^{\frac{1}{2}}f, M^*C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq \int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w) \leq B\langle f, f\rangle_{\mathscr{A}}.$$

Then F is continuous C-controlled M-frame for $\mathscr{H}$ with bounds $\frac{A}{\lambda}$ and B. □

The following results gives the invariance of a continuous C-controlled Bessel mapping by a adjointable operator.

Proposition 2.13. *Let $T \in End^*_{\mathscr{A}}(\mathscr{H})$ such that $TC = CT$ and F be a continuous C-controlled Bessel mapping with bound D. Then TF is also a continuous C-controlled Bessel mapping with bound $D\|T^*\|$.*

Proof. Assume that F is a continuous C-controlled Bessel mapping with bound D. Hence we have,

$$\int_{\Omega}\langle f, F(w)\rangle_{\mathscr{A}}\langle CF(w), f\rangle_{\mathscr{A}} d\mu(w) \leq D\langle f, f\rangle_{\mathscr{A}}, f \in \mathscr{H}.$$

We have,

$$\begin{aligned}\int_\Omega \langle f, TF(w)\rangle_{\mathscr{A}} \langle CTF(w), f\rangle_{\mathscr{A}} d\mu(w) &= \int_\Omega \langle T^* f, F(w)\rangle_{\mathscr{A}} \langle TCF(w), f\rangle_{\mathscr{A}} d\mu(w) \\ &= \int_\Omega \langle T^* f, F(w)\rangle_{\mathscr{A}} \langle CF(w), T^* f\rangle_{\mathscr{A}} d\mu(w) \\ &\leq D\langle T^* f, T^* f\rangle_{\mathscr{A}} \\ &\leq D\|T^*\|^2 \langle f, f\rangle_{\mathscr{A}}.\end{aligned}$$

The result holds.

□

Now, we study the invariance of a continuous C-controlled K-frame mapping by adjointable operator.

Theorem 2.14. *Let $C \in GL^+(\mathscr{H})$, and F be a continuous C-controlled K-frame for $\mathscr{H}$ with bounds A and B. If $T \in End^*_{\mathscr{A}}(\mathscr{H})$ with closed range such that $R(K^*T^*)$ is closed and C, K, T commute with each other. Then TF is a continuous C-controlled K-frame for $R(T)$.*

Proof. Assume that F is a continuous C-controlled K-frame with bounds A and B. Then,

$$A\langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f\rangle_{\mathscr{A}} \leq \int_\Omega \langle f, F(w)\rangle_{\mathscr{A}} \langle CF(w), f\rangle_{\mathscr{A}} \leq B\langle f, f\rangle_{\mathscr{A}}, f \in \mathscr{H}.$$

Since T has a closed range, then T has Moore-Penrose inverse $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger T T^\dagger = T^\dagger$, so $TT^\dagger_{/R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$.

We have,

$$\begin{aligned}\langle K^* C^{\frac{1}{2}} f, K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}} &= \langle (TT^\dagger)^* K^* C^{\frac{1}{2}} f, (TT^\dagger)^* K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}} \\ &= \langle (T^\dagger)^* T^* K^* C^{\frac{1}{2}} f, (T^\dagger)^* T^* K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}}.\end{aligned}$$

So,

$$\langle K^* C^{\frac{1}{2}} f, K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}} \leq \|(T^\dagger)^*\|^2 \langle T^* K^* C^{\frac{1}{2}} f, T^* K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}}. \tag{2.13}$$

Therfore,

$$\|(T^\dagger)^*\|^{-2} \langle K^* C^{\frac{1}{2}} f, K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}} \leq \langle T^* K^* C^{\frac{1}{2}} f, T^* K^* C^{\frac{1}{2}} f\rangle_{\mathscr{A}}. \tag{2.14}$$

Consequently, from theorem 1.4, and $R(T^*K^*) \subset R(K^*T^*)$, there exists some $\lambda \geq 0$ such that,

$$\langle T^*K^*C^{\frac{1}{2}}f, T^*K^*C^{\frac{1}{2}}f\rangle_{\mathscr{A}} \leq \lambda \langle K^*T^*C^{\frac{1}{2}}f, K^*T^*C^{\frac{1}{2}}f\rangle_{\mathscr{A}}. \tag{2.15}$$

