Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:28
https://doi.org/10.28919/jmcs/6826
ISSN: 1927-5307

SOME FIXED POINTS OF α-MEIR-KEELER CONTRACTION MAPPINGS IN G_b-METRIC SPACES

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Abstract. In this note, we define the notion of Meir-Keeler contraction in G_b-metric spaces. Further, by adding the concept of α-admissible, we introduce the definition of generalised α-Meir-Keeler contraction and used it for examining the existence of fixed points and uniqueness. Various results are also given as consequence of our results.

Keywords: Meir-Keeler contraction; α-admissible mappings; G_b-metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Banach fixed point theorem, commonly known as Banach contraction principle is the most important theorem in Metric fixed point theory. Due to its simplicity, easiness and applicability to various disciplines, Banach fixed point theorem has been extended and generalized in different directions. Extension of Banach fixed point theorem by changing the space i.e. metric space to other suitable space is one of the interest for many researchers. Some of the important works in this direction can be found through research papers in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein. Researchers also work on another direction to generalize

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Received September 26, 2021

Banach fixed point theorem. In this case, the contraction condition is generalized. There are large number of resesarch papers in literature working in this area. Some of the results related with our study can be found in [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

For our study it will be the combination of both ways. For the space to be used, we consider G_b-metric space [29] and for the contraction condition we use Meir-Keeler contraction [16].

2. Preliminaries

The definition of G_b-metric space is given by Aghajani et al. [29].

Definition 2.1. [29] *In a set $\Omega \neq \phi$, suppose $b \geq 1$ be a real number and $G : \Omega \times \Omega \times \Omega \to [0, +\infty)$ be a function satisfying*

1.: $G(\theta, \phi, \psi) = 0$ *if and only if* $\theta = \phi = \psi$ *for all* $\theta, \phi, \psi \in \Omega$*;*

2.: $0 < G(\theta, \theta, \phi)$ *for all* $\theta, \phi \in \Omega$ *with* $\theta \neq \phi$*;*

3.: $G(\theta, \theta, \phi) \leq G(\theta, \phi, \psi)$ *for all* $\theta, \phi, \psi \in \Omega$ *with* $\psi \neq \phi$*;*

4.: $G(\theta, \phi, \psi) = G(P[\theta, \phi, \psi])$ *where P is a permutation of θ, ϕ, ψ (symmetry in all three variables);*

5.: $G(\theta, \phi, \psi) \leq b[G(\theta, \mu, \mu) + G(\mu, \phi, \psi)]$ *for all* $\theta, \phi, \psi, \mu \in \Omega$ *(rectangular inequality)*

Here, G is said to be a G_b-metric and (Ω, G) is said to be a G_b-metric space.

Definition 2.2. [29] *In an G_b-metric space (Ω, G), a sequence $\{\theta_n\}$ is called*

(i): *G_b-convergent to a point $\theta \in \Omega$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(\theta_m, \theta_n, \theta) < \varepsilon$.*

(ii): *G_b-Cauchy if for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(\theta_m, \theta_n, \theta_l) < \varepsilon$.*

(iii): *Also, the space (Ω, G) is called complete G_b-metric space if every G_b-Cauchy sequence is G_b-convergent.*

We recall some types of α-admissible mappings in a metric space (X, d).

Definition 2.3. [19] *Let $A : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to [0, +\infty)$ are functions. Here, A is said to be an α-admissible if $\alpha(\theta, \phi) \geq 1$ implies $\alpha(A\theta, A\phi) \geq 1$ for all $\theta, \phi \in \Omega$.*

Definition 2.4. [20] *Let $A,B:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\to[0,+\infty)$ are functions. Here, the pair of mappings (A,B) is said to be an α-admissible if $\alpha(\theta,\phi)\geq 1$ implies $\alpha(A\theta,B\phi)\geq 1$ and $\alpha(B\theta,A\phi)\geq 1$ for all $\theta,\phi\in\Omega$.*

Definition 2.5. [21] *Let $A:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\to[0,+\infty)$ are functions. Here, A is known as triangular α-admissible, if:*

(i): $\alpha(\theta,\phi)\geq 1$, *implies* $\alpha(A\theta,A\phi)\geq 1,\ \theta,\phi\in\Omega$,

(ii): $\alpha(\theta,\phi)\geq 1$, $\alpha(\phi,\psi)\geq 1$, *implies* $\alpha(\theta,\psi)\geq 1,\ \theta,\phi,\psi\in\Omega$.

