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USING FUZZY LOGIC TO APPROXIMATE THE ACCURACY RATE IN SOLVING A SYSTEM OF LINEAR EQUATIONS

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Abstract: This paper proposes a numerical procedure for determining rate of accuracy of the numerical method used for solving a system of linear equations $Ax=b$. The idea of the method depends on the fuzzy value of the condition number and singularity rate instead of the crisp values. The procedure will be implemented on the mathematical code MATLAB and its Simulink features. Finally the applicability and efficiency of the procedure is illustrated by a numerical example.

Keywords: fuzzy set; fuzzy logic; crisp logic; shape functions; condition number; singular matrices.

2010 AMS Subject Classification: 68T37, 65N22, 26A33, 41A55.

1. INTRODUCTION

The term fuzzy logic was introduced with the 1965 proposal of fuzzy set theory by Lotfi Zadeh. [13] Fuzzy logic had, however, been studied since the 1920s, as infinite-valued logic-notably by Łukasiewicz and Tarski. [8, 16] Fuzzy logic provided mathematicians with an appropriate tool for modelling vagueness phenomenon and shed new light into the control theory for engineers.[4, 6, 7, 8, 12] Later, in 1974 it was applied and implemented practically for the first

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time by Mamdani [9] in steam engine control theory. Fuzzy sets and logic were made practically useful in the 1990s [16]. Several models based on the fuzzy sets, the fuzzy transform has been proposed and widely used as a numerical tools for solving differential equations [1, 2, 3]. Fuzzy logic allows us to model vague human language notions, especially, so called linguistic expressions (small, very big, more or less ... etc.). Fuzzy logic controllers have the many advantages over the conventional controllers [6]: they are cheaper to develop, they cover a wider range of operating conditions, and they are more readily customizable in natural language terms. [8, 14].

One way to measure the magnification factor is by means of the quantity $\|A\|\|A^{-1}\|$ called the condition number $\kappa(A)$ of the matrix A with respect to its inverse. The condition number determines the loss of precision due to round-off error in Gaussian elimination and can be used to estimate the accuracy of results obtained from matrix inversion and linear system solution. As a rule of thumb, if the condition number $\kappa(A) = 10^k$, then you may lose up to k digits of accuracy on top of what would be lost to the numerical method due to loss of precision from arithmetic methods. In Crisp logic, if $10^{k-1} < \kappa(A) < 10^k$ the number of loss of precision from arithmetic methods will be the same despite the significant difference in the values of the condition number, and consequently one action will be taken for different values of $\kappa(A)$ to keep the accuracy of the solution at a certain level. The values of $\kappa(A)$ close to the end of a certain interval and those close to the beginning of the next interval will give different number of loss of precision although they are almost equal. The fuzzy logic with the membership functions gives more reasonable values for the loss precisions. Fuzzy logic allows us also to model a non-specific mathematical and scientific language notions, especially, so called vague expressions (small rate, very small determinant, large condition number, nearly singular matrix ... etc.). It prevents different actions for almost same behaviours. [14]

The main aim of this paper is to overview the main concepts of fuzzy logic and its applications. Moreover, we will explain a construction of approximation technique with membership functions to use Fuzzy logic, condition number and degree of singularity of the matrix to approximate the decimal digits of accuracy in solving a System of linear equations. An example will be given with condition number and degree of singularity as input parameters, and the accuracy as output parameter.

2. BASIC CONCEPT OF FUZZY LOGIC

Fuzzy set and crisp set are the part of the distinct set theories, where the fuzzy set implements and allow the whole interval $[0, 1]$ to be the range of their characteristic functions which we call membership functions, and employ infinite-valued logic. The fuzzy membership functions enabled us to overcome the difficulty of having very different control actions for a small a change in the inputs. while crisp set employs bi-valued logic. [8, 13]

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Where A is a classical (crisp) set.

