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# COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS SATISFYING PROPERTY (E. A) USING AN INEQUALITY INVOLVING QUADRATIC TERMS IN MANGER SPACE FOR SIX MAPS

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**Abstract.** The purpose of the paper is to establish the existence of two fixed point theorems for six maps under the weaker concept of compatibility called occasionally weakly compatible in Menger space in which the contraction condition contains involving quadratic terms also. The obtained results are improved versions of some of the results obtain in the literature of Fixed Point Theory and Application by employing the property (E. A).

**Keywords:** fixed point; complete metric space; probabilistic metric space; menger spa compatible maps; compatible maps of type (A); weakly compatible; occasionally weakly compatible.

**2010 AMS Subject Classification:** 47H09, 47H10.

## 1. Introduction

Probabilistic metric space was first introduced by Menger [10]. Later, there are many authors who have some detailed discussions and applications of a probabilistic metric space, for example, we may see Schweizer and Sklar [1]. Fixed point theory has extensive applications in other region also. A generalization of Banach contraction principle on a complete Menger space which is a milestone in developing fixed point theorems in Menger space was obtained

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by Sehgal and Bharucha [20]. In 1982, Sessa [18] introduced weakly commuting mappings in metric space. Jungck [4] enlarged this concept to compatible mappings. The notion of compatible mappings in Menger space has been introduced by Mishra [17]. Jungck and Rhoades [6] introduced the notion of weak compatibility and showed that compatible mappings are weak compatible but the reverse is always not true.

In this paper, we consider the concept of occasionally compatible maps in Menger spaces to prove a common fixed point theorem for six mappings. Meanwhile, these results are improved by weak the completeness of the space to the property (E.A).

## 2. PRELIMINARIES

We recall some known definitions and results in Menger spaces.

**Definition 2.1.** [17] *A mapping $F:\mathbb{R}\to[0,1]$ is called a distribution function if it satisfies the conditions (i) F is nondecreasing (ii) F is left continuous, with* $\inf\{F(t):t\in\mathbb{R}\}=0$ *and* $\sup\{F(t):t\in\mathbb{R}\}=1$. *The Heaviside function H is a distribution function defined by*

$$H(x)=\begin{cases}0 & \text{if } x\le 0,\\ 1 & \text{if } x>0.\end{cases}$$

**Definition 2.2.** [17] *Let X be a non-empty set and let L denote the set of all distribution function defined on X, i.e., $L=\{F_{x,y}:x,y\in X\}$. An ordered pair $(X,F)$ is called a probabilistic metric spaces, where F is a mapping from $X\times X$ into L, if for every pair $(x,y)\in X$, a distribution function $F_{x,y}$ is assumed to satisfy the following four conditions:*

**(1):** $F_{x,y}(u)=1$, *forall* $u>0$,

**(2):** $F_{x,y}(u)=F_{y,x}(u)$,

**(3):** $F_{x,y}(u)=0$.

**(4):** *If* $F_{x,y}(u_1)=1$ *and* $F_{x,y}(u_2)=1$, *then* $F_{x,y}(u_1+u_2)=1$ *for all* $x,y,z\in X$ *and* $u_1,u_2\ge 0$.

**Definition 2.3.** [2] *A t-norm is function $t:[0,1]\times[0,1]\to[0,1]$ satisfying the following conditions:*

**(T1):** $t(a,1)=a,\ t(0,0)=0$,

**(T2):** $t(a,b)=t(b,a)$,

**(T3):** $t(c,d) \geq t(a,b)$ *for* $c \geq a,\ d \geq b$ $F_{x,y}(u) = 0$.

**(T4):** $t(t(a,b),c) = t(a,t(b,c))$ *for all* $a,b,c$ *in* $[0,1]$.

**Definition 2.4.** [2] *A Menger probabilistic metric space* $(X,F,t)$ *in ordered triple, where* $t$ *is a* $t$*-norm, and* $(X,F)$ *is a probabilistic metric space satisfying the condition* $F_{x,z}(u_1+u_2) \geq t(F_{x,z}(u_1), F_{x,z}(u_2))$ *forall* $x,y,z \in X$ *and* $u_1,u_2 \geq 0$.

Recently, Aamri and Moutawakil [13] defined the property (E. A) and proved some common fixed point theorems in metric spaces. Kubiaczyk and Sharma [12] defined the property (E. A) in PM-spaces and used it to prove results on common fixed points.

**Definition 2.5.** [12] *A pair of self-mappings*$(A,S)$ *of a Manger probabilistic metric space* $(X,F,\Delta)$ *is said to satisfy property (E. A.), if there exists a sequence* $\{x_n\}$ *in* $X$ *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

*for some* $t \in X$.

**Proposition 2.6.** [2] *In a Menger space* $(X,F,t)$, *if* $t(x,x) \geq x$, *forall* $x \in [0,1]$, *then* $t(a,b) = min\{a,b\}$, *for all* $a,b \in [0,1]$.

**Proposition 2.7.** [2] *A sequence* $\{x_n\}$ *in probabilistic metric space* $(X,F,t)$

**(a):** *is said to be converge to a point* $x \in X$ *if for every* $\varepsilon > 0$ *and* $\lambda > 0$, *there exists a positive integer* $N(\varepsilon,\lambda)$ *such that* $F_{x_n,x}(\varepsilon) > 1-\lambda$ *for all* $n,m \geq N(\varepsilon,\lambda)$.

**(b):** *is said to be a Cauchy sequence, if for every* $\varepsilon > 0$ *and* $\lambda > 0$,*there exists a positive integer* $N(\varepsilon,\lambda)$ *such that* $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ *for all* $n,m \geq N(\varepsilon,\lambda)$.

**Definition 2.8.** *Two self mappings* $A$ *and* $B$ *of a Menger space* $(X,F,t)$, *then the pair* $(A,B)$

**(i):** *is said to be* compatible[17] *if* $F_{ABx_n,BAx_n}(\varepsilon) \to 1$ *for all* $\varepsilon > 0$, *whenever* $\{x_n\}$ *is a sequence in* $X$ *such that* $ABx_n, BAx_n \to u$ *as* $n \to \infty$, *for some* $u \in X$ .

**(ii):** *is said to be* weakly compatible, *if* $Au = Bu$ *for some* $u \in X$, *then* $ABu = BAu$.

**(iii):** *is said to be* occasionally weakly compatible *(owc), if there is a point* $u \in X$ *such that* $Au = Bu$ *implies that* $ABu = BAu$.

**Remark 2.9.** *The implication of above definition is as follows compatible $\Rightarrow$ weakly compatible $\Rightarrow$ occasionally weakly compatible, its converses is not true.*

**Definition 2.10.** [21] *Let $(X,F,t)$ be a Menger space such that T-norm t is continuous and A, S be mapping X into itself compatible maps of type(A) if*

$$\lim_{n\to\infty} F_{SA_{x_n},AA_{x_n}}(t) = 1, \text{ and } \lim_{n\to\infty} F_{AS_{x_n},SS_{x_n}}(t) = 1,$$

*for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in X.$$

**Proposition 2.11.** [21] *Let $(X,F,T)$ be a Menger space such that T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$,and $A,S : X \to X$ be mappings. Let the pair $(A,S)$ be compatible maps of type (A) and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Then we have*

**(1):** $\lim_{n\to\infty} SAx_n = Az$ *if S is continuous at z.*

**(2):** $ASz = SAz$ *and* $Az = Sz$ *if A and S are continuous at z.*

Now, we give the some lemmas relevant to the proof of the out results in the next section.