Definition 2.6. [20] *Let $A,B:\Omega\to\Omega$ and $\alpha:\Omega\times\Omega\to[0,+\infty)$ are functions. Here, the pair (A,B) is said to be a triangular α-admissible, if*

(i): $\alpha(\theta,\phi)\geq 1$, *implies* $\alpha(A\theta,B\phi)\geq 1$ *and* $\alpha(B\theta,A\phi)\geq 1,\ \theta,\phi\in\Omega$,

(ii): $\alpha(\theta,\phi)\geq 1$, $\alpha(\phi,\psi)\geq 1$, *implies* $\alpha(\theta,\psi)\geq 1$, *for all* $\theta,\phi,\psi\in\Omega$.

Definition 2.7. [30] *Let $A:\Omega\to\Omega$ and $\alpha_G:\Omega\times\Omega\times\Omega\to[0,+\infty)$ are functions, then A is called α_G-admissible, if $\theta,\phi,\psi\in\Omega$, $\alpha_G(\theta,\phi,\psi)\geq 1$ implies $\alpha_G(A\theta,A\phi,A\psi)\geq 1$.*

Example 1. [6] *Consider $\Omega=[0,+\infty)$ and define $A:\Omega\to\Omega$ and $\alpha_G:\Omega\times\Omega\times\Omega\to[0,+\infty)$ by $A\theta=4\theta$, for all $\theta,\phi,\psi\in\Omega$, and*

$$\alpha_G(\theta,\phi,\psi)=\begin{cases} e^{\frac{\psi}{\theta\phi}}, & \text{if },\ \theta\geq\phi\geq\psi,\ \theta,\phi\neq 0\\ 0, & \text{if},\ \theta<\phi<\psi. \end{cases}$$

Then A is an α_G-admissible mapping.

Definition 2.8. [6] *Let $A,B:\Omega\to\Omega$ and $\alpha_G:\Omega\times\Omega\times\Omega\to[0,+\infty)$ are three functions. The pair (A,B) is called α_G-admissible if $\theta,\phi,\psi\in\Omega$ such that $\alpha_G(\theta,\phi,\psi)\geq 1$, then we have $\alpha_G(A\theta,A\phi,B\psi)\geq 1$ and $\alpha_G(B\theta,B\phi,A\psi)\geq 1$.*

3. Main Results

Here, we give various types of Meir-Keeler contractive mappings in order to extend various results of Selma et. al. [22] in G_b-metric space. Throughout this paper, assume (Ω,G) be an G_b-metric space, $b\geq 1$ be a real number and $A:\Omega\to\Omega$ be a mapping.

Definition 3.1. *An α_G-admissible mapping A in (Ω, G) is known as α_G-Meir-Keeler contraction mapping of type I, if there esists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \leq G(\theta, \phi, \psi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta, \phi, \psi)\; G(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{1}$$

for all $\theta, \phi, \psi \in \Omega$.

Definition 3.2. *An α_G-admissible mapping A in (Ω, G) is known as α_G-Meir-Keeler contraction mapping of type II, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \leq G(\theta, \phi, \phi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta, \phi, \phi)\; G(A\theta, A\phi, A\phi) < \frac{\varepsilon}{b} \tag{2}$$

for all $\theta, \phi \in \Omega$.

Remark 1. **(i):** *If A be an α_G-Meir-Keeler contraction of type I, then*

$$\alpha_G(\theta, \phi, \psi) G(A\theta, A\phi, A\psi) \leq \frac{G(\theta, \phi, \psi)}{b},$$

for all $\theta, \phi, \psi \in \Omega$ and equality is true, when $\theta = \phi = \psi$.