Hence, using (2.14) and (2.15) we have,

$$\begin{aligned}\int_{\Omega}\langle f, TF(w)\rangle_{\mathscr{A}}\langle CTF(w), f\rangle_{\mathscr{A}} d\mu(w) &= \int_{\Omega}\langle T^*f, F(w)\rangle_{\mathscr{A}}\langle TCF(w), f\rangle_{\mathscr{A}} d\mu(w)\\ &= \int_{\Omega}\langle T^*f, F(w)\rangle_{\mathscr{A}}\langle CF(w), T^*f\rangle_{\mathscr{A}} d\mu(w)\\ &\geq A\langle C^{\frac{1}{2}}K^*T^*f, C^{\frac{1}{2}}K^*T^*f\rangle_{\mathscr{A}}\\ &\geq \frac{A}{\lambda}\langle T^*C^{\frac{1}{2}}K^*f, T^*C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}},\end{aligned}$$

then,

$$\int_{\Omega}\langle f, TF(w)\rangle_{\mathscr{A}}\langle CTF(w), f\rangle_{\mathscr{A}} d\mu(w) \geq \frac{A}{\lambda}\|(T^{\dagger})^*\|^{-2}\langle C^{\frac{1}{2}}K^*f, C^{\frac{1}{2}}K^*f\rangle_{\mathscr{A}} \tag{2.16}$$

Using (2.16) and proposition2.13, the result holds.

□

Theorem 2.15. *Let $C \in GL^{\dagger}(\mathscr{H})$ and F be a continuous C-controlled K-frame for $\mathscr{H}$ with bounds A and B.*
If $T \in End^*_{\mathscr{A}}(\mathscr{H})$ is a isometry such that $R(T^*K^*) \subset R(K^*T^*)$ with $R(K^*T^*)$ is closed and C, K, T commute with each other, then TF is a continuous C-controlled K-frame for $\mathscr{H}$.

Proof. Using theorem 1.4, there exists some $\lambda \geq 0$ such that,

$$\|T^*K^*C^{\frac{1}{2}}f\|^2 \leq \lambda\|K^*T^*C^{\frac{1}{2}}f\|^2.$$

Assume A the lower bound for the continuous C-controlled K-frame F and T is an isometry then,

$$\begin{aligned}\frac{A}{\lambda}\|C^{\frac{1}{2}}K^*f\|^2 &= \frac{A}{\lambda}\|T^*C^{\frac{1}{2}}K^*f\|^2\\ &\leq A\|K^*T^*C^{\frac{1}{2}}f\|^2\\ &= A\|C^{\frac{1}{2}}K^*T^*f\|^2\\ &\leq \int_\Omega \langle T^*f, F(w)\rangle_{\mathscr{A}}\langle CF(w), T^*f\rangle_{\mathscr{A}} d\mu(w)\\ &= \int_\Omega \langle f, TF(w)\rangle_{\mathscr{A}}\langle TCF(w), f\rangle_{\mathscr{A}} d\mu(w),\end{aligned}$$

then,

$$\frac{A}{\lambda}\|C^{\frac{1}{2}}K^*f\|^2 \leq \int_\Omega \langle f, TF(w)\rangle_{\mathscr{A}}\langle CTF(w), f\rangle_{\mathscr{A}} d\mu(w). \tag{2.17}$$

Hence, from proposition 2.13 and inequality (2.17), we conclude that TF is a continuous C-controlled K-frame for $\mathscr{H}$ with bounds $\frac{A}{\lambda}$ and $B\|T^*\|^2$.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S. T. Ali, J. P. Antoine, J. P. Gazeau, Continuous frames in Hilbert spaces, Ann. Phys. 222 (1993), 1-37.

[2] L. Arambašić, On frames for countably generated Hilbert $\mathscr{C}^*$-modules, Proc. Amer. Math. Soc. 135 (2007), 469-478.

