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ROUGH CAUCHY SEQUENCES IN A CONE METRIC SPACE

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Abstract. In this paper, we have introduced the idea of rough Cauchyness of sequences in a cone metric space. We have also discussed several basic properties of rough Cauchy sequences in a cone metric space using the idea of Phu [8].

Keywords: rough convergence; rough Cauchy sequence; cone; cone metric space.

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1. Introduction

Metric spaces have been generalized in many ways. Huang and Zhang [5] introduced the idea of cone metric spaces where the idea of real numbers were replaced by an ordered Banach space and the idea of distance have been generalized to a vector through the idea of a cone defined in an ordered Banach space. Many works[4, 5] have been done in the setting of a cone metric space.

Phu [8] introduced the idea of rough convergence of sequences as a generalization of ordinary convergence of sequences in a normed linear space in 2001. There he also introduced the idea of rough Cauchyness of sequences as a generalization of Cauchyness of sequences.

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Phu [8] discussed about the usefulness of rough convergence and some of its basic properties. Phu [9] also extended the idea of rough convergence of sequences in an infinite dimensional normed linear space in 2003 and also in 2008 Ayter [1] introduced the idea of rough statistical convergence. Many works have been done by many authors [2, 6, 7] using the idea of Phu. Recently Banerjee and Mondal [3] studied the idea of rough convergence of sequences in a cone metric space. Here in this paper we define the idea of rough Cauchy sequences in a cone metric space and discuss some its basic properties.

2. Preliminaries

Definition 2.1.[8] Let $\{x_n\}$ be a sequence in a normed linear space $(X, \|.\|)$, and r be a non-negative real number. Then $\{x_n\}$ is said to be r-convergent to x if for any $\varepsilon > 0$, there exists a natural number k such that $\|x_n - x\| < r + \varepsilon$ for all $n \geq k$.

Definition 2.2. [5] Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if (i) P is closed nonempty, and $P \neq \{0\}$.

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ implies $x = 0$.

Let E be a real Banach space and P be a cone in E. Let us use the partial ordering [5] with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

Also by $x << y$, we mean $y - x \in intP$, the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $||x|| \leq K||y||$.

Definition 2.3. [5] Let X be a non empty set. If the mapping $d : X \times X \longrightarrow E$ satisfies the following three conditions

$(d1)$ $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

$(d2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;

$(d3)$ $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$; then d is called a cone metric on X, and (X,d) is called a cone metric space.

It is clear that a cone metric space is a generalization of metric spaces. Throughout (X,d) or simply X stands for a cone metric space which is associated with a real Banach space E with a cone P, $\mathbb{R}$ for the set of all real numbers, $\mathbb{N}$ for the set of all natural numbers, sets are always subsets of X unless otherwise stated.

Definition 2.4. [5] Let (X,d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for every $c \in E$ with $0 << c$ there is $k \in \mathbb{N}$ such that $d(x_n,x) << c$, whenever for all $n > k$.

We know that in a real Banach space E with cone P. If $x_0 \in intP$ and $c(>0) \in \mathbb{R}$ then $cx_0 \in intP$ and if $x_0 \in P$ and $y_0 \in intP$ then $x_0 + y_0 \in intP$. Hence we can also say that if $x_0, y_0 \in intP$ then $x_0 + y_0 \in intP$. Also it has been discussed in [3] that a real normed linear space is always connected and if E be a real Banach space with cone P then $0 \notin intP$.

Definition 2.5.[8, 3] Let (X,d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be r-convergent to x for some $r \in E$ with $0 << r$ or $r = 0$ if for every ε with $0 << \varepsilon$ there exists a $k \in \mathbb{N}$ such that $d(x_n,x) << r + \varepsilon$ for all $n \geq k$.

Let (X,d) be a cone metric space with normal cone P and normal constant k. Then from [3] we can say that for every $\varepsilon > 0$, we can choose $c \in E$ with $c \in intP$ and $k\|c\| < \varepsilon$ and also for each $c \in E$ with $0 << c$, there is a $\delta > 0$, such that $\|x\| < \delta$ implies $c - x \in intP$. We will use these ideas in the next section of our work.

3. Main Results

Definition 3.1. cf.[5] Let (X,d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if for every $(0 <<)\varepsilon$ there exists a natural number m such that $d(x_i,x_j) << \varepsilon$ for all $i, j \geq m$.