(ii): *If A be an α_G-Meir-Keeler contraction of type II, then*

$$\alpha_G(\theta, \phi, \phi) G(A\theta, A\phi, A\phi) \leq \frac{G(\theta, \phi, \phi)}{b},$$

for all $\theta, \phi \in \Omega$ and equality is true, when $\theta = \phi$.

Now, we introduce the following generalization of Meir-Keeler mappings.

Definition 3.3. *An α_G-admissible mapping A in (Ω, G) is known as generalized α_G-Meir-Keeler contraction mapping of type AI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \leq \Lambda(\theta, \phi, \psi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta,\phi,\psi)G(A\theta,A\phi,A\psi) < \frac{\varepsilon}{b} \tag{3}$$

where

$$\Lambda(\theta,\phi,\psi) = \max\{G(\theta,\phi,\psi), G(\theta,A\theta,A\theta), G(\phi,A\phi,A\phi), G(\psi,A\psi,A\psi)\}$$

for all $\theta,\phi,\psi \in \Omega$.

Definition 3.4. *An α_G-admissible mapping A in (Ω,G) is known as generalized α_G-Meir-Keeler contraction mapping of type AII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \le \Lambda(\theta,\phi,\phi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta,\phi,\phi)G(A\theta,A\phi,A\phi) < \frac{\varepsilon}{b} \tag{4}$$

where

$$\Lambda(\theta,\phi,\phi) = \max\{G(\theta,\phi,\phi), G(\theta,A\theta,A\theta), G(\phi,A\phi,A\phi)\}$$

for all $\theta,\phi \in \Omega$.

Definition 3.5. *An α_G-admissible mapping A in (Ω,G) is known as generalized α_G-Meir-Keeler contraction mapping of type BI, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \le \Lambda(\theta,\phi,\psi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta,\phi,\psi)G(A\theta,A\phi,A\psi) < \frac{\varepsilon}{b} \tag{5}$$

where

$$\begin{aligned}\Lambda(\theta,\phi,\psi) &= \max\Big\{G(\theta,\phi,\psi), G(\theta,A\theta,A\theta), G(\phi,A\phi,A\phi),\\ &\qquad G(\psi,A\psi,A\psi), \frac{1}{4}(G(\theta,A\theta,A\phi) + G(\phi,A\phi,A\psi) + G(\psi,A\psi,A\theta))\Big\}\end{aligned}$$

for all $\theta,\phi,\psi \in \Omega$.

Definition 3.6. *An α_G-admissible mapping A in (Ω, G) is known as generalized α_G-Meir-Keeler contraction mapping of type BII, if there exists $\delta > 0$ for all $\varepsilon > 0$ such that*

$$\varepsilon \leq \Lambda(\theta, \phi, \phi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta, \phi, \phi) G(A\theta, A\phi, A\phi) < \frac{\varepsilon}{b}, \tag{6}$$

where

$$\begin{aligned}\Lambda(\theta, \phi, \phi) &= \max\Big\{G(\theta, \phi, \phi), G(\theta, A\theta, A\theta), G(\phi, A\phi, A\phi),\\ &\quad \frac{1}{4}(G(\theta, A\theta, A\theta) + G(\theta, A\phi, A\phi) + G(\phi, A\theta, A\theta))\Big\}\end{aligned}$$

for all $\theta, \phi \in \Omega$.

Remark 2. **(i):** *Let $A : \Omega \to \Omega$ be a generalized α_s-Meir-Keeler contraction of type AI or BI. Then*

$$\alpha_s(\theta, \phi, \psi) G(A\theta, A\phi, A\psi) \leq \frac{\Lambda(\theta, \phi, \psi)}{b}$$

for all $\theta, \phi, \psi \in \Omega$, where the equality holds only when $\theta = \phi = \psi$.

(ii): *Let $A : \Omega \to \Omega$ be a generalized α_G-Meir-Keeler contraction of type AII or BII. Then*

$$\alpha_G(\theta, \phi, \phi) G(A\theta, A\phi, A\phi) \leq \frac{\Lambda G(\theta, \phi, \phi)}{b},$$

for all $\theta, \phi \in \Omega$, where the equality holds only when $\theta = \phi$.