In logic, fuzzy logic is a form of many-valued logic in which the truth value of variables may be any real number between 0 and 1 both inclusive. It is employed to handle the concept of partial truth, where the truth value may range between completely true and completely false. By contrast, in crisp logic, the truth values of variables may only be the integer values 0 or 1.[8, 15]

Fuzzy logic works with membership values in a way that mimics Crisp logic. To this end, replacements for basic operators and, or, not must be available. There are several ways to this. A common replacement is called the Zadeh operators [15]:

| | | | |
|-------------|---------------|---------------|-------------|
| Crisp logic | and(x, y) | or(x, y) | not (x) |
| Fuzzy logic | min(x, y) | max(x, y) | 1- x |

Let A and B be two fuzzy sets in D with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Then the union and intersection sets operations are defined in terms of their membership functions $\mu_A(x)$ as following,

Definition 1: The membership function of $A \cup B$ is denoted by $\mu_{A \cup B}(x)$, and defined pointwise for all $x \in D$ by,

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Definition 2: The membership function of $A \cap B$ is denoted by $\mu_{A \cap B}(x)$, and defined pointwise for all $x \in D$ by,

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Fuzzy logic is considered to be a basic control system that relies on the degrees of state of the input and the output depends on the state of the input and rate of change of this state. [5] In other words, a fuzzy logic system works on the principle of assigning a particular output (action)

depending on the probability of the state of the input. An interesting property is that the behaviour of a fuzzy system is not described using algorithms and formulas, but rather as a set of rules that may be expressed in natural language.

Numerical Steps:

1. Define the system of linear equations,
2. Use Gaussian Elimination method to reduce the coefficient matrix to upper triangular matrix (use a significant number of digits in the calculations),
3. From step 2, compute the inverse of the coefficient matrix, and the solution of the system of linear equations,
4. Compute the matrix and its inverse norms,
5. From step 4, compute the condition number of the coefficient matrix,
6. Compute the determinant of the coefficient matrix from step 2,
7. Consider the condition number and rate of singularity as input variables,
8. Define the descriptors of the input and output variables .
9. Define membership functions for each of input and output variables,
10. Form the logic rules based on the mathematical problem,
11. Evaluate the Rules,
12. Defuzzification,
13. Increase the number of computational digits used in solving the system of equations according to the rate of accuracy obtained from defuzzification in step 12.
14. Results

3. NUMERICAL COMPUTATIONS

According to figure one, we used MATLAB Simulink [10] features to describe and design a controller to determine the rate of accuracy of a system of linear equations $Ax=b$ solver as an output variable. Assume the input variables are the condition number $\kappa(A)$ and rate of singularity of the coefficient matrix. Four descriptor were used for the condition number, three descriptor for singularity rate input variables and five descriptor used for the output variable. The centroid defuzzification option and non-uniform triangular memberships in MATLAB [10] are used.

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The first input variable is considered to be the condition number of the coefficient matrix. Assume it's in % according to its largeness and defined as in the below table.

Table 1 The condition number rate approximate value.

| $\kappa(A)$ | $0 < \kappa(A) \leq 10^2$ | $10^2 < \kappa(A) \leq 10^4$ | $10^4 < \kappa(A) \leq 10^6$ | $10^6 < \kappa(A)$ |
|----------------------------------|---------------------------|------------------------------|------------------------------|--------------------|
| Crisp Condition No. Rate (CR) | $0\% < CR \leq 20\%$ | $20\% < CR \leq 40\%$ | $40\% < CR \leq 60\%$ | $CR > 60\%$ |

The fuzzy descriptor for the condition number worse rate variable is defined as:

SC: Small condition number,

MC: Medium condition number,

LC: Large condition number.

VLC: Very Large condition number.

The second input variable is considered to be the singularity rate of the coefficient matrix. Assume it's in % according to its worseness and defined as it's in the below table.

Table 2 The singularity rate approximate value.

| Determinant D(A) | $0 < D(A) \leq 10^{-2}$ | $10^{-2} < D(A) \leq 10^{-4}$ | $10^{-4} < D(A)$ |
|---------------------|-------------------------|-------------------------------|------------------|
| Singularity Rate | $0\% < SR \leq 20\%$ | $20\% < SR \leq 40\%$ | $40\% < SR$ |

The fuzzy descriptor for the singularity variable is defined as:

SS: Small singularity of the coefficient matrix,

MS: Medium singularity of the coefficient matrix,

LS: large singularity of the coefficient matrix.