Definition 3.2. A sequence $\{x_n\}$ in a cone metric space (X,d) is said to be a r-Cauchy sequence for some$(0 <<)r$ or $r = 0$ if for every $(0 <<)\varepsilon$ there exists a natural number m such that $d(x_i,x_j) << r + \varepsilon$ fior all $i, j \geq m$.

Example 3.3. Let $X = \mathbb{R}^2$ and $E = \mathbb{R}^2$ with $P = \{(x,y) \in E : x,y \geq 0\}$ and consider the function $d : X \times X \longrightarrow E$ be defined by:
$d(\eta,\xi) = (\|\eta - \xi\|, \|\eta - \xi\|)$ where $\|\eta - \xi\| = \sqrt{(\eta_1 - \xi_1)^2 + (\eta_2 - \xi_2)^2}$ for $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. It can be easily clarified that (X,d) is a cone metric space without being a metric space.

Now let us consider the sequence $\{\xi_n\} = \{((-1)^n, (-1)^n)\}$ in X . Clearly the sequence is not a Cauchy sequence in (X,d) since if we consider $\varepsilon = (1,1)$ then we can not able to find any $k \in \mathbb{N}$ for which we can write $d(\xi_m, \xi_n) << \varepsilon$ for all $m, n \geq k$. Because of the fact that

$$d(\xi_m, \xi_n) = (0,0); \text{ if both of } m,n \text{ are even}$$
$$d(\xi_m, \xi_n) = (0,0); \text{ if both of } m,n \text{ are odd}$$
$$d(\xi_m, \xi_n) = (2\sqrt{2}, 2\sqrt{2}); \text{ if } m \text{ is even and } n \text{ is odd}$$
$$d(\xi_m, \xi_n) = (2\sqrt{2}, 2\sqrt{2}); \text{ if } m \text{ is odd and } n \text{ is even}$$

But if we consider $r = (2\sqrt{2}, 2\sqrt{2})$ then for every $(0 << \varepsilon)$ we can write $d(\xi_m, \xi_n) << r + \varepsilon$ for all $n, m \geq 1$. Hence $\{\xi_n\}$ is not a Cauchy sequence in (X,d) but it is a r-Cauchy sequence in (X,d), where r is the degree of roughness. Also we should note that if we consider $r = (g,h)$, where $g \geq 2\sqrt{2}$ and $h \geq 2\sqrt{2}$ then $\{\xi_n\}$ is a r-Cauchy sequence in X.

It should be noted that when $r = 0$ then the idea of rough Cauchyness coincides with the notion Cauchyness.

Theorem 3.4. If a sequence $\{x_n\}$ is r-Cauchy in X then it is also p-Cauchy in X for some $(r <<)p$.

Proof. Let $\{x_n\}$ be a r-Cauchy sequence in the cone metric space X and let $(0 <<)\varepsilon$ be arbitrary. Now for $(0 <<)\varepsilon$ we can find a $k \in \mathbb{N}$ such that $d(x_i, x_j) << r + \varepsilon$ for all $i, j \geq k$. So we can write $0 << (r + \varepsilon) - d(x_i, x_j)$ for all $i, j \geq k$. We also have $0 << (p - r)$ and hence we can write $d(x_i, x_j) << p + \varepsilon$ for all $i, j \geq k$. Hence the result follows. □

Theorem 3.5. Every $\frac{r}{2}$-convergent sequence in a cone metric space (X,d) is r-Cauchy for every r as defined above.

Proof. Let $\{x_n\}$ be a $\frac{r}{2}$-convergent sequence in a cone metric space (X,d) and converges to x in X and consider a arbitrary $\varepsilon \in intP$. Now for the $(0 <<)\varepsilon$ there exists a natural number m such that $d(x_n, x) << \frac{r}{2} + \frac{\varepsilon}{2}$ for all $n \geq m$, so $[\frac{r}{2} + \frac{\varepsilon}{2}] - d(x_n, x) \in intP$ for all $n \geq m$. Now if $r \in intP$ then for any two natural numbers i, j we have $d(x_i, x_j) \leq d(x_i, x) + d(x_j, x) \longrightarrow (i)$. So $[d(x_i, x) + d(x_j, x)] - d(x_i, x_j) \in P$. Also for $i \geq m$ and $j \geq m$ we have $[\frac{r}{2} + \frac{\varepsilon}{2}] - d(x_i, x) \in intP$ and $[\frac{r}{2} + \frac{\varepsilon}{2}] - d(x_j, x) \in intP$. Hence $[r + \varepsilon] - [d(x_i, x) + d(x_j, x)] \in intP \longrightarrow (ii)$ Therefore using (i) and (ii) we have $d(x_i, x_j) << r + \varepsilon$. □