Lemma 3.1. *Let (Ω, G) be a G_b-metric space and $\{\theta_n\}$ be a sequence satisfying:*

(i): *$\theta_m \neq \theta_n$ for all $m \neq n$, $m, n \in \mathbb{N}$,*

(ii): *$G(\theta_n, \theta_{n+1}, \theta_{n+1}) \leq \frac{1}{b} G(\theta_{n-1}, \theta_n, \theta_n)$, for all $n \in \mathbb{N}$.*

Then, $\{\theta_n\}$ is a Cauchy sequence in (Ω, G).

Proof. In order to show that sequence $\{\theta_n\}$ is Cauchy, we must prove that $\lim_{n\to\infty} G(\theta_n, \theta_{n+k}, \theta_{n+k}) = 0$ for any $k \in \mathbb{N}$.

From (ii) we have

$$G(\theta_n, \theta_{n+1}, \theta_{n+1}) \leq \frac{1}{b^n} G(\theta_0, \theta_1, \theta_1), \text{ for all } n \in \mathbb{N}. \tag{7}$$

Applying limit as $n \to +\infty$ we get

$$\begin{aligned} 0 \leq \lim_{n\to\infty} G(\theta_n, \theta_{n+1}, \theta_{n+1}) &\leq \frac{1}{b^n} G(\theta_0, \theta_1, \theta_1) \\ \therefore \lim_{n\to\infty} G(\theta_n, \theta_{n+1}, \theta_{n+1}) &= 0. \end{aligned}$$

Now,

$$\begin{aligned} G(\theta_n, \theta_{n+k}, \theta_{n+k}) &\leq 2bG(\theta_n, \theta_{n+1}, \theta_{n+1}) + b^2 G(\theta_{n+1}, \theta_{n+k}, \theta_{n+k}) \\ &\leq 2bG(\theta_n, \theta_{n+1}, \theta_{n+1}) + 2b^3 G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2}) + b^4 G(\theta_{n+2}, \theta_{n+k}, \theta_{n+k}) \\ &\leq 2\Big\{ bG(\theta_n, \theta_{n+1}, \theta_{n+1}) + b^3 G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2}) + \dots \\ &\quad \dots + b^{2(k-1)+1} S(\theta_{n+k-1}, \theta_{n+k}, \theta_{n+k}) \Big\} \\ &\leq 2\Big\{ b\frac{G(\theta_0, \theta_1, \theta_1)}{b^n} + b^3 \frac{G(\theta_0, \theta_1, \theta_1)}{b^{n+1}} + \dots + b^{2(k-1)+1} \frac{G(\theta_0, \theta_1, \theta_1)}{b^{n+k-1}} \Big\} \\ &= \frac{2}{b^{n-1}} \{1 + b + \dots + b^k\} G(\theta_0, \theta_1, \theta_1) \\ &= \frac{2(b^k - 1)}{b^{n-1}(b-1)} G(\theta_0, \theta_1, \theta_1) \end{aligned}$$

$$\begin{aligned} \therefore \lim_{n\to\infty} G(\theta_n, \theta_{n+k}, \theta_{n+k}) &\leq \lim_{n\to\infty} \frac{2(b^k - 1)}{b^{n-1}(b-1)} G(\theta_0, \theta_1, \theta_1) \\ &= 0 \end{aligned}$$

Thus $\{\theta_n\}$ is a Cauchy sequence in G_b-metric space (Ω, G). □

Theorem 3.2. *Let (Ω, G) be a complete G_b-metric space and $\alpha_s : \Omega \times \Omega \times \Omega \to [0, +\infty)$ be a mapping. Let $A : \Omega \to \Omega$ satisfying:*

(i): *A is a generalized α_G-Meir-Keeler contraction mapping of type AI;*

(ii): *A is an α_G-admissible;*

(iii): *there is $\theta_0 \in \Omega$ so that $\alpha_G(\theta_0, \theta_0, A\theta_0) \geq 1$;*

(iv): *A is continuous.*

Then, there exists a fixed point of A in Ω.