**Lemma 2.12.** [17] *Let $x_n$ be a sequence in a Menger space $(X,F,t)$ with continuous t-norm and $t(x,x) > x$. Suppose, for all $k \in [0,1], \exists$ such that for all $\in > 0$ and $n \in N$,*

$$F_{x_n,x_{n+1}}(k\varepsilon)) \geq F_{x_{n-1},x_n}(\varepsilon),$$

*then $\{y_n\}$ is a cauchy sequence in $X$.*

**Lemma 2.13.** [17] *Let $(X,F,t)$ be a Menger space, if there exists $k \in (0,1)$ such that for $x,y \in X$ and $\in > 0$, $F_{x,y}(k\varepsilon) \geq F_{x,y}(\varepsilon)$. Then $x = y$.*

**Lemma 2.14.** [9] *Let $(X,F,\Delta)$ be a Menger space. If there exists some $K \in (0,1)$ which such the for any $p,q \in X$ and all $x > 0$,*

$$\int_0^{F_{p,q}(kx)} \phi(t)dt \geq \int_0^{F_(p,q)} \phi(t)dt$$

*where $\phi : [0,\infty) \to [0,\infty)$ is a summable nonnegative Lebesgue integrable function such that $\int_\varepsilon^1 \phi(t)dt > 0$ for each $\varepsilon \in [0,1)$,where $0 < k < 1$,then $p = q$.*

S.N. Mishra [17] proved the following theorem.

**Theorem 2.15.** [17] *Let $A,B,S,T,L$ and $M$ are self mappings of a complete Menger space $(X,F,t)$ with continuous t-norm that $t(x,x) \geq x$ for all $x \in [0,1]$ satisfying condition:*

**(i):** $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

**(ii):** $\forall\ p,q \in X, x > 0$, $\alpha \in (0,2)$ *and for some* $k \in (0,1)$ *such that,*

$$F_{Ap,Bq}(kx) \geq t(F_{Ap,\ Sp}(x), t(F_{Bq,Tq}(x), t(F_{Bq,Tq}(x), t(F_{Ap,Tq}(\alpha x), F_{Bq,Sp}((2-\alpha x))))))$$

**(iii):** *the pairs* $(A,S)$, $(B,T)$ *are compatible.*

**(iv):** *and $S$ and $T$ are continuous.*

*Then $A,B,S$ and $T$ have a unique common fixed point in $X$.*

In 2005, Singh and Jain [2] generalized the Theorem 2.12 to six mappings and generalize it in other respect. theorem.

**Theorem 2.16.** [2] *Let $A,B,S,T,L$ and $M$ are self mappings of a complete Menger space $(X,F,\Delta)$ with continuous t-norm that $t(x,x) \geq x$ for all $x \in [0,1]$ satisfying condition:*

**(i):** $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$,

**(ii):** $AB = BA$, $ST = TS$, $LB = BL$ *and* $MT = TM$,

**(iii):** *either $AB$ or $L$ is continuous,*

**(iv):** $(L,AB)$ *is compatible and* $(M,ST)$ *is weakly compatible.*

**(v):** *there exists* $k \in (0,1)$ *such that*

$$F_{Lp,Mq}(kX) \geq min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Mq}(\beta x), F_{ABp,Mp}(2-\beta)(x), F_{ABp,STq}(x)\}$$

*for all* $p,q \in X$, $\beta \in (0,2)$ *and* $x > 0$.

*Then $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.*

In 2013, Liu [22] extend and generalized the above result in some aspects.

**Theorem 2.17.** [22] *Let $A,B,S,T,L$ and $M$ are self mappings of a complete Menger space $(X,F,t)$ with continuous t-norm that $t(x,x) \geq x$ for all $x \in [0,1]$ satisfying condition:*

**(i):** $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$,

**(ii):** *$AB = BA$, $ST = TS$, $LB = BL$ and $MT = TM$,*

**(iii):** *the pairs $(L, AB)$, $(M, ST)$ are both weakly compatible,*

**(iv):** *there exists an upper semicontinuous function $\phi : [0,\infty] \to [0,\infty]$ with $\phi(0) = 0, \phi(x) < x$ for all $x > 0$ such that*

$$F_{Lp,Mq}(\phi(x)) \geq min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mp}(1+\beta)(x), F_{ABp,STq}(x)\}$$

*for all $p, q \in X$,, $\beta \geq 1$ and $x > 0$.*

**(v):** *$AB(X)$ and $ST(X)$ is closed in $X$.*

*Then $A, B, S, T, L$ and $M$ have a unique common fixed point in $X$.*

In 2011, Rashwan and Maustafa [15] has been proved a common fixed point theorem for four weakly compatible maps on a complete Menger space for four maps.

**Theorem 2.18.** [15] *Let $A, B, S$, and $T$ are self mappings of a complete Menger space $(X, F, t)$ where $t(x, y) = min(x, y)$ for all $x \in [0, 1]$ satisfying the following condition:*

**(i):** *$A(X)$,$B(X)$ are closed sets of $X$ and $A(X) \subset T(X)$, $B(X) \subset S(X)$,*

**(ii):** *the pairs $(A, S)$, $(B, T)$ are both weakly compatible,,*

**(iii):** *for all $x, y \in X$, $x > 0$ and $t > 0$, where $k \in (0, 1)$*

$$\begin{aligned}(F_{Ax,By}(ku))^2 \quad \geq \quad min\Big\{ & [F_{Sx,Ty}(t)]^2, F_{Sx,Ax}(t)F_{Ty,By}(u), F_{Sx,Ty}(u)F_{Sx,Ax}(u), \\ & F_{Sx,Ty}(u)F_{Ty,By}(u), F_{Sx,Ty}(u)F_{Sx,By}(2u), \\ & F_{Sx,Ty}(t)F_{Ty,Ax}(u), F_{Sx,By}(2u)F_{Ty,Ax}(u) \\ & F_{Sx,Ax}(u)F_{Ty,Ax}(u), F_{Sx,By}(2u)F_{Ty,By}(u)\Big\}\end{aligned}$$

*for all $x, y \in X$,, $x > 0$ and $t > 0$,where $k \in (0, 1)$.*

*Then $A, B, S$, and $T$ have a unique common fixed point in $X$.*

In this paper, we consider the concept of occasionally weakly compatible mappings (owc) in Menger spaces to prove a two commom fixed point theorems for six mappings without continuity of these mappings. These results are improved by weakening the completness of the space by using the property (E.A).These results partially extend the results of Singh and Jain [2].

## 3. Main Result

**Theorem 3.1.** *Let $A,B,S,T,L$ and $M$ are self mappings of a complete Menger space $(X,F,\Delta)$ with continuous t-norm that $t(x,x) \geq x$ for all $x \in [0,1]$ satisfying condition:*

**(i):** $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$, (3.1)

**(ii):** $AB = BA$, $ST = TS$, $LB = BL$ *and* $MT = TM$, (3.2)

**(iii):** *the pairs* $(L,AB)$, $(M,ST)$ *are owc,* (3.3)

**(iv):** *any one of the subspace either* $L(X)$ *or* $ST(X)$ *or* $M(X)$ *or* $AB(X)$ *is closed in* $X$, (3.4)

**(v):** *for all* $p,\ q \in X, t > 0$ *and for some* $k \in (0,1)$ *such that*

$$\begin{aligned}(F_{Lp,Mq}(kt))^2 \quad \geq \quad min\Big\{ & (F_{ABp,STq}(t))^2, F_{ABp,Lp}(t) \cdot F_{STq,Lp}(t), \\ & F_{ABp,Lp}(t) \cdot F_{STq,Mq}(t), \\ & F_{ABp,Mq}(t) \cdot F_{STq,Mq}(t), \\ & F_{ABp,Mq}(t) \cdot F_{STq,Lp}(t)\Big\}. \end{aligned} \tag{3.5}$$

Then $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.