The output variable is the action that should be taken due to the values of the inputs. The fuzzy descriptor for the accuracy action is defined as:

VSA: Very small increase in digits of calculations,

SA: A small increase in digits of calculations,

MA: Medium increase in digits of calculations,

XMA: Extra medium increase in digits of calculations,

LA: Large increase in digits of calculations,

VLA: Very increase in digits of calculations.

Fig. 1 The non-uniform triangular membership functions used to define the condition number input.

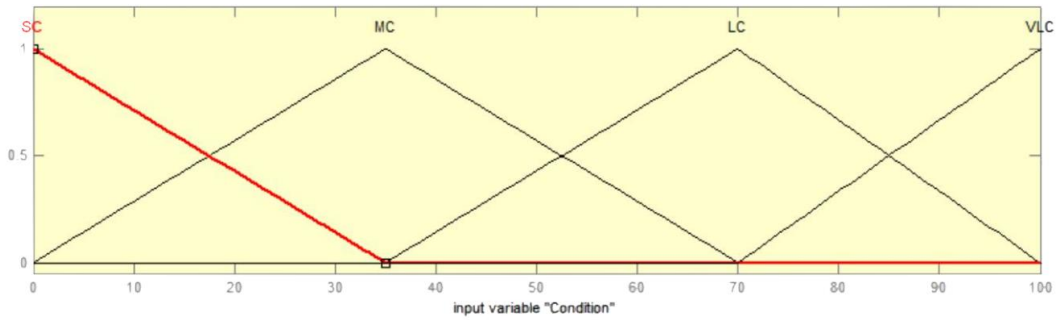


Fig. 2 The uniform triangular membership functions used to define the singularity input.

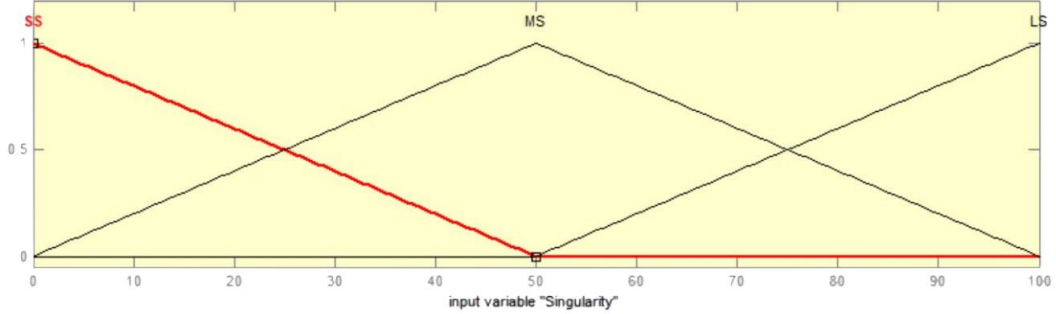


Fig. 3 The non-uniform triangular membership functions used to define the accuracy output.