Proof. Suppose $\theta_0 \in \Omega$ and $\alpha_s(\theta_0, \theta_0, A\theta_0) \geq 1$. Define the sequence $\{\theta_n\}$ in Ω as

$$\theta_{n+1} = A\theta_n, \text{ for all } n \in \mathbb{N}.$$

Suppose $\theta_{n_0} = \theta_{n_0+1}$ for some $n_0 \in \mathbb{N}$ that is $G(\theta_{n_0}, \theta_{n_0+1}, \theta_{n_0+1}) = 0$ implies that θ_{n_0} is a fixed point of A. Thus assume that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. From (ii), we have

$$\alpha_s(\theta_0, A\theta_0, A\theta_0) = \alpha_s(\theta_0, \theta_1, \theta_1) \geq 1$$

implies that

$$\alpha_s(A\theta_0, A\theta_1, A\theta_1) = \alpha_s(\theta_1, \theta_2, \theta_2) \geq 1 \tag{8}$$

continuing on the same lines, we have

$$\alpha_s(\theta_n, \theta_{n+1}, \theta_{n+1}) \geq 1, \ \forall\, n \in \mathbb{N}. \tag{9}$$

Here, we need to show that sequence $\{\theta_n\}$ satisfies the conditions of Lemma 3.1. If we put $\theta = \phi = \theta_n$ and $\psi = \theta_{n+1}$ in (3), for all $\varepsilon > 0$ there is $\delta > 0$ satisfying

$$\varepsilon \leq \Lambda(\theta_n, \theta_{n+1}, \theta_{n+1}) < \varepsilon + \delta$$

implies

$$\alpha_s(\theta_n, \theta_{n+1}, \theta_{n+1})\ G(A\theta_n, A\theta_{n+1}, A\theta_{n+1}) < \frac{\varepsilon}{b}, \tag{10}$$

where

$$\Lambda(\theta_n, \theta_{n+1}, \theta_{n+1}) = \max\{G(\theta_n, \theta_{n+1}, \theta_{n+1}), G(\theta_n, A\theta_n, A\theta_n), G(\theta_{n+1}, A\theta_{n+1}, A\theta_{n+1})\}.$$

From the Remark 2(ii), we have

$$\begin{aligned} G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2}) &= G(A\theta_n, A\theta_{n+1}, A\theta_{n+1}) \\ &\leq \alpha_s(\theta_n, \theta_{n+1}, \theta_{n+1})\ G(A\theta_n, A\theta_{n+1}, A\theta_{n+1}) \\ &\leq \frac{\Lambda(\theta_n, \theta_{n+1}, \theta_{n+1})}{b} \end{aligned}$$

due to the fact that $\theta_n \neq \theta_{n+1}$ we see that equality does not hold, hence

$$G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2}) < \frac{\Lambda(\theta_n, \theta_{n+1}, \theta_{n+1})}{b}. \tag{11}$$

If $\Lambda(\theta_n, \theta_{n+1}, \theta_{n+1}) = G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2})$ for some $n \in \mathbb{N}$, then (11) implies

$$G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2}) < \frac{G(\theta_{n+1}, \theta_{n+2}, \theta_{n+2})}{b}$$

which is not possible. Then $\Lambda(\theta_n,\theta_{n+1},\theta_{n+1}) = G(\theta_n,\theta_{n+1},\theta_{n+1})$ for all $n \in \mathbb{N}$, so that (11) yields

$$G(\theta_{n+1},\theta_{n+2},\theta_{n+2}) < \frac{G(\theta_n,\theta_{n+1},\theta_{n+1})}{b}, \tag{12}$$

which shows that Lemma 3.1(ii) is true.

Next, we consider the case for $\theta_n \neq \theta_m$ for all $n \neq m$.

If possible, let $\theta_n = \theta_m$ for some $m,n \in \mathbb{N}$. We have $G(\theta_n,\theta_{n+1},\theta_{n+1}) \geq 0$ for some $n \in \mathbb{N}$. In general let $m > n+1$.