*Proof.* Since $L(X) \subseteq ST(X)$, for any arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Lx_0 = STx_1 = y_0$, for this point $x_1 \in X$, then we can choose a point $x_2 \in X$ such that $Mx_1 = ABx_2 = y_1$. Inductively, we can define a sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$y_{2n} = L_{x_{2n}} = ST_{x_{2n+1}} \text{ and } y_{2n+1} = M_{x_{2n+1}} = AB_{x_{2n+2}},\ n = 0,1,2\dots\ . \tag{3.6}$$

Now, taking $p = x_{2n}$ and $q = x_{2n+1}$ in (3.5), we get

$$\begin{aligned}(F_{Lx_{2n},Mx_{2n+1}}(kt))^2 \quad \geq \quad min\Big\{ & (F_{ABx_{2n},STx_{2n+1}}(t))^2, F_{ABx_{2n},Lx_{2n}}(t) \cdot F_{STx_{2n+1},Lx_{2n}}(t), \\ & F_{ABx_{2n},Lx_{2n}}(t) \cdot F_{STx_{2n+1},Mx_{2n+1}}(t), \\ & F_{ABx_{2n},Mx_{2n+1}}(t) \cdot F_{STx_{2n+1},Mx_{2n+1}}(t), \\ & F_{ABx_{2n},Mx_{2n+1}}(t) \cdot F_{STx_{2n+1},Lx_{2n}}(t)\Big\}, \end{aligned}$$

$$
\begin{aligned}
(F_{y_{2n},y_{2n+1}}(kt))^2 \geq\ & min\Big\{(F_{y_{2n-1},y_{2n}}(t))^2,\\
& F_{y_{2n-1},y_{2n}}(t)\cdot F_{y_{2n},y_{2n}}(t), F_{y_{2n-1},y_{2n}}(t)\cdot F_{y_{2n},y_{2n+1}}(t),\\
& F_{y_{2n-1},y_{2n+1}}(t)\cdot F_{y_{2n},y_{2n+1}}(t), F_{y_{2n-1},y_{2n+1}}(t)\cdot F_{y_{2n},y_{2n}}(t)\Big\},\\
=\ & min\Big\{(F_{y_{2n-1},y_{2n}}(t))^2, F_{y_{2n-1},y_{2n}}(t)\cdot 1,\\
& F_{y_{2n-1},y_{2n}}(t)\cdot F_{y_{2n},y_{2n+1}}(t), F_{y_{2n-1},y_{2n+1}}(t)\cdot F_{y_{2n},y_{2n+1}}(t),\\
& F_{y_{2n-1},y_{2n+1}}(t)\cdot 1\Big\}. \qquad (3.7)
\end{aligned}
$$

It is clear that second, third and fifth terms(independently) in the RHS of the preceding inequality bigger than of $\min\{(F_{y_{2n-1},y_{2n}}(t))^2,(F_{y_{2n},y_{2n+1}}(t))^2\}$, and the fourth term follows as

$$
\begin{aligned}
F_{y_{2n-1},y_{2n+1}}(t)\cdot F_{y_{2n},y_{2n+1}}(t) &= t(F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t))\cdot F_{y_{2n},y_{2n+1}}(t)\\
&= min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}\cdot F_{y_{2n},y_{2n+1}}(t)\\
&= min\{F_{y_{2n-1},y_{2n}}(t)\cdot F_{y_{2n},y_{2n+1}}(t), (F_{y_{2n},y_{2n+1}}(t))^2\}\\
&\geq \min\{min\{(F_{y_{2n-1},y_{2n}}(t))^2, (F_{y_{2n},y_{2n+1}}(t))^2\}, (F_{y_{2n},y_{2n+1}}(t))^2\}\\
&\geq \min\{(F_{y_{2n-1},y_{2n}}(t))^2, (F_{y_{2n},y_{2n+1}}(t))^2\}.
\end{aligned}
$$

Finally, the inequality (3.7), reduced to the following form

$$
\begin{aligned}
(F_{y_{2n},y_{2n+1}}(kt))^2 &\geq min\Big\{(F_{y_{2n-1},y_{2n}}(t))^2, min\{(F_{y_{2n-1},y_{2n}}(t))^2,\ (F_{y_{2n},y_{2n+1}}(t))^2\}\Big\}\\
&= min\Big\{(F_{y_{2n-1},y_{2n}}(t))^2,\ (F_{y_{2n},y_{2n+1}}(t))^2\Big\}.
\end{aligned}
$$

Similarly, we get

$$
(F_{y_{2n+1},y_{2n+2}}(kt))^2 \geq min\Big\{(F_{y_{2n},y_{2n+1}}(t))^2,\ (F_{y_{2n+1},y_{2n+2}}(t))^2\Big\}.
$$

Then for all $n$ even or odd, we have

$$
(F_{y_n,y_{n+1}}(kt))^2 \geq min\Big\{(F_{y_{n-1},y_n}(t))^2,\ (F_{y_n,y_{n+1}}(t))^2\Big\}
$$

it follows that,

$$
F_{y_n,y_{n+1}}(kt) \geq min\Big\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}(t)\Big\}.
$$

Consequently,

$$F_{y_n,y_{n+1}}(t) \geq min\left\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-1}t)\right\}.$$

By repeated application of the above inequality, we get

$$F_{y_n,y_{n+1}}(t) \geq min\left\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-m}t)\right\}.$$

Since $F_{y_n,y_{n+1}}(k^{-m}t) \to 1$ as $m \to \infty$, it follows that

$$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t).$$

By Lemma 2.13, the sequence $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, then the Cauchy sequence $\{y_n\}$ converges to a point $z$(say) in $X$. Now, the subsequences are

$$\{Mx_{2n+1}\}, \{STx_{2n+1}\}, \{Lx_{2n}\}, \{ABx_{2n}\} \text{ of } \{y_n\} \to z \text{ as } n \to \infty. \tag{3.8}$$

Suppose that $L(X)$ be a closed subspace of $X$, then for every $t > 0$,there exists a point $v \in X$ such that $Lv = z$.

Again, for $L(X)$ contained in $ST(X)$, we have a point $u \in X$ such that

$$Lv = STu = z. \tag{3.9}$$

To prove u is a common coincidence point of $L$ and $ST$, we shall argue the following manner. For this put $p = x_{2n}$, and $q = u$ in (3.5), so, we get

$$\begin{aligned}(F_{Lx_{2n},Mu}(kt))^2 \geq min\Big\{&(F_{ABx_{2n},STu}(t))^2, F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STu,Lx_{2n}}(t),\\ &F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STu,Mu}(t), F_{ABx_{2n},Mu}(t)\cdot F_{STu,Mu}(t),\\ &F_{ABx_{2n},Mu}(t)\cdot F_{STu,Lx_{2n}}(t)\Big\}.\end{aligned}$$

Now,taking limit as $n \to \infty$ to both sides of the above inequality and using (3.8) and (3.9), we have

$$\begin{aligned}(F_{z,Mu}(kt))^2 &\geq min\Big\{(F_{z,z}(t))^2, F_{z,z}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,Mu}(t),\\ &\qquad F_{z,Mu}(t)\cdot F_{z,Mu}(t), F_{z,z}(t)\cdot F_{z,z}(t)\Big\}\\ &= (F_{z,Mu}(t))^2\end{aligned}$$