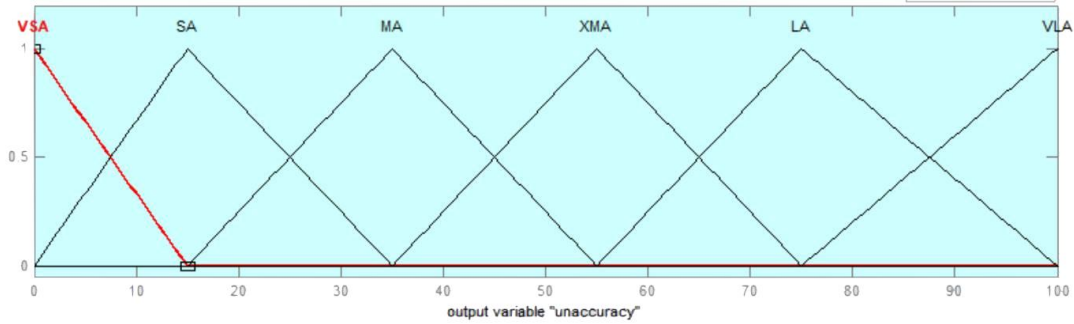


Fig. 4 A three dimensional surface figure for the input and output parameters

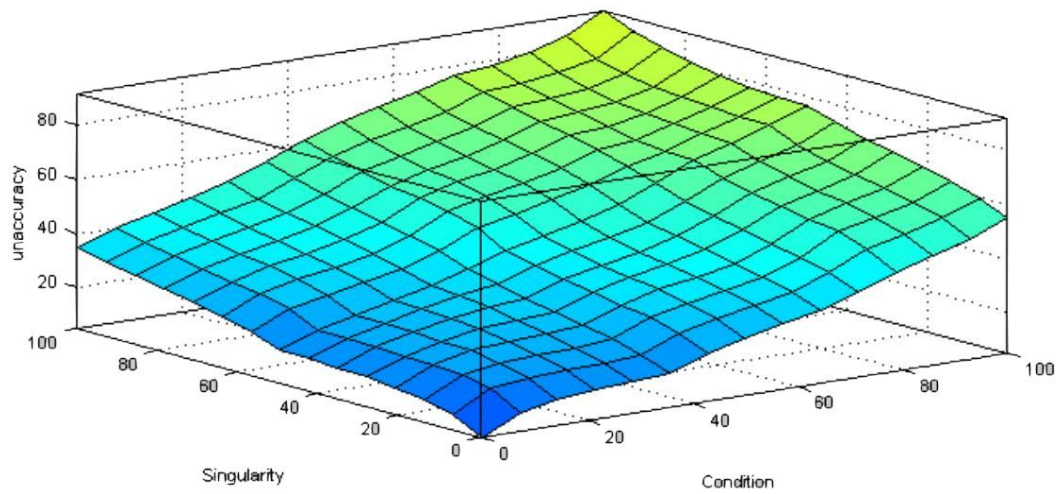


Table 3 The accuracy rate action output relative to the inputs .

| Condition No. Rate | Singularity Rate | Accuracy Rate Action | Condition No. Rate | Singularity Rate | Accuracy Rate Action |
|--------------------|------------------|----------------------|--------------------|------------------|----------------------|
| 10 | 30 | 22.5 | 45 | 60 | 48.9 |
| 25 | 30 | 27.5 | 70 | 80 | 68.6 |
| 40 | 40 | 35.1 | 75 | 65 | 63.9 |
| 30 | 35 | 32.3 | 80 | 20 | 53.3 |
| 65 | 40 | 46.6 | 80 | 75 | 68.8 |
| 34.8 | 65 | 41.6 | 35.2 | 65 | 41.8 |
| 50 | 49.7 | 43.9 | 50 | 50.3 | 44.1 |

Table 3 and fig. 4 show the accuracy action which should be taken according to the condition number and singularity rates. They indicate clearly that for large condition number rate with large singularity rate, a large accuracy action should be taken. For example, if the condition number and singularity rates 80% and 75% respectively, it's necessary to increase calculation digits by approximately 69%, while for the rates 80% and 20% , the action which should be taken is 53.3%, which is clearly less than the previous value due the lower rate of singularity. It's also clear that, for condition number rates 34.8 % and 35.2% with same singularity rate, approximately the same accuracy action should be taken although they belong to two separate intervals. While in crisp logic two completely different actions should be taken for almost the two close values.