We have $G(\theta_m,\theta_{m+1},\theta_{m+1}) = G(\theta_n,\theta_{n+1},\theta_{n+1})$, by the inequality (12) we have

$$\begin{aligned} G(\theta_n,\theta_{n+1},\theta_{n+1}) &= G(\theta_m,\theta_{m+1},\theta_{m+1}) \\ &< \frac{G(\theta_{m-1},\theta_m,\theta_m)}{b} \\ &< \frac{G(\theta_{m-2},\theta_{m-1},\theta_{m-1})}{b^2} \\ &\vdots \\ &< \frac{G(\theta_n,\theta_{n+1},\theta_{n+1})}{b^{m-n}} \end{aligned} \tag{13}$$

becomes impossible. Thus for some $m \neq n$, $\lambda_n = \lambda_m$ is not true and hence it must be $\theta_n \neq \theta_m$ for all $n \neq m$. So, due to Lemma 3.1, $\{\theta_n\}$ is a Cauchy sequence in (Ω, G). Thus, $\{\theta_n\}$ converges to an $u \in \Omega$ i.e.

$$\lim_{n\to\infty} G(\theta_n,u,u) = 0. \tag{14}$$

By the continuity of A, we have

$$\lim_{n\to\infty} G(A\theta_n,Au,Au) = \lim_{n\to\infty} G(\theta_{n+1},Au,Au) = 0,$$

so $\{\theta_n\}$ converges to Au. Since the limit is unique, hence $Au = u$. □

Theorem 3.3. *Let (Ω,G) be a complete G_b-metric space and $\alpha_G : \Omega\times\Omega\times\Omega \to [0,+\infty)$ be a mapping. Let $A : \Omega \to \Omega$ be a mapping such that*

(v): *for a pair of fixed points (θ,ϕ) of A, $\alpha_G(\theta,\phi,\phi) \geq 1$,*

together with the four conditions of Theorem 3.2, then A has a unique fixed point in Ω.

Proof. The existence of fixed point is proved in Theorem 3.2. Now, for uniqueness, consider θ and ϕ be two different fixed points of A in Ω.

By (3) we have

$$\varepsilon \leq \Lambda(\theta,\phi,\phi) < \varepsilon+\delta$$

implies

$$\alpha_G(\theta,\phi,\phi)G(A\theta,A\phi,A\phi) < \frac{\varepsilon}{b} \tag{15}$$

where

$$\begin{aligned}\Lambda(\theta,\phi,\phi) &= \max\{G(\theta,\phi,\phi),G(\theta,A\theta,A\theta),G(\phi,A\phi,A\phi)\}\\ &= \max\{G(\theta,\phi,\phi),0,0\}\\ &= G(\theta,\phi,\phi). \qquad (16)\end{aligned}$$

By (v), $\alpha_G(\theta,\phi,\phi) \geq 1$, since $G(\theta,\theta,\phi) > 0$, the Remark 2(ii) becomes

$$\begin{aligned}G(\theta,\phi,\phi) &= G(A\theta,A\phi,A\phi)\\ &\leq \alpha_G(\theta,\phi,\phi)G(A\theta,A\phi,A\phi)\\ &< \frac{\Lambda(\theta,\phi,\phi)}{b}\\ &= \frac{G(\theta,\phi,\phi)}{b} \qquad (17)\end{aligned}$$

which is a contradiction, hence $G(\theta,\phi,\phi) = 0$ i.e. $\theta = \phi$. Thus fixed point of A is unique. □

Definition 3.7. *In G_b-metric space (Ω,G), $\alpha_G : \Omega\times\Omega\times\Omega \to [0,+\infty)$ be a mapping. Then G_b-metric space (Ω,G) is known as an α-regular if for any sequence $\{\theta_n\}$, $\lim_{n\to\infty} G(\theta_n,\theta,\theta) = 0$ and $\alpha_G(\theta_n,\theta_{n+1},\theta_{n+1}) \geq 1$ for all $n\in\mathbb{N}$, we have $\alpha_G(\theta_n,\theta,\theta) \geq 1$ for all $n\in\mathbb{N}$.*