From Lemma 2.14, we get $z = Mu$. Hence from (3.9), we get

$$z = Mu = STu. \tag{3.10}$$

Since $(M, ST)$ is owc pair of maps. Therefore, so inequality (3.10) implies that

$$Mz = MSTu = STMu = STz. \tag{3.11}$$

Now, we show that $z$ is a common fixed point of $M$ and $ST$. Taking $p = x_{2n}$ and $q = z$ for $t > 0$ in (3.5), we get

$$\begin{aligned}(F_{Lx_{2n},Mz}(kt))^2 \ \geq \ min\Big\{ & (F_{ABx_{2n},STz}(t))^2, F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STz,Mz}(t), \\ & F_{ABx_{2n},Mz}(t)\cdot F_{STz,Mz}(t), F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STz,Lx_{2n}}(t), \\ & F_{ABx_{2n},Mz}(t)\cdot F_{STz,Lx_{2n}}(t)\Big\}.\end{aligned}$$

On letting $n \to \infty$ in the above inequality also using (3.8) and (3.11), we have

$$\begin{aligned}(F_{z,Mz}(kt))^2 \ &\geq \ min\Big\{(F_{z,Mz}(t))^2, F_{z,z}(t)\cdot F_{Mz,z}(t), F_{z,z}(t)\cdot F_{Mz,Mz}(t), \\ &\qquad F_{z,Mz}(t)\cdot F_{Mz,Mz}(t), F_{z,Mz}(t)\cdot F_{Mz,z}(t)\Big\}, \\ &= \ (F_{z,Mz}(t))^2.\end{aligned}$$

From Lemma 2.14, we get $Mz = z$ and using (3.11), we have

$$Mz = STz = z. \tag{3.12}$$

Now, we show that $z$ is common fixed point of $T$ also. For this, we taking $p = x_{2n}$, $q = Tz$ in (3.5),then we obtain that

$$\begin{aligned}(F_{Lx_{2n},MTz}(kt))^2 \ \geq \ min\Big\{ & (F_{ABx_{2n},STTz}(t))^2, F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STTz,Lx_{2n}}(t), \\ & F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STTz,MTz}(t), F_{ABx_{2n},MTz}(t)\cdot F_{STTz,MTz}(t), \\ & F_{ABx_{2n},MTz}(t)\cdot F_{STTz,Lx_{2n}}(t)\Big\}.\end{aligned}$$

On letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned}(F_{z,MTz}(kt))^2 \;\; \geq \;\; min\Big\{ & (F_{z,STTz}(t))^2, F_{z,z}(t) \cdot F_{STTz,z}(t),\\ & F_{z,z}(t) \cdot F_{STTz,MTz}(t), F_{z,MTz}(t) \cdot F_{STTz,MTz}(t),\\ & F_{z,MTz}(t) \cdot F_{STTz,z}(t)\Big\}. \quad (3.13)\end{aligned}$$

Using the property (ii) of the hypothesis in the Theorem and From (3.12) we have

$$MTz = TMz = Tz \text{ and } ST(Tz) = TS(Tz) = T(STz) = Tz. \qquad (3.14)$$

Now, using (3.14) in (3.13), we get

$$\begin{aligned}(F_{z,Tz}(kt))^2 \;\; &\geq \;\; min\{(F_{z,Tz}(t))^2, F_{z,z}(t) \cdot F_{Tz,z}(t), F_{z,z}(t) \cdot F_{Tz,Tz}(t),\\ &\qquad F_{z,Tz(t)} \cdot F_{Tz,Tz}(t), F_{z,Tz(t)} \cdot F_{Tz,z}(t)\},\\ &= \;\; (F_{z,Tz}(t))^2.\end{aligned}$$

By using Lemma 2.14, we have,

$$Tz = z, \text{ then from (3.12), we obtain that } z = Mz = STz = Sz. \qquad (3.15)$$

Hence, $z$ is a common fixed point of $M, S$ and $T$. Since $M(X) \subseteq AB(X)$, then there exists a point $v \in X$ such that

$$ABv = Mz(= z). \qquad (3.16)$$

On,taking $p = v$ and $q = x_{2n+1}$ in (3.5), we get

$$\begin{aligned}(F_{Lv,Mx_{2n+1}}(kt))^2 \;\; \geq \;\; min\Big\{ & (F_{ABv,STx_{2n+1}}(t))^2, F_{ABv,Lv}(t) \cdot F_{STx_{2n+1},Lv}(t),\\ & F_{ABv,Lv}(t) \cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ & F_{ABv,Mx_{2n+1}}(t) \cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ & F_{ABv,Mx_{2n+1}}(t) \cdot F_{STx_{2n+1},Lv}(t)\Big\}.\end{aligned}$$

Now, taking limits $n \to \infty$ to both sides and using (3.8) and (3.16), we get

$$\begin{aligned}(F_{Lv,z}(kt))^2 &\geq min\Big\{(F_{z,z}(t))^2, F_{z,Lv}(t)\cdot F_{z,Lv}(t),\\ &\qquad F_{z,Lv}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,Lv}(t)\Big\},\\ &\geq (F_{z,Lv}(t))^2.\end{aligned}$$

Again from Lemma 2.14 we get $z = Lv$. Therefore from (3.16), we obtain that

$$Lv = ABv = z. \tag{3.17}$$

Since the pair $(L, AB)$ is owc, then from (3.17), we get

$$Lz = LABv = ABLv = ABz. \tag{3.18}$$

Now, we show that $Lz = z$, for this purpose, taking $p = z$, $q = x_{2n+1}$ in (3.5), we get

$$\begin{aligned}(F_{Lz,Mx_{2n+1}}(kt)^2 &\geq min\Big\{(F_{ABz,STx_{2n+1}}(t))^2, F_{ABz,Lz}(t)\cdot F_{STx_{2n+1},Lz}(t),\\ &\qquad F_{ABz,Lz}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &\qquad F_{ABz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &\qquad F_{ABz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},Lz}(t)\Big\}.\end{aligned}$$

Again letting $n \to \infty$ to both sides, using (3.8) and (3.18), we get

$$\begin{aligned}(F_{Lz,z}(kt)^2 &\geq min\Big\{(F_{Lz,z}(t))^2, F_{z,Lz}(t)\cdot F_{z,Lz}(t), F_{z,Lz}(t)\cdot F_{z,z}(t),\\ &\qquad F_{z,z}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,Lz}(t)\Big\},\\ &\geq (F_{z,Lz}(t))^2.\end{aligned}$$

From Lemma 2.14, we get $Lz = z$. Hence from (3.18), we obtain that

$$Lz = ABz = z. \tag{3.19}$$

Finally, we show that $z$ is fixed point of $B$, taking $p = Bz$, $q = x_{2n+1}$ in (3.5), we get

$$\begin{aligned}(F_{LBz,Mx_{2n+1}}(kt))^2 \geq min\Big\{&(F_{ABBz,STx_{2n+1}}(t))^2, F_{ABBz,STx_{2n+1}}(t)\cdot F_{STx_{2n+1},LBz}(t),\\ &F_{ABBz,LBz}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &F_{ABBz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &F_{ABBz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},LBz}(t)\Big\}. \end{aligned} \tag{3.20}$$

Using the property (ii) of the hypothesis in the Theorem, we have

$$LBz = BLz = Bz \text{ and } AB(Bz) = BA(Bz) = B(ABz) = Bz. \tag{3.21}$$

. Letting $n \to \infty$ and using (3.21) in (3.20), we get

$$\begin{aligned}(F_{Bz,z}(kt))^2 &\geq min\Big\{(F_{Bz,z}(t))^2, F_{Bz,Bz}(t)\cdot F_{z,Bz}(t), F_{Bz,Bz}(t)\cdot F_{z,z}(t),\\ &\quad F_{Bz,z}(t)\cdot F_{z,z}(t), F_{Bz,z}(t)\cdot F_{z,Bz}(t)\Big\},\\ &= (F_{Bz,z}(t))^2.\end{aligned}$$

From Lemma 2.14, we get

$$Bz = z \text{ from (3.18) } z = Lz = ABz = Az. \tag{3.22}$$

From (3.15) and (3.22), we conclude that $z$ is a common fixed point of $A, B, S, T, L$ and $M$ in $X$. Uniqueness follows easily from (3.5).