The idea of boundedness of a sequence in cone metric spaces is similar as in the case of a metric space and it has been thoroughly discussed in [3]. We recall that a sequence $\{x_n\}$ in a cone metric space (X,d) is bounded if there is a $g \in intP$ such that $d(x_m,x_n) << g$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

Example 3.6. Let us consider the cone metric space (X,d) as defined in the example 3.1. Now if we consider the sequence $\{\xi_n\} = \{(1,2^n)\}$ in X then there does not exists any $0 << (r,s)$ such that $d(\xi_m,\xi_n) << (r,s)$ holds for every $m,n \in \mathbb{N}$. Because of the fact that for every $s \in \mathbb{R}$ there exists m,n such that $\|(1,2^n)-(1,2^m)\| > s$. Hence $\{\xi_n\}$ is not a bounded sequence in X.

Now if we consider $(0<<)\varepsilon = (p_1,p_2)$ then for every $k \in \mathbb{N}$ there exists $m,n > k(\in \mathbb{N})$ such that $\|(1,2^n)-(1,2^m)\| > p_2$ and hence for any $(0<<)\varepsilon$ we can not find any $k \in \mathbb{N}$ such that $d(\xi_m,\xi_n) << \varepsilon$ holds for every $m,n \geq k$. Therefore $\{\xi_n\}$ is not a Cauchy sequence in X.

Theorem 3.7. A bounded sequence in a cone metric space is always r-Cauchy for some r as defined above.

Proof. Let $\{x_n\}$ be a bounded sequence in a cone metric space (X,d). So there exists a $(0<<)s$ such that $d(x_n,x_m) << s$ for all $m,n \in \mathbb{N}$. Therefore $[s-d(x_n,x_m)] \in intP$ for all $m,n \in \mathbb{N}$. Hence for any $(0<<)\varepsilon$ we have $[s-d(x_n,x_m)]+\varepsilon \in intP$ for all $m,n \in \mathbb{N}$. Therefore $d(x_n,x_m) << s+\varepsilon$ for all $m,n \in \mathbb{N}$. So $\{x_n\}$ is a s-Cauchy sequence in X. ☐

Theorem 3.8. A sequence in a cone metric space is r-Cauchy then it is bounded in that space.

Proof. Let η_n be a r-Cauchy sequence in a cone metric space (X,d). Now since η_n is a r-Cauchy sequence for some $(0<<)\varepsilon$ we can find a $k \in \mathbb{N}$ such that $d(\eta_n,\eta_m) << r+\varepsilon$ for all $m,n \geq k$.

Now if we consider the sum $\sum d(\eta_n,\eta_m) = S$, where the summation runs over m,n not both greter equals to k then clearly $(0 \leq)S$. So $(0<<)S+r+\varepsilon$ because $0 << \varepsilon$. Now clearly $0 << (S+r+\varepsilon)-d(\eta_i,\eta_j)$ for i,j not both greater equals to k. Also $0 << (r+\varepsilon)-d(\eta_i,\eta_j)$ for all $i,j \geq k$.Hence we have $0 << S+(r+\varepsilon)-d(\eta_i,\eta_j)$ for all $i,j \geq k$ because $0 \leq S$. Therefore we can write $d(\eta_i,\eta_j) << S+(r+\varepsilon)$ for all $i,j \in \mathbb{N}$, where $0 << S+r+\varepsilon$. ☐

Theorem 3.9. Let (X,d) be a cone metric space with normal cone P and normal constant k and with the given condition $*$ as given bellow. If $\{x_n\}$ and $\{y_n\}$ be two $\frac{r}{2k^2}$-Cauchy sequence $(0 << r)$ in X then the sequence $\{d(x_n,y_n)\}$ is $||r||$-Cauchy.
$(*)$ If $p \leq c$ and $-p \leq c$ with $c \in P$ then $||p|| \leq k||c||$, where $p, c \in E$.