Theorem 3.4. *In a complete G_b-metric space (Ω,G), $b \geq 1$is a parameter and $\alpha_G : \Omega\times\Omega\times\Omega \to [0,+\infty)$ be an α_G-admissible mapping. Let $A : \Omega\to\Omega$ be a generalized α_G-Meir-Keeler contraction of type AI satisfying:*

- **(i):** *There is $\theta_0 \in \Omega$ so that $\alpha_G(\theta_0,A\theta_0,A\theta_0) \geq 1$;*
- **(ii):** *The G_b-metric space (Ω,G) is an α-regular, then there exists a fixed point of A in Ω;*
- **(iii):** *For all pairs of fixed points, $\theta,\phi \in \Omega$, $\alpha_G G(\theta,\phi,\phi) \geq 1$;*

Then A has unique fixed point.

Proof. Suppose $\theta_0 \in \Omega$ be such that $\alpha_G(\theta_0, A\theta_0, A\theta_0) \geq 1$. Define a sequence $\{\theta_n\} \in \Omega$ such that $\theta_{n+1} = A\theta_n$ for all $n \in \mathbb{N}$ and converges to $u \in \Omega$ uniquely.

As (Ω, G) is α_G-regular, $\alpha_G(\theta_n, u, u) \geq 1$.

By (3), we have

$$\varepsilon \leq \Lambda(\theta_n, u, u) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta_n, u, u) G(A\theta_n, Au, Au) < \frac{\varepsilon}{b} \tag{18}$$

where

$$\Lambda(\theta_n, u, u) = \max\{G(\theta_n, u, u), G(\theta_n, A\theta_n, A\theta_n), G(u, Au, Au)\}. \tag{19}$$

On the other hand, from the Remark 2(ii), we have

$$\begin{aligned} G(\theta_{n+1}, Au, Au) &= G(A\theta_n, Au, Au) \\ &\leq \alpha_G(\theta_n, u, u) G(A\theta_n, Au, Au) \\ &< \frac{\Lambda(\theta_n, u, u)}{b} \end{aligned} \tag{20}$$

We have

$$\lim_{n\to\infty} G(\theta_{n+1}, Au, Au) = G(u, Au, Au).$$

Also

$$\begin{aligned} \lim_{n\to\infty} \Lambda(\theta_n, u, u) &= \lim_{n\to\infty} \max\{G(\theta_n, u, u), G(\theta_n, A\theta_n, A\theta_n), G(u, Au, Au)\} \\ &= G(u, Au, Au) \end{aligned}$$

Taking limit as $n \to +\infty$ in (20), we have

$$G(u, Au, Au) \leq \frac{G(u, Au, Au)}{b}$$

which conclude that $G(u, Au, Au) = 0$.

Uniqueness part is identical to Theorem 3.3. □

Note: Theorem 3.2, 3.3, 3.4 will be true for generalized α_G-Meir-Keeler contraction mapping of type BI and BII.

4. Consequences:

Here, we consider some consequences of Theorem 3.2, 3.3 and 3.4.

Corollary 4.1. *Let (Ω, G) be complete G_b-metric space and $A : \Omega \to \Omega$ be an α_G-admissible mapping satisfying:*

(i): *for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq N(\theta, \phi, \psi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta, \phi, \psi) G(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{21}$$

where

$$N(\theta, \phi, \psi) = \max\left\{G(\theta, \phi, \psi), \frac{1}{3}[G(\theta, A\theta, A\theta) + G(\phi, A\phi, A\phi) + G(\psi, A\psi, A\psi)]\right\} \tag{22}$$

for all $\theta, \phi, \psi \in \Omega$.

(ii): *There exists $\theta_0 \in \Omega$ such that $\alpha_G(\theta_0, A\theta_0, A\theta_0) \geq 1$.*

(iii): *A is continuous or G_b-metric space (Ω, G) is α_s-regular.*

Then, A has a fixed point in Ω.