Example

In this example the rate of accuracy will be evaluated for 60% as condition number worse rate and 30% as singularity rate. From figs. 1-3, the input and output membership functions μ 's can be represented mathematically as:

$$\mu_{SC}(x) = \frac{35-x}{35}, \quad 0 \leq x \leq 35,$$

$$\mu_{MC}(x) = \begin{cases} \frac{x}{35}, & 0 \leq x \leq 35, \\ \frac{70-x}{35}, & 35 \leq x \leq 70, \end{cases}$$

$$\mu_{LC}(x) = \begin{cases} \frac{x-35}{35}, & 35 \leq x \leq 70, \\ \frac{100-x}{30}, & 70 \leq x \leq 100, \end{cases}$$

$$\mu_{VLC}(x) = \frac{x-70}{30}, \quad 70 \leq x \leq 100.$$

$$\mu_{SS}(y) = \frac{50-y}{50}, \quad 0 \leq y \leq 50,$$

$$\mu_{MS}(y) = \begin{cases} \frac{y}{50}, & 0 \leq y \leq 50, \\ \frac{100-y}{50}, & 50 \leq y \leq 100, \end{cases}$$

$$\mu_{LS}(y) = \frac{y-50}{50}, \quad 50 \leq y \leq 100.$$

$$\mu_{VSA}(w) = \frac{15-w}{15}, \quad 0 \leq w \leq 15,$$

$$\mu_{SA}(x) = \begin{cases} \frac{w}{15}, & 0 \leq w \leq 15, \\ \frac{35-w}{20}, & 15 \leq w \leq 35, \end{cases}$$

$$\mu_{MA}(x) = \begin{cases} \frac{w-15}{20}, & 15 \leq w \leq 35, \\ \frac{55-w}{20}, & 35 \leq w \leq 55, \end{cases}$$

$$\mu_{XMA}(w) = \begin{cases} \frac{w-35}{20}, & 35 \leq w \leq 55, \\ \frac{75-w}{20}, & 55 \leq w \leq 75, \end{cases}$$

$$\mu_{LA}(w) = \begin{cases} \frac{w-55}{20}, & 55 \leq w \leq 75, \\ \frac{100-w}{25}, & 75 \leq w \leq 100, \end{cases}$$

$$\mu_{VL}(w) = \frac{w-75}{25}, \quad 75 \leq w \leq 100.$$

The fuzzy rules are considered as in table 4.

Table 4 The applied fuzzy rules.

| x \ y | SS | MS | LS |
|-------|-----|-----|-----|
| SC | VSA | SA | MA |
| MC | SA | MA | XMA |
| LC | MA | XMA | LA |
| VLC | XMA | LA | VLA |

Assume the condition number large by 60% , singularity by 30%

$$\mu_{MC}(x) = \frac{70-x}{35}, \quad \mu_{LC}(x) = \frac{x-35}{35},$$

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$$\begin{aligned}\mu_{SS}(y) &= \frac{50-y}{50}, & \mu_{MS}(y) &= \frac{y}{50}, \\ \mu_{MC}(60) &= \frac{70-60}{35} = \frac{2}{7}, & \mu_{LC}(60) &= \frac{60-35}{35} = \frac{5}{7}, \\ \mu_{SS}(30) &= \frac{50-30}{50} = \frac{2}{5}, & \mu_{MS}(30) &= \frac{y}{50} = \frac{3}{5},\end{aligned}$$

The above four equations leads to four rules need to evaluate

Rule 1: Condition number medium and small singularity,

Rule 2: Condition number medium and medium singularity,

Rule 3: Condition number large and small singularity,

Rule 4: Condition number large and medium singularity.

Since the antecedent part of each of the above rules are connected by and operator, we use the minimum operator to evaluate strength of each rule

$$\begin{aligned}\text{Strength of rule 1: } S1 &= \min(\mu_{MC}(60), \mu_{SS}(30)) \\ &= \min\left(\frac{2}{7}, \frac{2}{5}\right) = \frac{2}{7},\end{aligned}$$

$$\begin{aligned}\text{Strength of rule 2: } S2 &= \min(\mu_{MC}(60), \mu_{MS}(30)) \\ &= \min\left(\frac{2}{7}, \frac{3}{5}\right) = \frac{2}{7},\end{aligned}$$

$$\begin{aligned}\text{Strength of rule 3: } S3 &= \min(\mu_{LC}(60), \mu_{SS}(30)) \\ &= \min\left(\frac{5}{7}, \frac{2}{5}\right) = \frac{2}{5},\end{aligned}$$

$$\begin{aligned}\text{Strength of rule 4: } S4 &= \min(\mu_{LC}(60), \mu_{MS}(30)), \\ &= \min\left(\frac{5}{7}, \frac{3}{5}\right) = \frac{3}{5}\end{aligned}$$