Proof. Let $\varepsilon > 0$ be preassigned real number. So by using the property of a normal cone there exists a $c \in intP$ such that $k||c|| < \varepsilon$. Since $c \in intP$, we have $\frac{c}{2} \in intP$ and also $\frac{r}{2k^2} \in intP$. Now there exists two positive integers k_1 and k_2 such that $d(x_i,x_j) << \frac{r}{2k^2} + \frac{c}{2}$ for all $i, j \geq k_1$ and $d(y_i,y_j) << \frac{r}{2k^2} + \frac{c}{2}$ for all $i, j \geq k_2$. If k be the maximum of k_1 and k_2 then by using the property of normal cone we have $||d(x_i,x_j)|| \leq k||\frac{r}{2k^2} + \frac{c}{2}||$ and $||d(y_i,y_j)|| \leq k||\frac{r}{2k^2} + \frac{c}{2}||$ for all $i, j \geq k$.

Now $d(x_i,y_i) - d(x_j,y_j) \leq d(x_i,x_j) + d(y_i,y_j)$ and $d(x_j,y_j) - d(x_i,y_i) \leq d(x_i,x_j) + d(y_i,y_j)$. Since $[d(x_i,x_j) + d(y_i,y_j)] \in P$, by the given condition we have $||d(x_i,y_i) - d(x_j,y_j)|| \leq k||d(x_i,x_j) + d(y_i,y_j)|| \leq k||d(x_i,x_j)|| + k||d(y_i,y_j)||$. Hence for all $i, j \geq k$ we have $||d(x_i,y_i) - d(x_j,y_j)|| \leq ||r|| + k||c|| < ||r|| + \varepsilon$. □

Theorem 3.10. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a cone metric space (X,d) such that $d(x_n,y_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $\{x_n\}$ is r-Cauchy if and only if $\{y_n\}$ is r-Cauchy.

Proof. Let $\{x_n\}$ be a r-Cauchy sequence in (X,d). Then for $0(<< \varepsilon)$ there exists two positive integers k_1 and k_2 such that $d(x_i,x_j) << r + \frac{\varepsilon}{3}$ for all $i, j \geq k_1$ and $d(x_i,y_i) << \frac{\varepsilon}{3}$ for all $i \geq k_2$. Now $d(y_i,y_j) \leq d(x_i,y_i) + d(x_i,y_j)$ and also $d(x_i,y_j) \leq d(x_i,x_j) + d(x_j,y_j)$. Hence $d(x_j,y_j) \leq d(x_i,y_i) + d(x_i,x_j) + d(x_j,y_j)$. If $k = max(k_1,k_2)$, then for all $i, j \geq k$ we have $d(x_i,y_j) << r + \varepsilon$ and hence $\{y_n\}$ is r-Cauchy.

Conversely if $\{y_n\}$ is r-Cauchy then similarly we can show that $\{x_n\}$ is r-Cauchy. □

Theorem 3.11. Let (X,d) be a cone metric space with normal cone P and normal constant k. If a sequence $\{x_n\}$ in (X,d) is r-Cauchy and also converges to x in X then the sequence $\{d(x_n,x) - r\}$ is converges to 0 in E, provided that $\{d(x_n,x) - r\}$ is a sequence in P.

Proof. Let $\{x_n\}$ be a r-Cauchy sequence and converges to x in X. Let $\varepsilon > 0$ be preassigned. Then we have an element $c \in E$ with $0 << c$ and $k\|c\| < \varepsilon$. Now for $0 << c$ there exists $k_1, k_2 \in \mathbb{N}$ such that $d(x_n,x_m) << r + \frac{c}{2}$ for all $n, m \geq k_1$ and $d(x_m,x) << \frac{c}{2}$ for all $m \geq k_2$.

Hence $0 \leq d(x_n, x) - r << c$ for all $n \geq k$, k is the maximum of k_1 and k_2. Now by using the property of a normal cone we have $\|d(x_n, x) - r\| \leq k\|c\| < \varepsilon$ for all $n \geq k$. Therefore $\{d(x_n, x) - r\}$ converges to 0 in E. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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