Also

(iv): *For every pair of fixed points (θ, ϕ) of A, if $\alpha_G(\theta, \phi, \phi) \geq 1$;*

Then, fixed point of A is unique in Ω.

Proof. As $N(\theta, \phi, \psi) \leq \Lambda(\theta, \phi, \psi)$ for all $\theta, \phi, \psi \in \Omega$, proof is obvious from Theorems 3.2, 3.3 and 3.4. □

Corollary 4.2. *Let (Ω, G) be complete G_b-metric space and $A : \Omega \to \Omega$ be an α_G-Meir-Keeler contraction of type I, that is, there exists $\delta > 0$ for every $\varepsilon > 0$ souch that*

$$\varepsilon \leq G(\theta, \phi, \psi) < \varepsilon + \delta$$

implies

$$\alpha_G(\theta,\phi,\psi)G(A\theta,A\phi,A\psi) < \frac{\varepsilon}{b} \tag{23}$$

for all $\theta,\phi,\psi \in \Omega$.

If A is continuous or G_b-metric space (Ω,G) is α-regular, then A has a fixed point. Further, with condition (v) in Theorem 3.3 the fixed point of A is unique.

Proof. The proof follows easily from the relation $G(\theta,\phi,\psi) \le \Lambda(\theta,\phi,\psi)$ for all $\theta,\phi,\psi \in \Omega$. □

Taking $\alpha(\theta,\phi,\psi) = 1$ in Theorem 3.4, we get

Corollary 4.3. *Let (Ω,G) be a complete G_b-metric space and $A : \Omega \to \Omega$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le \Lambda(\theta,\phi,\psi) < \varepsilon + \delta$$

implies

$$G(A\theta,A\phi,A\psi) < \frac{\varepsilon}{b} \tag{24}$$

where

$$\Lambda(\theta,\phi,\psi) = \max\Big\{G(\theta,\phi,\psi), G(\theta,A\theta,A\theta), G(\phi,A\phi,A\phi), G(\psi,A\psi,A\psi)\Big\} \tag{25}$$

for all $\theta,\phi,\psi \in \Omega$. Then, fixed point of A is unique.

Corollary 4.4. *Let (Ω,G) be a complete G_b-metric space and $A : \Omega \to \Omega$ be a continuous mapping. If there exists $\delta > 0$ for every $\varepsilon > 0$ such that*

$$\varepsilon \le N(\theta,\phi,\psi) < \varepsilon + \delta$$

implies

$$G(A\theta,A\phi,A\psi) < \frac{\varepsilon}{b} \tag{26}$$

where

$$N(\theta,\phi,\psi) = \max\Big\{G(\theta,\phi,\psi), \frac{1}{3}\Big[G(\theta,A\theta,A\theta) + G(\phi,A\phi,A\phi) + G(\psi,A\psi,A\psi)\Big]\Big\} \tag{27}$$

for all $\theta, \phi, \psi \in \Omega$. *Then, A has a unique fixed point.*

The Meir-Keeler contraction can be stated on G_b-metric spaces as follows.

Corollary 4.5. *Let* (Ω, G) *be a complete* G_b*-metric space and* $A : \Omega \to \Omega$ *be a continuous Meir-Keeler mapping. If there exists* $\delta > 0$ *for every* $\varepsilon > 0$ *such that*

$$\varepsilon \leq G(\theta, \phi, \psi) < \varepsilon + \delta$$

becomes

$$G(A\theta, A\phi, A\psi) < \frac{\varepsilon}{b} \tag{28}$$

for all $\theta, \phi, \psi \in \Omega$. *Then A has a unique fixed point.*

5. Conclusion

In this article, we define Meir-Keeler contraction in G_b-metric spaces using the concept of α-admissible mapping. Further, we define generalized α_G-Meir-Keeler contraction. Using these definitions of contractive mappings we prove theorems for the existence and uniqueness of fixed points. We show that obtained results are potential generalizations of various results in the literature.

Acknowledgements

Author is thankful to the editor and anonymous referees for their valuable comments and suggestions.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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