**Uniqueness:** Let us assume that $z$ and $z_1$ be two distinct common fixed point of the given six self maps. Now,put $p = z$ and $q = z_1$ in (3.5), we obtain

$$\begin{aligned}(F_{Lz,Mz_1}(kt))^2 \geq min\Big\{&(F_{ABz,STz_1}(t))^2, F_{ABz,Lz}(t)\cdot F_{STz_1,Lz}(t),\\ &F_{ABz,Lz}(t)\cdot F_{STz_1,Mz_1}(t),\\ &F_{ABz,Mz_1}(t)\cdot F_{STz_1,Mz_1}(t),\\ &F_{ABz,Mz_1}(t)\cdot F_{STz_1,Lz}(t)\Big\},\end{aligned}$$

Since $z$ and $z_1$ are fixed points of $A,B,S,T,L$ and $M$,then we have

$$\begin{aligned}(F_{z,z_1}(kt))^2 &\geq min\Big\{(F_{z,z_1}(t))^2, F_{z,z}(t)\cdot F_{z_1,z}(t), F_{z,z}(t)\cdot F_{z_1,z_1}(t),\\ &\qquad F_{z,z_1}(t)\cdot F_{z_1,z_1}(t), F_{z,z_1}(t)\cdot F_{z_1,z}(t)\Big\},\\ &= min\Big\{(F_{z,z_1}(t))^2, 1\cdot F_{z_1,z}(t), 1\cdot 1,\\ &\qquad F_{z,z_1}(t)\cdot 1, F_{z,z_1}(t)\cdot F_{z_1,z}(t)\\ &= (F_{z,z_1}(t))^2.\end{aligned}$$

It is clear $z = z_1$ from Lemma 2.14. Hence, $A$, $B$, $S$, $T$, $L$ and $M$ have a unique common fixed point in $X$. □

**Theorem 3.2.** *Let A, B,S,T,L and M are self mappings of a complete Menger space* $(X,F,t)$ *with continuous t-norm that* $t(x,x) \geq x$ *for all* $x \in [0,1]$ *satisfying conditions* (3.1),(3.2),(3.4), (3.5) *and the pairs* $(L,AB)$, $(M,ST)$ *are compatible maps of type (A) .Then* $A,B,S,T,L$ *and M have a unique common fixed point in X.*

*Proof.* The proof is same lines the condition 3.6 to 3.8 in Theorem 3.1. From the hypothesis, assume $(M,ST)$ is compatible maps of type (A) and $M$ and $ST$ is continuous, so by proposition 2.12, we get

$$Mz = STz. \tag{3.23}$$

Now, we show that $z$ is a common fixed point of $M$ and $ST$. Taking $p = x_{2n}$ and $q = z$ for $t > 0$ in (3.5), we get

$$\begin{aligned}(F_{Lx_{2n},Mz}(kt))^2 &\geq min\Big\{(F_{ABx_{2n},STz}(t))^2, F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STz,Mz}(t),\\ &\qquad F_{ABx_{2n},Mz}(t)\cdot F_{STz,Mz}(t), F_{ABx_{2n},Lx_{2n}}(t)\cdot F_{STz,Lx_{2n}}(t),\\ &\qquad F_{ABx_{2n},Mz}(t)\cdot F_{STz,Lx_{2n}}(t)\Big\}.\end{aligned}$$

On letting $n \to \infty$ in the above inequality also using (3.8) and (3.23), we have

$$\begin{aligned}(F_{z,Mz}(kt))^2 &\geq min\Big\{(F_{z,Mz}(t))^2, F_{z,z}(t)\cdot F_{Mz,z}(t), F_{z,z}(t)\cdot F_{Mz,Mz}(t),\\ &\qquad F_{z,Mz}(t)\cdot F_{Mz,Mz}(t), F_{z,Mz}(t)\cdot F_{Mz,z}(t)\Big\},\\ &= (F_{z,Mz}(t))^2.\end{aligned}$$

From Lemma 2.14, we get $Mz = z$ and using (3.23), we have

$$Mz = STz = z. \tag{3.24}$$

Rest of the proof is same line from the condition 3.12 to 3.15. Hence, $z$ is a common fixed point of $M, S$ and $T$. Since the pair $(L, AB)$ isis compatible maps of type (A) and $L$ and $AB$ is continuous, so by proposition 2.12, we get

$$Lz = ABz. \tag{3.25}$$

Now, we show that $Lz = z$, for this purpose, taking $p = z$, $q = x_{2n+1}$ in (3.5), we get

$$\begin{aligned}(F_{Lz,Mx_{2n+1}}(kt)^2 &\geq min\Big\{(F_{ABz,STx_{2n+1}}(t))^2, F_{ABz,Lz}(t)\cdot F_{STx_{2n+1},Lz}(t),\\ &\qquad F_{ABz,Lz}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &\qquad F_{ABz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},Mx_{2n+1}}(t),\\ &\qquad F_{ABz,Mx_{2n+1}}(t)\cdot F_{STx_{2n+1},Lz}(t)\Big\}.\end{aligned}$$

Again letting $n \to \infty$ to both sides, using (3.8) and (3.25), we get

$$\begin{aligned}(F_{Lz,z}(kt)^2 &\geq min\Big\{(F_{Lz,z}(t))^2, F_{z,Lz}(t)\cdot F_{z,Lz}(t), F_{z,Lz}(t)\cdot F_{z,z}(t),\\ &\qquad F_{z,z}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,Lz}(t)\Big\},\\ &\geq (F_{z,Lz}(t))^2.\end{aligned}$$

From Lemma 2.14, we get $Lz = z$. Hence from (3.25), we obtain that

$$Lz = ABz = z. \tag{3.26}$$

Rest of the proof is same lines from the condition (3.19) onward in Theorem 3.1.

□

## 4. Corollary and Example

**Corollary 4.1.** *Let A, B,S,T,L and M are self mappings of a complete Menger space* $(X,F,t)$ *with continuous t-norm that* $t(x,x) \geq x$ *for all* $x \in [0,1]$ *satisfying conditions* (3.1),(3.2),(3.4), (3.5) *and the pairs* $(L,AB)$, $(M,ST)$ *are either compatible or compatible maps of type (A) or weakly competible.Then* $A,B,S,T,L$ *and* $M$ *have a unique common fixed point in* $X$.