Table 5 Fuzzy rules for condition number rate 60% and singularity rate by 30%.

| x \ y | SS | MS | LS |
|-------|---------------|---------------|-----|
| SC | VSA | SA | MA |
| MC | $\frac{2}{7}$ | $\frac{2}{7}$ | XMA |
| LC | $\frac{2}{5}$ | $\frac{3}{5}$ | LA |
| VLC | XMA | LA | VLA |

Since we used mean of maximum, the defuzzification technique is:

$$\begin{aligned} \text{Maximum strength} &= \text{Max}(S1, S2, S3, S4) \\ &= \frac{3}{5} \end{aligned}$$

Which correspond to rule 4, Condition number is large and singularity is medium and has maximum strength $\frac{3}{5}$. To find out the final defuzzified value, we now take average of $\mu_{XMA}(w)$,

$$\mu_{XMA}(w) = \begin{cases} \frac{w-35}{20}, & 35 \leq w \leq 55 \\ \frac{75-w}{20}, & 55 \leq w \leq 75 \end{cases}$$

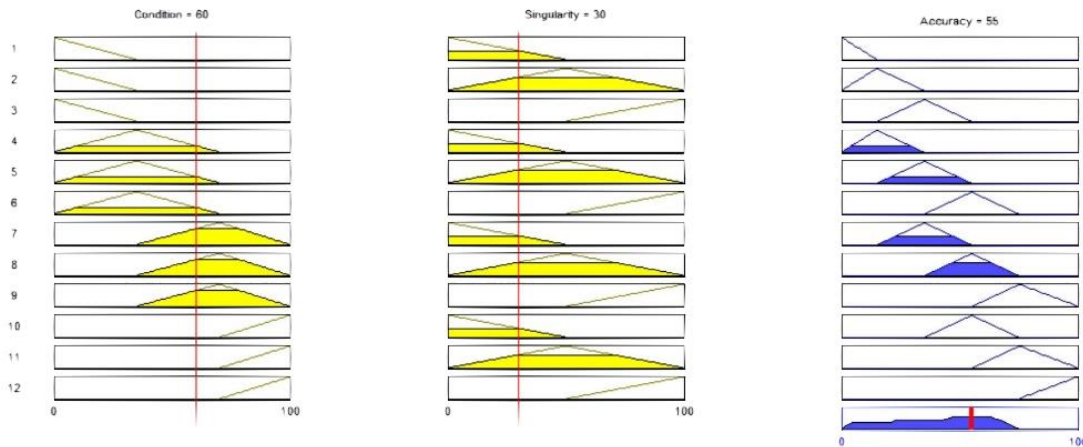
$$\frac{3}{5} = \frac{w-35}{20}, \quad \frac{3}{5} = \frac{75-w}{20}$$

$$w = 47, \quad w = 63$$

So the mean defuzzified value for the accuracy rate for the condition number rate 60% and singularity rate 30% is:

$$\frac{47+63}{2} = 55, \text{ which is the same results obtained by MATLAB in fig. 4.}$$

Fig. 5 MATLAB result for the required accuracy action at 60% and 30% inputs



4. CONCLUSIONS

In this paper, we presented a method which use fuzzy logic, condition numbers, rate of singularity and non-uniform triangular membership functions to approximate the accuracy of the solution of the system of linear equations $Ax=b$. The proposed method is applicable rather than the other existing methods which depends mainly on crisp values of the inputs. The presented method can introduce a new and fundamental change in dealing with mathematical fuzzy terms and vagueness phenomenon (ie. Large, small, very small,...etc.). It provided us with a tool to

avoid different actions for almost same input values. The method can be easily used and implemented in different mathematical and physical problems where there are fuzzy data.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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