[3] A. Askari-Hemmat, M. A. Dehghan, M. Radjabalipour, Generalized frames and their redundancy, Proc. Amer. Math. Soc. 129 (2001), 1143-1147.

[4] P. Balazs, J-P. Antoine and A. Grybos, Wighted and Controlled Frames. Int. J. Walvelets Multi. Inf. Process. 8 (2010), 109-132.

[5] O. Christensen, An Introduction to Frames and Riesz bases, Brikhouser, (2016).

[6] J.B. Conway, A Course In Operator Theory, AMS, V.21, 2000.

[7] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986), 1271-1283.

[8] F. R. Davidson, $\mathscr{C}^*$-algebra by example, Fields Ins. Monog. 1996.

[9] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic fourier series, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[10] J. P. Gabardo and D. Han, Frames associated with measurable space, Adv. Comp. Math. 18 (2003), 127-147.

[11] D. Gabor, Theory of communications, J. Elec. Eng. 93 (1946), 429-457.

[12] L. Gavruta, Frames for operators, Appl. Comput. Harmon. Anal. 32 (2012), 139-144.

[13] S. Kabbaj, M. Rossafi, $*$-operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Wavelet Linear Algebra, 5 (2018), 1-13.

[14] G. Kaiser, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.

[15] M. R. Kouchi and A. Rahimi, On controlled frames in Hilbert$\mathscr{C}^{*}$modules, Int. J. Walvelets Multi. Inf. Process. 15 (2017), 1750038.

[16] E. C. Lance, Hilbert C^{*}-Modules: A Toolkit for Operator Algebraist, 144 pages, vol. 210 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK, (1995).

[17] M. Nouri, A. Rahimi and Sh. Najafzadeh, Controlled K-frames in Hilbert Spaces, J. Ramanujan Soc. Math. Math. Sc. 4 (2015), 39-50.

[18] W. Paschke, Inner product modules over B^{*}-algebras, Trans. Amer. Math. Soc. 182 (1973), 443-468.

[19] M. Rossafi, S. Kabbaj, $*$-K-operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Asian-Eur. J. Math. 13 (2020), 2050060.

[20] M. Rossafi, A. Bourouihiya , H. Labrigui and A. Touri, The duals of $*$-operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Asia Math. 4 (2020), 45-52.

[21] M. Rossafi, A. Touri, H. Labrigui and A. Akhlidj, Continuous $*$-K-G-Frame in Hilbert C^{*}-Modules, J. Funct. Spaces, 2019 (2019), Article ID 2426978.

[22] M. Rossafi, S. Kabbaj, Operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, J. Linear Topol. Algebra, 8 (2019), 85-95.

[23] S. Kabbaj, M. Rossafi, $*$-operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Wavelet Linear Algebra, 5 (2018), 1-13.

[24] M. Rossafi, S. Kabbaj, $*$-K-g-frames in Hilbert $\mathscr{A}$-modules, J. Linear Topol. Algebra, 7 (2018), 63-71.

[25] M. Rossafi, S. Kabbaj, $*$-g-frames in tensor products of Hilbert C^{*}-modules, Ann. Univ. Paedagog. Crac. Stud. Math. 17 (2018), 17-25.

[26] M. Rossafi, S. Kabbaj, K-operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Asia Math. 2 (2018), 52-60.

[27] M. Rossafi, S. Kabbaj, Frames and Operator Frames for $B(\mathscr{H})$, Asia Math. 2 (2018), 19-23.

[28] M. Rossafi, A. Akhlidj, Perturbation and Stability of Operator Frame for $End_{\mathscr{A}}^{*}(\mathscr{H})$, Math-Recherche Appl. 16 (2017-2018), 65-81.

[29] M. Rossafi, S. Kabbaj, Generalized Frames for $B(\mathscr{H},\mathscr{K})$, Iran. J. Math. Sci. Inf. accepted.

[30] L. C. Zhang, The factor decomposition theorem of bounded generalized inverse modules and their topological continuity, J. Acta Math. Sin. 23 (2007), 1413-1418.