**Example 4.2.** *Let* $(X,d)$ *be a metric space and* $(X,F,\Delta)$ *is a complete Menger space where* $t(a,b) = min\{a,b\}$ *for all* $a,b \in [0,1]$. *Let* $X = [0,6]$*be the set with the metric* $d$ *defined by* $d(x,y) = |x-y|$ *and for each* $t \in [0,1]$. *Define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & if\, t > 0 \\ 0 & if\, t = 0 \end{cases}$$

*for all* $x,y \in X$. *Let* $A,B,S,T,L$ *and* $M$ *be self mappings on* $X$ *defined as* $L(X) = M(X) = 1$,

$$A(x) = \begin{cases} \frac{x}{5}+\frac{4}{5} & ,if\, 0 \leq x \leq 1, \\ 0 & ,if\, 1 < x < 6. \end{cases} \quad B(x) = \begin{cases} \frac{1}{6}(2x+4) & ,if\, 0 \leq x \leq 1, \\ 0 & ,if\, 1 < x < 6. \end{cases}$$

$$S(x) = \begin{cases} 1 & ,\text{if } 0 \leq x \leq 1, \\ \frac{2}{3} & ,\text{if } 1 < x < 6. \end{cases} \quad T(x) = \begin{cases} 1 & ,\text{if } 0 \leq x \leq 1, \\ \frac{1}{5} & ,\text{if } 1 < x < 6. \end{cases}$$

Now we varifying the conditions of Theorem 3.1.

(1) $M(X) = \{1\} \subseteq AB(X) = \{0\} \cup [1, \frac{20}{15})$, $L(X) = \{1\} \subseteq ST(X) = \{1\}$.

(2) $ST = TS = \{1\}$, $AB = BA = \frac{x}{15} + \frac{14}{15}$, $LB = BL = \{1\}$, $TM = MT = \{1\}$.

(3) The pair of maps $(L,AB)$ and $(M,ST)$ are satisfying owc condition at the point 1.

(4) Finally, it is easy to varify that for any $x,y \in X$, $RHS \geq LHS$ in (3.4) for all possible cases.

Hence, all the hypothesis of Theorem 3.1 satisfied and 1 is a unique common fixed point of $A,B,S,T,L$ and $M$ in $X$

**Theorem 4.3.** *Let A, B,S,T,L and M are self mappings of a Menger space* $(X,F,t)$ *with continuous t-norm that* $t(x,x) \geq x$ *for all* $x \in [0,1]$ *satisfying condition* (3.1), (3.2), (3.5) *and*

**(i):** *the pairs* $(L,AB)$ *and* $(M,ST)$ *are both owc,* (4.1)

**(ii):** *either* $L(X)$ *or* $ST(X)$ *is closed in* $X$ *and* $(L,AB)$ *satisfies the property* $(E.A)$ *or* $M(X)$ *or* $AB(X)$ *is closed in* $X$ *and* $(M,ST)$ *satisfies the property* $(E.A)$ (4.2)

*Then $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.*

*Proof.* From the hypothesis, assume $L(X)$ is closed in $X$ and $(L,AB)$ satisfies the property $(E,A)$. By the definition of property $(E.A)$ there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} ABx_n = z$ for some $t \in X$. Since $L(X) \subseteq ST(X)$, there exist a sequence $\{y_n\}$ such that $Lx_n = STy_n$. Hence $\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} STy_n = z$. Now,we show that $\lim_{n\to\infty} My_n = z$. Indeed, in view of (3.5), taking $p = x_n$ and $q = y_n$ we get

$$\begin{aligned}(F_{Lx_n,My_n}(kt))^2 \;\geq\; min\Big\{ & (F_{STy_n,Lx_n}(t))^2, F_{ABx_n,Lx_n}(t)F_{STy_n,Lx_n}(t), \\ & F_{ABx_n,Lx_n}(t)F_{STy_n,My_n}(t), \\ & F_{ABx_n,My_n}(t)F_{STy_n,My_n}(t), \\ & F_{ABx_n,My_n}(t)F_{STy_n,Lx_n}(t)\Big\}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}(F_{z,My_n}(kt))^2 \;&\geq\; min\Big\{(F_{z,z}(t))^2, F_{z,z}(t)\cdot F_{z,z}(t), F_{z,z}(t)\cdot F_{z,My_n}(t), \\ &\qquad F_{z,My_n}(t)\cdot F_{z,My_n}(t), F_{z,My_n}(t)\cdot F_{z,z}(t)\Big\}, \\ &= (F_{z,My_n}(t))^2.\end{aligned}$$

From Lemma 2.14, we get $My_n = z$. Therefore, we concluded that

$$Lx_n, ABx_n \text{ and } My_n, STy_n \to z \text{ as } n \to \infty. \tag{4.3}$$

Since $L(X)$ is closed in $X$, then there exists $v \in X$ such that $Lv = z$. Since $L(X) \subseteq ST(X)$. There is a point $u \in X$ such that

$$Lv = STu = z.$$

Rest of the proof is same lines from the condition (3.9) onward in Theorem 3.1. □

**Corollary 4.4.** *Let $A,B,S,T,L$ and $M$ are self mappings of a Menger space $(X,F,\Delta)$ with continuous t-norm that $t(x,x) \geq x$ for all $x \in [0,1]$ satisfying conditions* (3.1), (3.2), (3.5)*, the pairs* $(L,AB)$, $(M,ST)$ *are either compatible or compatible of maps type (A) or weakly compatible and*

**a:** *either $ST(X)$ is closed in $X$ and $(L,AB)$ satisfies the property $(E.A)$*

*or $AB(X)$ is closed in $X$ and $(M,ST)$ satisfies the property $(E.A)$* (4.4)

*Then $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.*

**Example 4.5.** *Let $(X,d)$ be a metric space and $(X,F,\Delta)$ is a complete Menger space where $t(a,b)=min\{a,b\}$ for all $a,b\in[0,1]$.*

*Let $X=[0,2]$ with the metric $d$ defined by $d(x,y)=|x-y|$ and define*

*$F_{x,y}=H(u-d(x,y))$ for all $x,y\in X$ and $u>0$.*

*Let $A,B,S,T,L$ and $M$ be self mappings on $X$ defined as $L(X)=M(X)=\frac{3}{2}$,*

$$A(x)=\begin{cases} x & , \text{if } 0\le x\le \frac{3}{2}, \\ 1 & , \text{if } \frac{3}{2}<x\le 2. \end{cases} \quad B(x)=\begin{cases} \frac{3}{2} & , \text{if } 0\le x\le \frac{3}{2}, \\ 0 & , \text{if } \frac{3}{2}<x\le 2. \end{cases}$$

$$S(x)=\begin{cases} \frac{3}{2} & , \text{if } 0\le x\le \frac{3}{2}, \\ 1 & , \text{if } \frac{3}{2}<x\le 2. \end{cases} \quad T(x)=\begin{cases} \frac{3}{2} & , \text{if } 0\le x\le \frac{3}{2}, \\ \frac{x}{4} & , \text{if } \frac{3}{2}<x\le 2. \end{cases}$$

Now, verification of Theorem 3.4 hypothesis (1)$L(X)=\{\frac{3}{2}\}\subseteq ST(X)=\{\frac{3}{2}\}$, $M(X)=\{\frac{3}{2}\}\subseteq AB(X)=\{0,\frac{3}{2}\}$.

(2) The condition (3.2) is also satisfying.

(3) Clearly the pairs $(L,AB)$ $(M,ST)$ satisfying owc condition at the point $\frac{3}{2}$. (4) It is easy to verify that any $x,y\in X$, $RHS\ge LHS$ in (3.4) for all possible cases.Hence, $\frac{3}{2}$ is the unique common fixed point of $A,B,S,T,L$ and $M$ in $X$.

## 5. APPLICATION

**(i) Application to Volterra type integral equation:**

A differential equation can be replaced by an integral equation with the help of initial and boundary conditions. Integral equation were first encountered in the theory of Fourier integrals. Actual development of the theory of integrals equations began with the works of the Italian mathematician V.Volterra (1896) and then studied by Traian Lalescu in his thesis (1908) and the Swedish mathematician I.Fredholm (1990).

The general type of linear integral is of the from

$$y(x)=F(x)+\lambda\int_a^b K(x,t)y(t)dt,$$

where $F(x)$ and $K(x,t)$ are known functions while $y(x)$ is to be determined. The function $K(x,t)$ is called the Kernel of the integral equation.

If $a$ and $b$ are constants, the equation is known as Fredholm integral equation.

If $a$ is a constant while $b$ is a variable, it is called a Volterra integral equation.

A Volterra integral equation of the first kind is an integral eqution of the from.

$$F(x) = \int_a^x K(x,t)y(t)dt,$$

A Volterra integral equation of the second kind is an integral eqution of the from.

$$y(x) = F(x) + \int_a^x K(x,t)y(t)dt,$$

An integral equation is called homogeneous if $F(x) = 0$.

As an application of our results, we will consider the following Volterra type integral equation

$$p(t) = g(t) + \int_0^t \Omega(t,s,p(s))ds, \tag{5.1}$$

for all $t \in [0,K]$, where $k > 0$

Let $C([0,K],\mathbb{R})$ be the space of all continuous functions defined on $[0,K]$ endowed with the b-metric

$$d(x,y) = \max_{t\in[0,K]} \left|F_{x(t)-y(t)}\right|^2,\ x,y \in C([0,K],\mathbb{R}) \tag{5.2}$$

Alternatively the space $C([0,K],\mathbb{R})$ can be endowed with the b-metric

$$d_B(x,y) = \max_{t\in[0,K]} (\left|F_{x(t)-y(t)}\right|^2 e^{-2It}),\ x,y \in C([0,K],\mathbb{R}), I > 0, \tag{5.3}$$

and the induced metric $d_B(x,y) = \|x-y\|_B$, for all $x,y \in C([0,K],\mathbb{R})$.

We define the mapping $F : C([0,K],\mathbb{R}) \times C([0,K],\mathbb{R}) \to \mathbb{D}$ by

$$F_{x,y} = H(t - d_B(x,y)),\ x,y \in C([0,K],\mathbb{R}), t > 0. \tag{5.4}$$

We know that the space $C([0,K],\mathbb{R},F,T_M)$ is the $(\varepsilon,\lambda)-$complete Menger space induced by the Banach space $C([0,K],\mathbb{R}$. Also, one can easily show that in the space $C([0,K],\mathbb{R},F,T_M)$, the convergence in norms $\|.\|_B$ and $\|.\|_\infty$ are equivalent each other,with respect to $(\varepsilon,\lambda)-$ topology.

Now,we discuss the existence of solution for Volterra type integral equation 5.1

**Theorem 5.1.** *Let $C([0,K],\mathbb{R},F,T_M)$ be the Menger PM-space induced by the Banach space $C([0,K],\mathbb{R})$ and let $\Omega \in C([0,K]\times[0,K]\times\mathbb{R},\mathbb{R})$ be an operator satisfying the following conditions:*

**(i):** $\|\Omega\| = \sup_{t,s\in[0,k],\mathbb{R}} |\Omega(t,s,p(s)| < \infty$

**(ii):** *there exist $I>0$ such that for all $p,q \in C([0,K],\mathbb{R})$ and all $t,s\in[0,K]$ , we get*

$$\begin{aligned}|\Omega(t,s,Lp(s)-\Omega(t,s,Mq(s)| \quad \leq \quad \frac{I}{\sqrt{2}} I\, min\Big\{ & |ABp-STp|^2, \\ & |ABp-Lp|\cdot|STq-Mq|, \\ & |ABp-Mq|\cdot|STq-Mq|, \\ & |ABp-Mq|\cdot|STq-Lp| \Big\}.\end{aligned}$$

where $A,B,S,T,L,M : C([0,K],\mathbb{R})$ is dedined by

$$p(t) = g(t) + \int_0^t \Omega(t,s,p(s))ds, \; g \in C([0,K],\mathbb{R}).$$

Then, the Volterra type integral equation 5.1 has a unique solution $x^* \in C([0,K],\mathbb{R})$

*Proof.* For each $p,q\in C([0,K],\mathbb{R})$ we consider $(d_B(x,y)) = \max_{t,\in[0,k]}(|p(t)-q(t)|^2)e^{-2It}$,

where $I$ satisfies condition (ii). As we mentioned above $C([0,K],\mathbb{R},F,T_N)$ is a completed PM-space

$$d_B(Lp,Mq) \leq \max_{t\in[0,K]}(\left|F_{Lp(t)-Mq(t)}\right|^2)e^{-2It}$$

$$d_B(p,q) \leq \max_{t\in[0,K]}\left(\int_0^t |\Omega(t,s,Lp(s))-\Omega(t,s,Mq(s)ds|^2 e^{-2It}\right)$$

$$\begin{aligned}\leq \frac{I^2}{2} \qquad & max\Big\{ d_B(ABp,STp)^2, d_B(ABp,Lp)\cdot d_B(ST,Lp), \\ & d_B(ABp,Lp)\cdot d_B(STp,Mq), d_B(ABp,Mq)\cdot d_B(STp,Mq), \\ & d_B(ABp,Mq)\cdot d_B(STp,Lp)\Big\} \\ & max_{t\in[0,K]}\left(\int_0^t e^{I(s-t)}ds\right)^2\end{aligned}$$

$$= \frac{1}{2}(1-e^{-Ik'})^2 \qquad max\Big\{ d_B(ABp,STp)^2, d_B(ABp,Lp)\cdot d_B(ST,Lp),$$
$$d_B(ABp,Lp)\cdot d_B(STp,Mq), d_B(ABp,Mq)\cdot d_B(STp,Mq),$$
$$d_B(ABp,Mq)\cdot d_B(STp,Lp)\Big\}.$$

Putting $c^2=(1-e^{-Ik'})^2$ , by using (5.4) , for any $r\geq 0$ and $k\in\mathbb{N}$ we drive

$$(F_{Lp,Mq}(r))^2 \qquad = H^2\Big(r^2 - c^2\ max\Big\{ d_B(ABp,STp)^2, d_B(ABp,Lp)\cdot d_B(ST,Lp),$$
$$d_B(ABp,Lp)\cdot d_B(STp,Mq), d_B(ABp,Mq)\cdot d_B(STp,Mq),$$
$$d_B(ABp,Mq)\cdot d_B(STp,Lp)\Big\}\Big)$$
$$H^2\left(\left(\frac{r}{c}\right)^2 - max\Big\{ d_B(ABp,STp)^2, d_B(ABp,Lp)\cdot d_B(ST,Lp),\right.$$
$$d_B(ABp,Lp)\cdot d_B(STp,Mq), d_B(ABp,Mq)\cdot d_B(STp,Mq),$$
$$\left. d_B(ABp,Mq)\cdot d_B(STp,Lp)\Big\}\right)$$
$$= min\left\{ \left(F_{ABp,STq}\left(\frac{r}{c}\right)\right)^2, F_{ABp,Lp}\left(\frac{r}{c}\right)\cdot F_{STq,Lp}\left(\frac{r}{c}\right), F_{ABp,Lp}\left(\frac{r}{c}\right)\cdot F_{STq,Mq}\left(\frac{r}{c}\right),\right.$$
$$\left. F_{ABp,Mq}\left(\frac{r}{c}\right)\cdot F_{STq,Mq}\left(\frac{r}{c}\right), F_{ABp,Mq}\left(\frac{r}{c}\right)\cdot F_{STq,Lp}\left(\frac{r}{c}\right)\right\}.$$

for all $p,q\in C([0,K],\mathbb{R})$. Therefore by theorem 3.1. We deduce that all operator F has a unique fixed point $x^*\in C([0,K],\mathbb{R})$ , which is the unique solution of the integral equation 5.1. □

**(ii) Application to fixed point for metric spaces:**

We propose corresponding commom fixed point theorem in metric spaces. In fact,every metric space $(X,d)$ can be taken as a particular Menger space by $F: X\times X\to R$ defined by $F_{x,y}(t)=H(t-d(x,y))$ for all $x,y\in X$.

**Theorem 5.2.** *Let $A,B,S,T,L$ and $M$ are six self mappings of a complete metric space $(X,d)$ for all $x\in[0,1]$ satisfying conditions (3.1),(3.2),(3.3) and (3.4).*

$(i)$ *for all* $p,\ q \in X$, *and for some* $k \in (0,1)$ *such that*

$$\begin{aligned} d^2(Lp,Mq) \;\; &\leq \;\; k\, max\Big\{ d^2(ABp,STq), d(ABp,Lp)\cdot d(STq,Lp), \\ & d(ABp,Lp)\cdot d(STq,Mq), \\ & d(ABp,Mq)\cdot d(STq,Mq), \\ & d(ABp,Mq)\cdot d(STq,Lp)\Big\}. \end{aligned} \tag{5.5}$$

Then $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.

*Proof.* Define $F_{x,y}(t) = H(t-d(x,y))$, $t(a,b) = min\{a,b\}$, for all $a,b \in [0,1]$. Then this metric space can be taken as a Menger space. Theorem 5.2 enjoys the assumption of Theorem 3.1, including inequality (5.5) reduces to inequality (3.5) in Theorem 3.1. □

**Corollary 5.3.** *Let* $A,B,S,T,L$ *and* $M$ *are self mappings of a complete metric space* $(X,d)$ *for all* $x \in [0,1]$ *satisfying conditions* $(3.1),(3.2),(3,4)$ *and* $(5.5)$*, the pairs* $(L,AB)$*,* $(M,ST)$ *are either compatible or compatible maps of typeg (A).*

**Theorem 5.4.** *Let* $A,B,S,T,L$ *and* $M$ *are six self mappings of a complete metric space* $(X,d)$ *for all* $x \in [0,1]$ *satisfying conditions (3.1),(3.2),(3.3) and (3.4).*

**(i):** *for all* $p,\ q \in X$, *and for some* $t \in (0,1)$ *such that*

$$\int_0^{d^2(Lp,Mq)} \phi(t)dt \leq \int_0^{m(p,q)} \phi(t)dt \quad (5.6)$$

*where* $\phi : [0,\infty) \to [0,\infty)$ *is a summable nonnegative Lebesgue integrable function such that* $\int_\varepsilon^1 \phi(t)dt > 0$ *for each* $\varepsilon \in [0,1)$*,where* $0 < t < 1$*, and*

$$\begin{aligned} m(p,q) \;\; &= \;\; min\Big\{ (d^2(ABp,STq), d(ABp,Lp)\cdot d(STq,Lp), \\ & d(ABp,Lp)\cdot d(STq,Mq), \\ & d(ABp,Mq)\cdot d(STq,Mq), \\ & d(ABp,Mq)\cdot d(STq,Lp)\Big\}. \end{aligned}$$

Then the pairs $(L,AB)$ and $(M,ST)$ have a coincidence point each and $A,B,S,T,L$ and $M$ have a unique common fixed point in $X$.

*Proof.* Define $F_{x,y}(t) = H(t - d(x,y))$, $t(a,b) = min\{a,b\}$, for all $a,b \in [0,1]$. Then this metric space can be taken as a Menger space. Theorem 5.4 enjoys the assumption of Theorem 3.1, including inequality (5.6) reduces to inequality (3.5) in Theorem 3.1. To accomplish this notice that for any $x,y \in X$ and $t > 0$, $F_{Lp,Mq}(kt)^2 = 1$ provided $kt > d(Lp,Mq)$ which hold (5.6). Otherwise,if $kt \leq d(Lp,Mq)$,then

$$t \leq \ min \ \Big\{(d^2(ABp,STq), d(ABp,Lp)\cdot d(STq,Lp), d(ABp,Lp)\cdot d(STq,Mq), d(ABp,Mq)\cdot d(STq,Mq), d(ABp,Mq)\cdot d(STq,Lp)\Big\}.$$

and hence in all the cases, condition (5.6) holds. □

**Corollary 5.5.** *Let $A,B,S,T$, and $L$ are five self mappings of a complete metric space $(X,d)$ for all $x \in [0,1]$ satisfying conditions.*

- **(i):** *$L(X) \subseteq ST(X), L(X) \subseteq AB(X)$,*
- **(ii):** *$AB = BA$, $ST = TS$, $LB = BL$ and $LT = TL$,*
- **(iii):** *the pairs $(L,AB)$, $(L,ST)$ are owc,*
- **(iv):** *any one of the subspace either $L(X)$ or $ST(X)$ or $AB(X)$ is closed in $X$,*
- **(v):** *for all $p,\ q \in X$, and for some $t \in (0,1)$ such that*

$$\int_0^{d^2(Lp,Lq)} \phi(t)dt \leq \int_0^{m(p,q)} \phi(t)dt \quad (5.7)$$

*where $\phi : [0,\infty) \to [0,\infty)$ is a summable nonnegative Lebesgue integrable function such that $\int_\varepsilon^1 \phi(t)dt > 0$ for each $\varepsilon \in [0,1)$,where $0 < t < 1$, and*

$$\begin{aligned} m(p,q) \ = \ min\Big\{ & (d^2(ABp,STq), d(ABp,Lp)\cdot d(STq,Lp), \\ & d(ABp,Lp)\cdot d(STq,Lq), \\ & d(ABp,Lq)\cdot d(STq,Lq), \\ & d(ABp,Lq)\cdot d(STq,Lp)\Big\}. \end{aligned}$$

for all $p,q \in X$, and $t > 0$. Then $A,B,S,L$ and $T$ have a unique common fixed point in $X$.

**Corollary 5.6.** *Let $(X,d)$ be a metric space and $A,B,S,T : X \to X$ be four self mappings satisfying the following condition:*

- **(i):** *$A(X)$,$B(X)$ are closed sets of $X$ and $A(X) \subset T(X)$, $B(X) \subset S(X)$,*
- **(ii):** *the pairs $(A,S)$, $(B,T)$ are both owc,*
- **(iii):** *for all $p,q \in X$, and $t > 0$,*

$$\int_0^{d^2(Ap,Bq)} \phi(t)dt \leq \int_0^{m(p,q)} \phi(t)dt \quad (5.8)$$

*where $\phi : [0,\infty) \to [0,\infty)$ is a summable nonnegative Lebesgue integrable function such that $\int_\varepsilon^1 \phi(t)dt > 0$ for each $\varepsilon \in [0,1)$,where $0 < t < 1$, and*

$$\begin{aligned} m(p,q) \quad = \quad min\Big\{ & d^2(Sp,Tq)], d(Sp,Ap) \cdot d(Tq,Bq), d(Sp,Tq) \cdot d(Sp,Ap), \\ & d(Sp,Tq) \cdot d(Tq,Bq), d(Sp,Tq) \cdot d(Sp,Bq), \\ & d(Sp,Tq) \cdot d(Tq,Ap) \Big\} \end{aligned}$$

*for all $p,q \in X$, and $t > 0$.*

*Then $A,B,S$, and $T$ have a unique common fixed point in $X$.*

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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